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An Extension of a General Model for Matroids on Closure Spaces to Spaces

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Abstract: A matroid is one of the most important structures in combinatorics. Many researchers never stop the expansion and discussion for classical matroids. Recently, Faigle and Fujishige present a general model for matroids.
The general matroids of Faigle and Fujishige based on closure spaces and co-closure spaces respectively. This article will improve and generalize the general matroids of Faigle and Fujishige from closure spaces to that on spaces, and from co-closure spaces to co-spaces. As well known, most of spaces are not closure spaces and most of co-spaces are not co-closure spaces. Hence, all these works provide a cornerstone for the applications of the theory of the general model of Faigle and Fujishige.

Keywords: matroid, graded poset, Jordan-Dedekind chain condition, base,rank

1 Introduction and preliminaries

A matroid, one of the most important structures in combinatorics, was produced by Whitney [1] in 1935. It has been used in information system and information retrieval (see [2,3,4]), formal concept analysis (see [5,6, 7,8,9,10,11]), and so on (see. [12,13,14,15]). In order to generalize the applied fields of matroid and search out much more matroidal approaches for information system and data mining, we may first extent the model of matroids. Many researchers never stop the expansion and discussion for classical matroids (see [16,17,18]). Recently, Faigle and Fujishige present a general model for matroids in [19].

One of the important and central results [19] exists in [[19], Section 5]. As Section 5 in [19] shows, the general matroids are stood on closure spaces and the "dual" of the general matroids is based on co-closure spaces.

As well known, most of spaces are not closure spaces and most of co-spaces are not co-closure spaces. However, the general matroids and the "dual" model of Faigle and Fujishige based on closure spaces and co-closure spaces respectively. In addition, we find that the applied fields of the new model are seldom explored to date. To make the general model apply in much more fields, the first step is to improve and generalize the general model so as to satisfy the requirement in real life. This paper will general the central results in [[19], Section 5], that is [[19], Theorem 5.1] and [[19], Theorem 5.2], from closure spaces to spaces and from co-closure spaces to co-spaces. We hope that with the new matroids provided in this paper, some new methods may be found in the future.

Before presenting and recalling some preliminary knowledge needed later on, we assume throughout to be given on a finite ground set E.

We notice that a basis, the basic definition [19], is given as follows.

(4I) "Let $\mathscr{F}$ be a non-empty family of subsets of E. $\mathscr{F}$ is called *constructible* if for all $F \in \mathscr{F}$: (C) either $F = \emptyset$ or $F \setminus e \in \mathscr{F}$ for some $e \in F$.

Note that (C) implies $\emptyset \in \mathscr{F}$. For any $F \in \mathscr{F}$, we set $\Gamma(F) = \{e \in E \setminus F \mid F \cup e \in \mathscr{F}\}$, and call F a *basis* of $\mathscr{F}$ if $\Gamma(F) = \emptyset$."

According to (4I), [19] continues to give the following assertion.

(4II) "So the bases of $\mathscr{F}$ are exactly the (inclusion-wise) maximal members of $\mathscr{F}$."

We may infer that (4II) is not true. The reason is described as follows.

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We suppose $E = \{a,b,c,d,e\}$ and $\mathscr{F} = \{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,c,d\}, \{a,d\}\}$. Then $\mathscr{F}$ is a constructible family on E by the definition shown in (4I). Since $\Gamma(\{a,d\}) = \emptyset$ and $\Gamma(\{a,b,c,d\}) = \emptyset$, both $\{a,d\}$ and $\{a,b,c,d\}$ are bases of $\mathscr{F}$ in light of the definition of basis in [19] (see (4I)). But $\{a,d\}$ is not a maximal member of $\mathscr{F}$.

According to our analysis, this paper will define the basis as:

let $\mathscr{F}$ be a constructible. For any $F \in \mathscr{F}$, we call F a *basis* of $\mathscr{F}$ if F is a maximal (inclusion-wise) members of $\mathscr{F}$. Denote by $\mathscr{B} = \mathscr{B}(\mathscr{F})$ the collection of bases of $\mathscr{F}$.

In light of the discussion in [19], our definition of basis is reasonable and confident. It is not very difficult to prove that if the definition of basis is adopted with ours, the relative consequences in [19] are true and the relative results are not needed to be discussed and proved again. In other words, our definition of basis will not effect to our goal of this paper. Hence, we confirm that the definitions of basis in [19] and ours are a technically speaking. Additionally, the model and other results in [19] are great contributions to matroid theory. Therefore, we still use some definitions and properties in [19] such as the definition of constructible (see (4I)) and so on. We recall some basic facts as follows.

Definition 1.1. [19] Let $\mathscr{F}$ be a non-empty family of subsets of E and $\mathscr{H}$ be a family of subsets of E with $\emptyset, E \in \mathscr{H}$.

(1) The *restriction* of $\mathscr{F}$ to a subset $S \subseteq E$ is the family $\mathscr{F}(S) = \{F \in \mathscr{F} \mid F \subseteq S\}$.

(2) Let $\mathscr{F}$ be a constructible. $\mathscr{F}$ induces a *basis rank function* ρ on the collection of subsets of E via $\rho(S) = \max\limits_{B \in \mathscr{B}} |S \cap B| = \max\limits_{F \in \mathscr{F}} |S \cap F|$.

(3) A constructible family $\mathscr{F}$ is said to form an *independence system relative to* $\mathscr{H}$ (or an *$\mathscr{H}$-independence system*) if for all $H \in \mathscr{H}$, there exists some $F \in \mathscr{F}(H)$ such that $|F| = \rho(H)$.

An $\mathscr{H}$-independence system $\mathscr{F}$ is called an *$\mathscr{H}$-matroid* if for all $H \in \mathscr{H}$,

(M) all the bases B of the restriction $\mathscr{F}(H)$ have the same cardinality $|B| = \rho(H)$.

Lemma 1.1. [19] (1) A constructible family $\mathscr{F}$ is an $\mathscr{H}$-independence system if and only if $\rho(H) = \rho_H(H)$ for all $H \in \mathscr{H}$, where ρ_H is the basis rank function of $\mathscr{F}(H)$.

(2) Every restriction of a constructible family is constructible.

More detail for poset theory and lattice theory, please refer to [20]. The following lists some of them used frequently in the sequel.

Definition 1.2. [[20], p.5] In a poset P of finite length with 0, the *height* $h[x]$ of an element $x \in P$ is, by definition, the least upper bound of the lengths of the chains $0 = x_0 < x_1 < \ldots < x_l = x$ between 0 and x.

A poset P is a *graded poset* if there is a function $g : P \to \mathbb{N}$ from P to the chain of all integers (in their natural order) satisfies:

(G1) $x > y$ implies $g[x] > g[y]$.

(G2) If x covers y, then $g[x] = g[y] + 1$.

We may indicate that for a finite poset P, (G1) in Definition 1.2 is redundant because (G1) is followed from (G2). This only skillfully express. Hence, we accept Definition 1.2 in what follows.

Lemma 1.2. [[20], p.5] (1) Jordan-Dedekind chain condition: in a poset P, all maximal chains between the same endpoints have the same finite length. All graded poset satisfies Jordan-Dedekind chain condition.

(2) Let P be any poset with 0 in which all chains are finite. Then P satisfies the Jordan-Dedekind chain condition if and only if it is graded by $h[x]$.

2 Matroids on spaces

As stated above, this section extends [[19], Theorem 5.1, Theorem 5.2].

As a generalization of closure spaces, we give the definitions relative to spaces.

Definition 2.1. Let $\mathscr{T}$ be a family of subsets of E with $\emptyset, E \in \mathscr{T}$. The pair $(E, \mathscr{T})$ is called a *space*. We say that $Y \in \mathscr{T}$ *covers* $X \in \mathscr{T}$ in $(\mathscr{T}, \subseteq)$ (denoted $X \prec Y$) if $X \subset Y$ holds, and in addition, for all $Z \in \mathscr{T}$, there is $X \subset Z \subseteq Y \Rightarrow Z = Y$.

Remark 2.1. Comparing Definition 2.1 with the definition of a closure space in [[19], Section 5], we may easily confirm that a closure space is a space, but not vice versa.

Let $\mathscr{H}$ be a family of subsets of E with $\emptyset, E \in \mathscr{H}$. The authors point [19] that an $\mathscr{H}$-greedoid $(E, \mathscr{F})$ (i.e. if $\mathscr{F}$ is a constructible family with the equicardinality property (M)) can produce a system $(E, \widetilde{\mathscr{F}})$, where $\mathscr{B}$ is a base system, and $\widetilde{\mathscr{F}} = \{B \cap H \mid B \in \mathscr{B}, H \in \mathscr{H}\}$. In addition, even $\mathscr{H}$ is graded, $\widetilde{\mathscr{F}}$ is an $\mathscr{H}$-independence system, but not necessarily an $\mathscr{H}$-matroid. Evidently, this view is also true if it adopts our definition of basis.

Here, we provide a method in Subsection 2.1 to produce an $\mathscr{H}$-matroid only if $\mathscr{H}$ is graded. Additionally, we generalize the utilization of this idea to obtain matroids from rank function on spaces. Further, in Subsection 2.2, the similar results are provided for matroids on co-spaces.

2.1 Matroids from rank function

Let $\mathscr{H}$ be a family of subsets of E with $\emptyset, E \in \mathscr{H}$. Let $(\mathscr{H}, \subseteq)$ satisfy Jordan-Dedekind chain condition. Then by

Lemma 1.2, $(\mathscr{H},\subseteq)$ is a graded poset which is graded by its height function h. In addition, h satisfies the following (2.1.0) and (2.1.1). Let $X,Y\in\mathscr{H}$

(2.1.0) $h(X)=0\Leftrightarrow X=\emptyset$.

(2.1.1) $X\prec Y\Rightarrow h(Y)=h(X)+1$.

According to $\emptyset\in\mathscr{H}$, (2.1.0) and (2.1.1), we follow that for any $H\in\mathscr{H}$, if $h(H)=k$, then $k\le|H|$.

We define a family $\mathscr{F}$ of subsets of E associated with h by inductive way, in which every $F\in\mathscr{F}$ is presented by the following sketch of procedure:

Let $\mathscr{H}_k=\{M\in\mathscr{H}\mid h(M)=k\},(k=0,1,\dots,h(E))$.

(i) We define $\mathscr{F}_0=\{\emptyset\}$.

(ii) Let $\mathscr{H}_1=\{H_1,\dots,H_n\}$.

If $n=1$.

we find that there is $u_1\in H_1$. Hence, we define $\mathscr{F}_1=\{X\mid X=\{u_1\}\}$.

If $2\le n$.

To consider with $(\mathscr{H},\subseteq)$ satisfying Jordan-Dedekind chain condition, with inductive method, we may use inductive method easily prove that there are $u_{H_i}\in H_i,(i=1,2,\dots,n)$ satisfying $u_{H_p}\neq u_{H_q},(p\neq q;p,q=1,2,\dots,n)$.

Let $\mathscr{F}_{H_i}=\{X\mid X=\{u_{H_i}\}\},(i=1,2,\dots,n)$.

Therefore, we define $\mathscr{F}_1=\mathscr{F}_{H_1}\cup\dots\cup\mathscr{F}_{H_n}$.

(iii) Let $\mathscr{H}_k=\{M_1,\dots,M_t\}$ where $2\le k\le h(E)$.

Let $\mathfrak{A}_{M_i}=\{X\mid X\in\mathscr{H}$ and $X\prec M_i\}$

and $\mathscr{A}_{\mathfrak{A}_{M_i}}=\{F\in\mathscr{F}_{k-1}\mid F\subseteq Y$ for some $Y\in\mathfrak{A}_{M_i}\}$, $(i=1,\dots,t)$.

In fact, we may easily be assured $\mathfrak{A}_{M_i}=\{X\mid X\in\mathscr{H}_{k-1}$ and $X\prec M_i\}$, $(i=1,\dots,t)$.

We continue our definition for $\mathscr{F}$.

(iv) For any $i\in\{1,\dots,t\}$, we define $\mathscr{F}_{M_i}$ as follows.

If $M_i=\bigcup_{F\in\mathscr{A}_{\mathfrak{A}_{M_i}}}F$.

Since $h(M_i)=h(X)+1=|F|+1$ for any $X\in\mathfrak{A}_{M_i}$ and $F\in\mathscr{A}_{\mathfrak{A}_{M_i}}$,

we may easily decide that there is $|\mathscr{A}_{\mathfrak{A}_{M_i}}|\ge 2$. Hence,

let $\mathscr{F}_{M_i}=\{X\mid X=F_X\cup a$ for $F_X\in\mathscr{A}_{\mathfrak{A}_{M_i}}$ and $a\in\bigcup_{F\in\mathscr{A}_{\mathfrak{A}_{M_i}}}F\setminus F_X\}$.

If $M_i\neq\bigcup_{F\in\mathscr{A}_{\mathfrak{A}_{M_i}}}F$.

Then, we may easily decide that there is $a_{M_i}\in M_i\setminus\bigcup_{F\in\mathscr{A}_{\mathfrak{A}_{M_i}}}F$. Hence,

Let $\mathscr{F}_{M_i}=\{X\mid X=a_{M_i}\cup F,F\in\mathscr{A}_{\mathfrak{A}_{M_i}}\}$.

Therefore, we define $\mathscr{F}_k=\bigcup_{i=1}^{t}\mathscr{F}_{M_i}$.

(v) Combining (i)-(iv), we define $\mathscr{F}=\mathscr{F}_0\cup\mathscr{F}_1\cup\dots\cup\mathscr{F}_k\cup\dots\cup\mathscr{F}_{h(E)}$.

Next, for constructing algorithms and analyzing with $(E,\mathscr{F})$, we present a lemma.

Lemma 2.1. Let $\mathscr{H}$ be a family of subsets of E with $\emptyset,E\in\mathscr{H}$ and h be the height function of $(\mathscr{H},\subseteq)$. If $(\mathscr{H},\subseteq)$ satisfies Jordan-Dedekind chain condition. Then there is an $\mathscr{H}$-independence system $\mathscr{F}$ such that $M=(E,\mathscr{F})$ is an $\mathscr{H}$-matroid whose rank function agrees with h on $\mathscr{H}$.

Proof. It immediately follows from the definition $\mathscr{F}$ in the sketch of procedure above that $\mathscr{F}$ is constructible. In light of (2.1.1) and $(\mathscr{H},\subseteq)$ satisfying Jordan-Dedekind chain condition, we find that each basis B of the restriction $\mathscr{F}(H)$ has cardinality $|B|=h(H)$ for every $H\in\mathscr{H}$. This follows $|B|=\rho(H)$ for every $H\in\mathscr{H}$, where ρ is the basis rank function ρ on $\mathscr{F}$. Thus, we obtain that $\mathscr{F}$ is an $\mathscr{H}$-independence system. So, $\mathscr{F}$ has the property (M).

Additionally, for every $H\in\mathscr{H}$, $|B|=h(H)=\rho(H)$ holds according to the above discussion where B is a basis of $\mathscr{F}(H)$. Therefore, we demonstrate that the rank function of $(E,\mathscr{F})$ agrees with h on $\mathscr{H}$.

We find that the most important and central results in [19] are Theorem 5.1 and Theorem 5.2. If we extend [[19], Theorem 5.1] from closure space to space, we can obtain our goal. Essential to this aim is a function as follows. Let $\mathscr{H}$ be a family of subsets of E and $\emptyset,E\in\mathscr{H}$.

Let $r:\mathscr{H}\to\mathbb{N}$ satisfy: for all $X,Y\in\mathscr{H}$,

(2.1.2) $r(X)=0$ if $X=\emptyset$.

(2.1.3) $X\prec Y\Rightarrow r(X)\le r(Y)\le r(X)+1$.

Actually, if we define $r(X)\equiv 0$ for any $X\in\mathscr{H}$, then the function r on $\mathscr{H}$ satisfies (2.1.2) and (2.1.3). Therefore, we may state that the definition of r is effective.

Afterwards, we construct a family $\mathscr{D}$ of subsets of E associated with $(\mathscr{H},\subseteq)$ and r. Every member in $\mathscr{D}$ is presented in the sketch as an algorithmic procedure in the following.

Step 1. Let $\mathscr{H}=\{H_0=\emptyset,H_1,H_2,\dots,H_{n-1},H_n=E\}$, $\mathscr{D}=\{\emptyset\}$ and $j=0$.

Step 2. Let $j=j+1$.

Step 3. If $j>n$, then stop. Otherwise, go to Step 4.

Step 4. Let $\mathscr{S}_j=\{M\mid M$ is a maximal chain from $\emptyset$ to H_j in $(\mathscr{H},\subseteq)\}$ and $\mathscr{S}=\emptyset$.

Step 5. $\mathscr{S}=\mathscr{S}_j$.

Step 6. Choose $\mathfrak{M}\in\mathscr{S}$,

i.e. $\mathfrak{M}=\{H_0=A_0\prec A_1\prec\dots\prec A_{k_{\mathfrak{M}}}=H_j\}$, where $A_i\in\mathscr{H},(i=1,\dots,k_{\mathfrak{M}})$.

Step 7. Let $\mathfrak{O}_{H_j}=\mathfrak{M}$ and $s=k_{\mathfrak{M}}$.

Step 8. Choose $A_t\in\mathfrak{O}_{H_j}$ which satisfies $A_t\prec A_{t+1}\prec\dots\prec A_s,r(A_t)=r(A_s)$

but $r(A_{t-1})=r(A_s)-1$.

Step 9. Let $\mathscr{D}=\mathscr{D}\cup A_t$.

Step 10. Let $\mathfrak{O}_{H_j}=\mathfrak{O}_{H_j}\setminus\{A_{t+1}\prec\dots\prec A_s\}$. Let $s=t$.

Step 11. If $s\neq 0$, then go to Step 8. Otherwise, $\mathscr{S}=\mathscr{S}\setminus\mathfrak{M}$ and go to Step 12.

Step 12. If $\mathscr{S}\neq\emptyset$, then go to Step 6. Otherwise, go to Step 2.

We analyze with the above algorithmic procedure and obtain (α).

(α) It is easily seen $\emptyset \in \mathscr{D}$. For any $X, Y \in \mathscr{D}$, if $X \prec Y$ holds in $(\mathscr{D}, \subseteq)$, then $r(Y) = r(X) + 1$ hods. We may state that r on $(\mathscr{D}, \subseteq)$ satisfies (G2).

If $r : \mathscr{H} \to \mathbb{N}$ satisfies (2.1.2), (2.1.3) and the following expression

(2.1.4) $h(D) = r(D)$ for any $D \in \mathscr{D}$, where h is the height function of $(\mathscr{D}, \subseteq)$.

Then, considering with the property of r satisfying (G2), we indicate that $(\mathscr{D}, \subseteq)$ is graded by its height function. Thus, we decide that $(\mathscr{D}, \subseteq)$ satisfies Jordan-Dedekind chain condition in view of Lemma 1.2. Therefore, we obtain the following expression.

(2.1.5) for every $D \in \mathscr{D}$, all maximal chains from $\emptyset$ to D in $(\mathscr{D}, \subseteq)$ have the same cardinality.

Furthermore, we demonstrate that $(\mathscr{D}, \subseteq)$ satisfies the following (2.1.6) and (2.1.7). Let $X, Y \in \mathscr{D}$ and h be the height function of $(\mathscr{D}, \subseteq)$.

(2.1.6) $X \prec Y \Rightarrow h(X) + 1 = h(Y)$.

(2.1.7) for any $H \in \mathscr{H}$, there exists $Z \subseteq H$ satisfying $Z \in \mathscr{D}$ and $r(Z) = r(H)$.

Additionally, let $E \in \mathscr{D}$. If r satisfies (2.1.2), (2.1.3) and (2.1.4), then we also obtain the descriptions (β) and (γ).

(β) By Lemma 2.1, we find that there is a $\mathscr{D}$-independence system $\mathscr{F}_{\mathscr{D}}$ such that $M_{\mathscr{D}} = (E, \mathscr{F}_{\mathscr{D}})$ is a $\mathscr{D}$-matroid whose rank function agrees with r on $\mathscr{D}$.

Furthermore, according to Lemma 2.1, (2.1.4) and (2.1.5) and (2.1.6), we may be assured $r(D) = \rho(D) = h(D)$ for any $D \in \mathscr{D}$, where ρ is the basis rank function induced by $\mathscr{F}_{\mathscr{D}}$.

(γ) Let $H \in \mathscr{H} \setminus \mathscr{D}$. By (2.1.7), we obtain that there is a $D_H \in \mathscr{D}$ satisfying $D_H \subseteq H$ and $r(D_H) = r(H)$. Additionally, for any basis B in D_H, considered the construction of $\mathscr{F}_{\mathscr{D}}$, we may point that B is also a basis in H. Thus, we produce $\rho(H) = r(H) = r(D_H) = |B|$.

If $E \notin \mathscr{D}$. We will construct $\mathscr{D}'$ from $\mathscr{D}$ as the following description (ω).

(ω) Let $\mathscr{D}' = \mathscr{D} \cup E$ and define $r' : \mathscr{H} \to \mathbb{N}$ as:

$$r'(H) = \begin{cases} r(E) + 1, & \text{if } H = E \\ r(H), & \text{if } H \in \mathscr{H} \setminus E \end{cases}$$

In fact, if B is a maximal element in $(\mathscr{D}, \subseteq)$, then there is $r'(E) = r(B) + 1$. Thus, we can state that r' satisfies the expressions (2.1.2), (2.1.3) and (2.1.4) because r satisfies them.

Hence, considering the above discussion $(\alpha) - (\gamma)$, we find that there is a $\mathscr{D}'$-independence system $\mathscr{F}_{\mathscr{D}'}$ such that $M' = (E, \mathscr{F}_{\mathscr{D}'})$ is a $\mathscr{D}'$-matroid whose rank function agrees with r on $\mathscr{D}' \setminus E$ and $r(E) + 1$ on E.

$(\alpha) - (\gamma)$, (ω) and the item (3) in Definition 1.1 together will clearly provide the following result.

Theorem 2.1. Let $\mathscr{H}$ be a family of subsets of E with $\emptyset, E \in \mathscr{H}$. Let $r : \mathscr{H} \to \mathbb{N}$ satisfy (2.1.2), (2.1.3) and (2.1.4). Let $\mathscr{D}$ be produced in the above algorithmic procedure. Then

(1) If $E \in \mathscr{D}$.

Then, there is an $\mathscr{H}$-independence system $\mathscr{F}_{\mathscr{D}}$. In addition, $M = (E, \mathscr{F}_{\mathscr{D}})$ is an $\mathscr{H}$-matroid whose rank function agrees with r on $\mathscr{H}$.

(2) If $E \notin \mathscr{D}$.

Let $\mathscr{D}' = \mathscr{D} \cup E$. Then there is an $\mathscr{H}$-independence system $\mathscr{F}_{\mathscr{D}'}$. In addition, $M = (E, \mathscr{F}_{\mathscr{D}'})$ is an $\mathscr{H}$-matroid whose rank function agrees with r on $\mathscr{H} \setminus E$ and $r(E) + 1$ on E.

we may compare Theorem 2.1 with [[19], Theorem 5.1] as follows.

The pre-conditions in [[19], Theorem 5.1] are the following:

(**5.1**.1) $(E, \mathscr{H})$ is a closure space, i.e. $\mathscr{H}$ is intersection-closed.

(**5.1**.2) $(\mathscr{H}, \subseteq)$ is a lattice with $H_1 \wedge H_2 = H_1 \cap H_2$ and $H_1 \vee H_2 = \cap \{H \in \mathscr{H} | H_1 \cup H_2 \subseteq H\}$.

(**5.1**.3) The function $r : \mathscr{H} \to \mathbb{N}$ satisfies (2.1.2), (2.1.3) and

$(\clubsuit)$ $H_1 \prec (H_1 \vee H_2) \Rightarrow r(H_1 \wedge H_2) + r(H_1 \vee H_2) \leq r(H_1) + r(H_2)$.

However, the approaches presented in Theorem 2.1 are:

(**1**.1) $(E, \mathscr{H})$ is a space.

(**1**.2) The function $r : \mathscr{H} \to \mathbb{N}$ satisfies (2.1.2), (2.1.3) and (2.1.4).

Let $\mathscr{J} \subseteq \mathscr{H}$. We may easily explore that if $\mathscr{H}$ satisfies the condition $(\clubsuit)$, then $\mathscr{J}$ satisfies the condition (2.1.4); however, if $\mathscr{J}$ satisfies (2.1.4), then we can not confirm that $\mathscr{H}$ satisfies the condition $(\clubsuit)$. Combining this assertion with the above comparisons, we may obtain the following descriptions.

(δ) Theorem 2.1 holds for some spaces which do not have to be closure. But [[19], Theorem 5.1] holds for some closure spaces only.

(ε) Theorem 2.1 does not ask $(\mathscr{H}, \subseteq)$ to be a lattice. But [[19],Theorem 5.1] asks $(\mathscr{H}, \subseteq)$ to be a lattice.

(ζ) Theorem 1 points that the condition (2.1.4) only needs to satisfy by $\mathscr{D}$ which is a subset of $\mathscr{H}$. But the condition $(\clubsuit)$ in [[19], Theorem 5.1] must be satisfied by $\mathscr{H}$.

Summarizing the above, we confirm that Theorem 2.1 is an extension of [[19], Theorem 5.1]. Equivalently to say, Theorem 2.1 is a simpler and weaker version comparing with [[19], Theorem 5.1].

2.2 Matroids on co-spaces

We may easily reveal that [[19], Theorem 5.2] is not only a result about matroids on co-closure spaces, but also another main result in [19]. We will extend [[19], Theorem 5.2] to co-spaces. This is another goal of this paper.

First, we define co-space.

Definition 2.2 Let $\mathscr{C} = (E, \mathscr{T})$ be a space. Let $\mathscr{T}^* = \{E \setminus T \mid T \in \mathscr{T}\}$. Then the structure $\mathscr{C}^* = (E, \mathscr{T}^*)$ is a so-called *co-space*.

Remark 2.2. Comparing Definition 2.2 with the definition of a co-closure space in [[19], Section 5.2] and Remark 2.1, we may be assured that every co-closure space is a co-space, but not vice versa.

We may easily find that $(\mathscr{T}^*, \subseteq)$ is anti-isomorphic to the poset $(\mathscr{T}, \subseteq)$. Additionally, $(\mathscr{T}^*, \subseteq)$ is a space in view of Definition 2.1. Therefore, using [20], we may declare that the poset corresponding to $\mathscr{T}^*$ is Jordan-Dedekind if and only if the poset corresponding to $\mathscr{T}$ is.

Considering Theorem 2.1 with the above expression, we may easily obtain the following result.

Theorem 2.2. Let $(E, \mathscr{H}^*)$ be a co-space and $r : \mathscr{H}^* \to \mathbb{N}$ be a function with property (2.1.2), (2.1.3) and (2.1.4). Corresponding to $\mathscr{H}^*$ and r, let $\mathscr{D}$ be defined as the algorithmic procedure.

(1) If $E \in \mathscr{D}$. Then there is an $\mathscr{H}^*$-independence system $\mathscr{F}_{\mathscr{D}}$. In addition, $M = (E, \mathscr{F}_{\mathscr{D}})$ is an $\mathscr{H}^*$-matroid whose rank function agrees with r on $\mathscr{H}$.

(2) If $E \notin \mathscr{D}$.

Let $\mathscr{D}' = \mathscr{D} \cup E$. Then there is an $\mathscr{H}^*$-independence system $\mathscr{F}_{\mathscr{D}'}$ In addition, $M = (E, \mathscr{F}_{\mathscr{D}'})$ is an $\mathscr{H}^*$-matroid whose rank function agrees with r on $\mathscr{H}^* \setminus E$ and $r(E) + 1$ on E.

Similar to the discussions in regards the relationships between Theorem 2.1 and [[19], Theorem 5.1], we affirm that Theorem 2.2 is an extension of [[19], Theorem 5.2]; Theorem 2.2 is simpler and weaker than [[19], Theorem 5.2]. Additionally, from the view of obtaining Theorem 2.2 and the process of obtaining Theorem 5.2 in [19], we may describe that Theorem 2.2 is a “corollary” of Theorem 1, but [[19], Theorem 5.2] is not a “corollary” of [[19], Theorem 5.1].

Therefore, we may state that the results in this paper are the extensions that in [19] on closure spaces to spaces.

3 Conclusion

The authors [19] characterize $\mathscr{H}$-matroids with greedy algorithm. But they do not characterize closure spaces of $\mathscr{H}$-matroids, even matroids, with greedy algorithm. Perhaps, the key difficulty to solve this problem need to generalize closure spaces to spaces which is discussed in this paper. Hence, we hope discover the answer of the problem in the future work.

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