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Convergence Analysis of Jacobi Pseudo-Spectral Method for the Volterra Delay Integro-Differential Equations

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Abstract: The Jacobi pseudo-spectral method for the Volterra delay integro-differential equations of the second kind is proposed in this paper. We provide a rigorous error analysis for the proposed method, which indicates that the numerical errors (in the $L^2_{\omega_{\alpha,\beta}}$-norm and the L^∞-norm) will decay exponentially provided that the source function is sufficiently smooth. Numerical examples are given to illustrate the theoretical results.

Keywords: Volterra delay integro-differential equation, Pseudo-spectral method, Convergence.

1 Introduction

In this paper, we consider the Volterra delay integro-diffe rential equations of the form

$$y^{'}(x) = a(x)y(x)+b(x)y(qx)+c(x)+\int_0^x K_1(x,\zeta)y(\zeta)d\zeta + \int_0^{qx} K_2(x,\delta)y(\delta)d\delta, \ \ 0<x\leq T. \tag{1.1}$$

with the given initial condition $y(0) = y_0$, where the unknown function $y(x)$ is defined in $0<x\leq T<\infty$, $0<q<1$. $a(x), b(x), c(x)$ are three given functions and $K_1(x,s)$, $K_2(x,s)$ are two given kernels.

Equations of this type arise in scientific fields such as physics, biology, ecology, control theory and so on. Due to the wide application of these equations, they must be solved successfully with efficient numerical methods. For these problems, many numerical approaches can be applied directly, such as collocation methods, which have been provided [[10]-[11]], Sine-collocation method see, e.g., [13] and references therein. But the main approach used there is the spectral-collocation method which is similar to a finite-difference approach. Consequently, the corresponding error analysis is more tedious as it does not fit in a unified framework. However, with a finite-element type approach, as will be performed in this work, it is natural to put the approximation scheme under the general Jacobi-Galerkin type framework. As demonstrated in the recent book of Shen etc. [15], there is a unified theory with Jacobi polynomials to approximate numerical solutions for differential and integral equations. It is also rather straightforward to derive the pseudo-spectral method from the corresponding continuous version. The relevant convergence theories under the unified framework, as will be seen from Sects. 4, are cleaner and more reasonable than those obtained in [17].

The purpose of this work is to provide numerical methods for the Volterra delay integro-differential equations
based on pseudo-spectral methods. Spectral methods are a class of techniques used in applied mathematics and scientific computing to numerically solve certain partial differential equations (PDEs) (see e.g.([1],[2],[3])and the references therein), often involving the use of the Fast Fourier Transform. Where applicable, spectral methods have excellent error properties, with the so-called ”exponential convergence” being the fastest possible.

The paper is organized as follows. In Section 2, we introduce the Jacobi pseudo-spectral approaches for the Volterra delay integro-differential equations (1.1). Some

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preliminaries and useful lemmas are provided in Section 3. In Section 4, the convergence analysis is given. We prove the error estimates in the $L^2_{\omega_{\alpha,\beta}}$-norm and L^∞-norm. Numerical experiments are carried out in Section 5, which will be used to verify the theoretical results obtained in Section 4. The final section contains conclusions.

2 Jacobi pseudo-spectral method

In this section, we formulate the Jacobi pseudo-spectral schemes for problem (1.1). For this purpose, Let $\omega_{\alpha,\beta} = (1-t)^\alpha(1+t)^\beta$ be a weight function in the usual sense, for $\alpha,\beta > -1. J_k^{\alpha,\beta}(t),\ k = 0,1,\cdots,$ denote the Jacobi polynomials. The set of Jacobi polynomials $\{J_k^{\alpha,\beta}\}_{k=0}^\infty$ forms a complete $L^2_{\omega_{\alpha,\beta}}(-1,1)$-orthogonal system.

For the sake of applying the theory of orthogonal polynomials, we make the linear transformations $x = \frac{T(1+t)}{2}, \zeta = \frac{T(1+s)}{2}, \delta = \frac{T(1+\eta)}{2}$. Then the problem (1.1) becomes

$$u'(t) = \widetilde{a}(t)u(t) + \widetilde{b}(t)u(qt+q-1) + \widetilde{c}(t) + \int_{-1}^{t} \widetilde{K}_1(t,s)u(s)ds + \int_{-1}^{qt+q-1} \widetilde{K}_2(t,\eta)u(\eta)d\eta, \quad -1 < t \le 1. \tag{2.1}$$

where

$$\begin{aligned} &u(t) = y(\frac{T}{2}(1+t)),\ \widetilde{a}(t) = \frac{T}{2}a(\frac{T}{2}(1+t)),\\ &\widetilde{b}(t) = \frac{T}{2}b(\frac{T}{2}(1+t)),\ \widetilde{c}(t) = \frac{T}{2}c(\frac{T}{2}(1+t)),\\ &\widetilde{K}_1(t,s) = \frac{T^2}{4}K_1(\frac{T}{2}(1+t),\frac{T}{2}(1+s)),\\ &\widetilde{K}_2(t,\eta) = \frac{T^2}{4}K_2(\frac{T}{2}(1+t),\frac{T}{2}(1+\eta)).\end{aligned}$$

Before using pseudo-spectral methods, we need to restate problem (2.1). The usual way to deal with the original problem is: writing $v(t) = u'(t), w(t) = u(qt+q-1)$, (2.1) is equivalent to a linear Volterra integral equations of the second kind with respect to u, v, w.

$$\begin{cases} u(t) = y_0 + \int_{-1}^{t} v(s)ds,\\ w(t) = y_0 + q\int_{-1}^{t} v(qs+q-1)ds = y_0 + \int_{-1}^{qt+q-1} v(\lambda)d\lambda,\\ v(t) = g(t) + \int_{-1}^{t}(\widetilde{K}_1(t,s)u(s) + \widetilde{a}(t)v(s) + \overline{K}_2(t,s)w(s))ds\\ \qquad +q\int_{-1}^{t}\widetilde{b}(t)v(qs+q-1)ds. \end{cases} \tag{2.2}$$

where $\overline{K}_2(t,s) = q\widetilde{K}_2(t,qs+q-1), g(t) = y_0(\widetilde{a}(t)+\widetilde{b}(t)) + \widetilde{c}(t)$.

Now, let N be any positive integer and $\mathscr{P}_N(\Lambda)$ be the set of all algebraic polynomials of degree at most N. Obviously, the Jacobi polynomials $J_0^{\alpha,\beta}(t), J_1^{\alpha,\beta}(t),\cdots,J_N^{\alpha,\beta}(t)$ are the basis functions of $\mathscr{P}_N(\Lambda)$.

Next, we denote the collocation points by $\{t_i\}_{i=0}^N$, which is the set of (N+1) Jacobi Gauss point. We also define the Jacobi interpolating polynomial $I_N^{\alpha,\beta}v \in \mathscr{P}_N(\Lambda)$, satisfying

$$I_N^{\alpha,\beta}v(t_i) = v(t_i), \quad 0 \le i \le N.$$

It can be written as an expression of the form

$$I_N^{\alpha,\beta}v(t) = \sum_{i=0}^{N} v(t_i)F_i(t) \tag{2.3}$$

where $F_i(t)$ is the Lagrange interpolation basis function associated with the Jacobi collocation points $\{t_i\}_{i=0}^N$.

Now we describe the Jacobi pseudo-spectral Galerkin method. For this purpose,set $s = s(t,\theta) = \frac{t-1}{2} + \frac{t+1}{2}\theta, \quad \lambda = \lambda(t,\theta) = (\frac{q(t+1)}{2}-1) + \frac{q(t+1)}{2}\theta,\ \ \theta \in [-1,1]$. We define that

$$\mathscr{M}u(t) = \int_{-1}^{t} u(s)ds = \int_{-1}^{1}(\frac{t+1}{2})u(s(t,\theta))d\theta, \tag{2.4}$$

$$\overline{\mathscr{M}}u(t) = \int_{-1}^{t} \widetilde{a}(t)u(s)ds = \int_{-1}^{1}(\frac{t+1}{2})\widetilde{a}(t)u(s(t,\theta))d\theta \tag{2.5}$$

$$\begin{aligned}(\widetilde{\mathscr{M}}u)(t) &= \int_{-1}^{t} \widetilde{K}_1(t,s)u(s)ds\\ &= \int_{-1}^{1}(\frac{t+1}{2})\widetilde{K}_1(t,s(t,\theta))u(s(t,\theta))d\theta.\end{aligned} \tag{2.6}$$

$$\begin{aligned}\widehat{\mathscr{M}}u(t) &= \int_{-1}^{t} \overline{K}_2(t,s)u(s)ds\\ &= \int_{-1}^{1}(\frac{t+1}{2})\overline{K}_2(t,s(t,\theta))u(s(t,\theta))d\theta\end{aligned} \tag{2.7}$$

$$\begin{aligned}\mathscr{M}^1u(t) &= q\int_{-1}^{t} u(qs+q-1)ds = \int_{-1}^{qt+q-1} u(\lambda)d\lambda\\ &= \int_{-1}^{1} \frac{q(t+1)}{2}u(\lambda(t,\theta))d\theta\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\overline{\mathscr{M}}^1u(t) &= q\int_{-1}^{t} \widetilde{b}(t)u(qs+q-1)ds\\ &= \int_{-1}^{1} \frac{q(t+1)}{2}\widetilde{b}(t)u(\lambda(t,\theta))d\theta\end{aligned} \tag{2.9}$$

Using (N +1)-point Gauss-Jacobi quadrature formula with weight $\omega_{\alpha,\beta}$ to approximate (2.4)-(2.9) yields

$$\mathscr{M}u(t) \approx \mathscr{M}_N u(t) := \sum_{j=0}^{N}(\frac{t+1}{2})u(s(t,\theta_j))\omega_{-\alpha,-\beta}(\theta_j)\omega_j, \tag{2.10}$$

$$\overline{\mathscr{M}}u(t) \approx \overline{\mathscr{M}}_N u(t) := \sum_{j=0}^{N}(\frac{t+1}{2})\widetilde{a}(t)u(s(t,\theta_j))\omega_{-\alpha,-\beta}(\theta_j)\omega_j \tag{2.11}$$

$$\widetilde{\mathscr{M}}u(t) \approx \widetilde{\mathscr{M}}_N u(t): \\ = \sum_{j=0}^{N} (\frac{t+1}{2}) \widetilde{K}_1(t, s(t,\theta_j)) u(s(t,\theta_j)) \omega_{-\alpha,-\beta}(\theta_j)\omega_j \tag{2.12}$$

$$\widehat{\mathscr{M}}u(t) \approx \widehat{\mathscr{M}}_N u(t): \\ = \sum_{j=0}^{N} (\frac{t+1}{2}) \overline{K}_2(t, s(t,\theta_j)) u(s(t,\theta_j)) \omega_{-\alpha,-\beta}(\theta_j)\omega_j \tag{2.13}$$

$$\mathscr{M}^1 u(t) \approx \mathscr{M}_N^1 u(t) := \sum_{j=0}^{N} \frac{q(t+1)}{2} u(\lambda(t,\theta_j)) \omega_{-\alpha,-\beta}(\theta_j)\omega_j \tag{2.14}$$

and

$$\overline{\mathscr{M}}^1 u(t) \approx \overline{\mathscr{M}}_N^1 u(t) := \sum_{j=0}^{N} \frac{q(t+1)}{2} \widetilde{b}(t) u(\lambda(t,\theta_j)) \omega_{-\alpha,-\beta}(\theta_j)\omega_j \tag{2.15}$$

where $\{\theta_j\}_{j=0}^N$are the (N + 1)-degree Jacobi-Gauss points associated with $\omega_{\alpha,\beta}$, and $\{\omega_j\}_{j=0}^N$ are the corresponding Jacobi weights. On the other hand, instead of the continuous inner product, the discrete inner product will be implemented by the following equality,

$$(u,v)_N = \sum_{j=0}^{N} u(\theta_j) v(\theta_j) \omega_j. \tag{2.16}$$

As a result,

$$(u,v)_{\omega_{\alpha,\beta}} = (u,v)_N, \quad \text{if } uv \in \mathscr{P}_{2N}(\Lambda).$$

By the definition of $I_N^{\alpha,\beta}$, we have

$$(u,v)_N = (I_N^{\alpha,\beta} u, v)_N. \tag{2.17}$$

The Jacobi pseudo-spectral Galerkin method is to find

$$u_N(t) = \sum_{j=0}^{N} \widetilde{u}_j J_j^{\alpha,\beta}(t), \quad v_N(t) = \sum_{j=0}^{N} \widetilde{v}_j J_j^{\alpha,\beta}(t), \\ w_N(t) = \sum_{j=0}^{N} \widetilde{w}_j J_j^{\alpha,\beta}(t) \in \mathscr{P}_N(\Lambda),$$

such that

$$\begin{cases} (u_N, v_1)_N = (y_0 + \mathscr{M}_N v_N, v_1)_N \\ (w_N, v_2)_N = (y_0 + \mathscr{M}_N^1 v_N, v_2)_N \\ (v_N, v_3)_N = (g(t) + \widetilde{\mathscr{M}}_N u_N + \overline{\mathscr{M}}_N v_N \\ + \widehat{\mathscr{M}}_N w_N + \overline{\mathscr{M}}_N^1 v_N, v_3)_N, \end{cases} \tag{2.18}$$
$$\forall v_1, v_2, v_3 \in \mathscr{P}_N(\Lambda).$$

where $\{\widetilde{u}_j\}_{j=0}^N, \{\widetilde{v}_j\}_{j=0}^N$and $\{\widetilde{w}_j\}_{j=0}^N$ are determined by

$$\begin{cases} \sum_{j=0}^{N} \{(J_j^{\alpha,\beta}, J_i^{\alpha,\beta})_N\}\widetilde{u}_j - \sum_{j=0}^{N} (\mathscr{M}_N J_j^{\alpha,\beta}, J_i^{\alpha,\beta})_N \widetilde{v}_j = (y_0, J_i^{\alpha,\beta})_N \\ -\sum_{j=0}^{N} (\mathscr{M}_N^1 J_j^{\alpha,\beta}, J_i^{\alpha,\beta})_N \widetilde{v}_j + \sum_{j=0}^{N} \{(J_j^{\alpha,\beta}, J_i^{\alpha,\beta})_N\}\widetilde{w}_j = (y_0, J_i^{\alpha,\beta})_N \\ -\sum_{j=0}^{N} (\widetilde{\mathscr{M}}_N J_j^{\alpha,\beta}, J_i^{\alpha,\beta})_N \widetilde{u}_j + \sum_{j=0}^{N} (J_j^{\alpha,\beta} - \overline{\mathscr{M}}_N J_j^{\alpha,\beta} - \overline{\mathscr{M}}_N^1 J_j^{\alpha,\beta}, J_i^{\alpha,\beta})_N \widetilde{v}_j \\ -\sum_{j=0}^{N} (\widehat{\mathscr{M}}_N J_j^{\alpha,\beta}, J_i^{\alpha,\beta})_N \widetilde{w}_j = (g(t), J_i^{\alpha,\beta})_N. \end{cases} \tag{2.19}$$

Denoting

$$\widetilde{X} = [\widetilde{u}_0, \widetilde{u}_1, \cdots, \widetilde{u}_N, \widetilde{v}_0, \widetilde{v}_1, \cdots, \widetilde{v}_N, \widetilde{w}_0, \widetilde{w}_1, \cdots, \widetilde{w}_N]^\top, \tag{2.19}$$

yields a equation of the matrix form

$$A\widetilde{X} = g_N, \tag{2.20}$$

where

$$g_N(i) = \begin{cases} (y_0, J_i^{\alpha,\beta})_N, & 0 \le i \le N, \\ (y_0, J_{i-N-1}^{\alpha,\beta})_N, & N+1 \le i \le 2N+1, \\ (g(t), J_{i-2N-2}^{\alpha,\beta})_N, & 2N+2 \le i \le 3N+2. \end{cases}$$

$$A(i,j) = \begin{cases} (J_j^{\alpha,\beta}, J_i^{\alpha,\beta})_N, \ 0 \le i \le N, 0 \le j \le N, \\ 0, \ N+1 \le i \le 2N+1, 0 \le j \le N, \\ -(\widetilde{\mathscr{M}}_N J_j^{\alpha,\beta}, J_{i-2N-2}^{\alpha,\beta})_N, \\ 2N+2 \le i \le 3N+2, 0 \le j \le N, \\ -(\mathscr{M}_N J_{j-N-1}^{\alpha,\beta}, J_i^{\alpha,\beta})_N, \\ 0 \le i \le N, N+1 \le j \le 2N+1, \\ -(\mathscr{M}_N^1 J_{j-N-1}^{\alpha,\beta}, J_{i-N-1}^{\alpha,\beta})_N, \\ N+1 \le i \le 2N+1, N+1 \le j \le 2N+1, \\ (J_{j-N-1}^{\alpha,\beta} - (\overline{\mathscr{M}}_N + \overline{\mathscr{M}}_N^1) J_{j-N-1}^{\alpha,\beta}, J_{i-2N-2}^{\alpha,\beta})_N, \\ 2N+2 \le i \le 3N+2, N+1 \le j \le 2N+1, \\ 0, \ 0 \le i \le N, 2N+2 \le j \le 3N+2, \\ (J_{j-2N-2}^{\alpha,\beta}, J_{i-N-1}^{\alpha,\beta})_N, \\ N+1 \le i \le 2N+1, 2N+2 \le j \le 3N+2, \\ -(\widehat{\mathscr{M}}_N J_{j-2N-2}^{\alpha,\beta}, J_{i-2N-2}^{\alpha,\beta})_N, \\ 2N+2 \le i \le 3N+2, 2N+2 \le j \le 3N+2. \end{cases}$$

3 Some useful lemmas

We first introduce some Hilbert spaces. For simplicity, denote $\partial_t v(t) = (\partial/\partial_t)v(t)$, etc. For a nonnegative integer m, define

$$H^m_{\omega_{\alpha,\beta}}(-1,1) := \{v : \partial_t^k v(t) \in L^2_{\omega_{\alpha,\beta}}(-1,1), \ \ 0 \le k \le m\},$$

with the semi-norm and the norm as

$$|v|_{L^2_{\omega_{\alpha,\beta}}} = \|\partial_t^m v(t)\|_{L^2_{\omega_{\alpha,\beta}}}, \quad \|v\|_m = (\sum_{k=0}^{m} \|\partial_t^k v(t)\|^2_{L^2_{\omega_{\alpha,\beta}}})^{\frac{1}{2}},$$

respectively. It is convenient sometime to introduce the semi-norms

$$|v|_{H^{m,N}_{\omega_{\alpha,\beta}}(\Lambda)} = (\sum_{k=\min(m,N+1)}^{m} |\partial_t^k v(t)|_{L^2_{\omega_{\alpha,\beta}}(\Lambda)})^{\frac{1}{2}}.$$

For bounding some approximation error of Jacobi polynomials, we need the following nonuniformly-weighted Sobolev spaces:

$$H^m_{\omega_{\alpha,\beta},*}(-1,1) := \{v : \partial_t^k v(t) \in L^2_{\omega_{\alpha+k,\beta+k}}(-1,1), \ \ 0 \le k \le m\},$$

equipped with the inner product and the norm as

$$(u,v)_{m,*} = \sum_{k=0}^{m} (\partial_t^k u, \partial_t^k v)_{\omega_{\alpha+k,\beta+k}}, \quad \|v\|_{m,*} = \sqrt{(v,v)_{m,*}}.$$

Next, we define the orthogonal projection $P_N : L^2(\Lambda) \longrightarrow \mathscr{P}_N(\Lambda)$ as

$$(u - P_N u, v) = 0, \quad \forall v \in \mathscr{P}_N(\Lambda).$$

P_N possesses the following approximation properties ((5.4.11), (5.4.12) and (5.4.24) on pp. 283-287 in Ref. ([6]):

$$\|u - P_N u\|_{L^2(\Lambda)} \le cN^{-m}\|u\|_{H^m(\Lambda)} \tag{3.1}$$

and

$$\|u - P_N u\|_{L^\infty} \le cN^{\frac{3}{4}-m}\|u\|_{m,\infty} \tag{3.2}$$

We have the following optimal error estimate for the interpolation polynomials based on the Jacobi Gauss points(cf.[17]).

Lemma 3.1 For any function v satisfying $v \in H^m_{\omega_{\alpha,\beta},*}(-1,1)$, we have

$$\|v - I_N^{\alpha,\beta} v\|_{L^2_{\omega_{\alpha,\beta}}(\Lambda)} \le cN^{-m}\|\partial_t^m v\|_{L^2_{\omega_{\alpha+m,\beta+m}}}, \tag{3.3}$$

for the Jacobi Gauss points and Jacobi Gauss-Radau points.

Lemma 3.2 If $v \in H^m_{\omega_{\alpha,\beta},*}(-1,1)$, for some $m \ge 1$ and $\phi \in \mathscr{P}_N(\Lambda)$, then for the Jacobi Gauss and Jacobi Gauss-Radau integration we have(cf.[17])

$$\begin{aligned} |(v,\phi)_{\omega_{\alpha,\beta}} - (v,\phi)_N| &\le \|v - I_N^{\alpha,\beta} v\|_{L^2_{\omega_{\alpha,\beta}}} \|\phi\|_{L^2_{\omega_{\alpha,\beta}}} \\ &\le cN^{-m}\|\partial_t^m v\|_{L^2_{\omega_{\alpha+m,\beta+m}}} \|\phi\|_{L^2_{\omega_{\alpha,\beta}}}. \end{aligned} \tag{3.4}$$

We have the following result on the Lebesgue constant for the Lagrange interpolation polynomials associated with the zeros of the Jacobi polynomials; (cf.[17]).

Lemma 3.3 Let $\{F_j(t)\}_{j=0}^N$ be the N-th Lagrange interpolation polynomials associated with the Gauss, or Gauss-Radau, or Gauss-Lobatto points of the Jacobi polynomials. Then

$$\begin{aligned} \|I_N^{\alpha,\beta}\|_{L^\infty} &:= \max_{t\in[-1,1]} \sum_{j=0}^N |F_j(t)| \\ &= \begin{cases} c\log N & -1 < \alpha,\beta \le -\frac{1}{2}, \\ cN^{\gamma+\frac{1}{2}}, & \gamma = \max(\alpha,\beta), \text{otherwise}. \end{cases} \end{aligned} \tag{3.5}$$

We now introduce some notation. For $r \ge 0$ and $\kappa \in [0,1]$, $C^{r,\kappa}([-1,1])$will denote the space of functions whose r-th derivatives are Hölder continuous with exponent κ, endowed with the usual norm $||\cdot||_{r,\kappa}$. When $\kappa = 0, C^{r,0}([-1,1])$ denotes the space of functions with r continuous derivatives on [-1, 1], also denoted by $C^r([-1,1])$, and with norm $||\cdot||_r$.

We will make use of a result of Ragozin ([7],[8]), which states that,for each nonnegative integer r and $\kappa \in [0,1]$, there exists a constant $C_{r,\kappa} > 0$ such that for any function $v \in C^{r,\kappa}([-1,1])$, there exists a polynomial function $\tau_N v \in \mathscr{P}_N$ such that

$$\|v - \tau_N v\|_{L^\infty} \le C_{r,\kappa} N^{-(r+\kappa)}\|v\|_{r,\kappa}. \tag{3.6}$$

where $||\cdot||_\infty$ is the norm of the space $L^\infty([-1,1])$, and when the function $v \in C([-1,1])$. Actually,τ_N is a linear operator from $C^{r,\kappa}([-1,1])$ to $\mathscr{P}_N$.

We will need the fact that$\widetilde{\mathscr{M}}$,which be defined by (2.6), is compact as an operator from $C([-1,1])$ to $C^{r,\kappa}([-1,1])$ for any $0 < \kappa < 1$. (see [18].)

Lemma 3.4 Let $0 < \kappa < 1$. then, for any function $v \in C([-1,1])$,there exists a positive constant C such that

$$\frac{|\widetilde{\mathscr{M}}v(t') - \widetilde{\mathscr{M}}v(t'')|}{|t'-t''|^\kappa} \le c \max_{-1\le t\le 1}|v(t)|.$$

Proof. We only need to prove that $\widetilde{\mathscr{M}}$ is Hölder continuous. For any$t', t'' \in [-1,1]$ and $t' \ne t''$,

$$\frac{|\widetilde{\mathscr{M}}v(t') - \widetilde{\mathscr{M}}v(t'')|}{|t'-t''|^\kappa} = \frac{|\int_{t'}^{t''}\widetilde{K}_1 v(\theta)d\theta - \int_0^{t'}\partial_t\widetilde{K}_1(t'-t'')v(\theta)d\theta)|}{|t'-t''|^\kappa} \le c\max_{-1\le t\le 1}|v(t)|.$$

This implies that

$$\|\widetilde{\mathscr{M}}v\|_{0,\kappa} \le C\|v\|_{L^\infty}, \quad 0 < \kappa < 1. \tag{3.7}$$

Clearly, $\mathscr{M}, \mathscr{M}^1, \overline{\mathscr{M}}, \overline{\mathscr{M}}^1$ and $\widehat{\mathscr{M}}$ also satisfy (3.7).

To prove the error estimate, we will apply the standard Gronwall Lemma. We call such a function $v = v(t)$ locally integrable on the interval $[-1,1]$ if for each $t \in [-1,1]$, its Lebesgue integral $\int_{-1}^1 v(s)ds$ is finite.

Lemma 3.5 Suppose that $v(t), w_*(t)$ are nonnegative and

$$v(t) \le w_*(t) + c\int_{-1}^t v(s)ds, \quad t \in [-1,1]$$

Then

$$v(t) \le w_*(t) + \widetilde{c}\int_{-1}^t w_*(s)ds, \quad t \in [-1,1].$$

Due to Lemma 3.5, we have the following Lemma.

Lemma 3.6 Suppose that $E(t), H(t)$ are nonnegative and $0 < q < 1$,

$$E(t) \le H(t) + c_1\int_{-1}^t E(s)ds + c_2\int_{-1}^{qt+q-1} E(s)ds, \quad t \in [-1,1]$$

Then

$$E(t) \le H(t) + c\int_{-1}^t H(s)ds, \quad t \in [-1,1].$$

Proof. Obviously,

$$\int_{-1}^{qt+q-1} E(\theta)d\theta \le \int_{-1}^t E(\theta)d\theta$$

which implies that

$$E(t) \le H(t) + (c_1 + c_2)\int_{-1}^t E(s)ds$$

This leads to desired result with the help of Lemma 3.5.

In our analysis, we will need the following estimate for the Lagrange interpolation associated with the Jacobi Gaussian collocation points.

Lemma 3.7 For every bounded function v, there exists a constant C independent of v such that

$$\|I_N^{\alpha,\beta}v(t)\|_{L^\infty} = \|\sum_{j=0}^{N}|v(t_j)F_j(t)|\|_{L^\infty} \leq \begin{cases} c\log N\|v\|_{L^\infty} & -1<\alpha,\beta\leq -\frac{1}{2}, \\ cN^{\gamma+\frac{1}{2}}\|v\|_{L^\infty}, & \gamma=\max(\alpha,\beta), \text{otherwise.} \end{cases}$$

where $F_j(t)$ is the Lagrange interpolation basis function associated with the Jacobi collocation points $\{t_j\}_{j=0}^N$.

Proof. It is obvious that

$$\|I_N^{\alpha,\beta}v(t)\|_{L^\infty} = \|\sum_{j=0}^{N}|v(t_j)F_j(t)|\|_{L^\infty} \leq \max_{t\in[-1,1]}\sum_{j=0}^{N}|v(t_j)||F_j(t)| \leq (\max_{t\in[-1,1]}\sum_{j=0}^{N}|F_j(t)|)\|v\|_{L^\infty}.$$

By the Lemma 3.3, we obtain the desired result.

Lemma 3.8 For every bounded function v, there exists a constant C independent of v such that

$$\|I_N^{\alpha,\beta}v(t)\|_{L^2_{\omega_{\alpha,\beta}}} \leq c\|v\|_{L^\infty},$$

where $F_j(t)$ is the Lagrange interpolation basis function associated with the Jacobi collocation points $\{t_j\}_{j=0}^N$.

Proof. It is obvious that

$$\|I_N^{\alpha,\beta}v(t)\|^2_{L^2_{\omega_{\alpha,\beta}}} = \int_{-1}^{1}(I_N^{\alpha,\beta}v)^2\omega^{\alpha,\beta}dt = \sum_{j=0}^{N}v^2(t_j)\omega_j \leq \|v\|^2_{L^\infty}\sum_{j=0}^{N}\omega_j = \gamma_0\|v\|^2_{L^\infty}.$$

where $\gamma_0 = c_0(J_0^{\alpha,\beta},J_0^{\alpha,\beta})_{\omega_{\alpha,\beta}}$. As a consequence,

$$\sup_N\|I_N^{\alpha,\beta}v(t)\|_{L^2_{\omega_{\alpha,\beta}}} \leq C\|v\|_{L^\infty},$$

with $C=\sqrt{\gamma_0}$.

4 Convergence for Jacobi pseudo-spectral method

As $I_N^{\alpha,\beta}$ is the interpolation operator which is based on the (N + 1)-degree Jacobi-Gauss points with weight $\omega_{\alpha,\beta}$, in terms of (2.16) and (2.17), the pseudo-spectral Galerkin solution u_N, v_N, w_N satisfies

$$\begin{cases} (u_N,v_1)_{\omega_{\alpha,\beta}} - (I_N^{\alpha,\beta}\mathscr{M}_N v_N, v_1)_{\omega_{\alpha,\beta}} = (I_N^{\alpha,\beta}y_0, v_1)_{\omega_{\alpha,\beta}}, \\ (w_N,v_2)_{\omega_{\alpha,\beta}} - (I_N^{\alpha,\beta}\mathscr{M}_N^1 v_N, v_2)_{\omega_{\alpha,\beta}} = (I_N^{\alpha,\beta}y_0, v_2)_{\omega_{\alpha,\beta}}, \\ (v_N - I_N^{\alpha,\beta}(\widetilde{\mathscr{M}}_N u_N + \overline{\mathscr{M}}_N v_N + \widehat{\mathscr{M}}_N w_N + \overline{\mathscr{M}}_N^1 v_N), v_3)_{\omega_{\alpha,\beta}} \\ = (I_N^{\alpha,\beta}g(t), v_3)_{\omega_{\alpha,\beta}}, \qquad \forall v_1,v_2,v_3 \in \mathscr{P}_N(\Lambda). \end{cases} \tag{4.1}$$

where

$$\mathscr{M}_N v_N = \mathscr{M}v_N - (\mathscr{M}v_N - \mathscr{M}_N v_N) = \mathscr{M}v_N - Q(t),$$

with

$$\begin{aligned} Q(t) &= \mathscr{M}v_N - \mathscr{M}_N v_N = \int_{-1}^{1}(\frac{t+1}{2})v_N(s(t,\theta))d\theta \\ &- \sum_{j=0}^{N}(\frac{t+1}{2})v_N(s(t,\theta_j))\omega_{-\alpha,-\beta}(\theta_j)\omega_j \\ &= ((\frac{t+1}{2})\omega_{-\alpha,-\beta}, v_N(s(t,\cdot)))_{\omega_{\alpha,\beta}} \\ &- ((\frac{t+1}{2})\omega_{-\alpha,-\beta}, v_N(s(t,\cdot)))_N, \end{aligned} \tag{4.2}$$

$$\begin{aligned} Q_1(t) &= (\frac{q(t+1)}{2}\omega_{-\alpha,-\beta}, v_N(\lambda(t,\cdot)))_{\omega_{\alpha,\beta}} \\ &- (\frac{q(t+1)}{2}\omega_{-\alpha,-\beta}, v_N(\lambda(t,\cdot)))_N, \end{aligned}$$

in which $(\cdot,\cdot)_{\omega_{\alpha,\beta}}$ represents the continuous inner product with respect to θ, and $(\cdot,\cdot)_N$ is the corresponding discrete inner product defined by the Gauss-Jacobi quadrature formula. Similar to (4.2),we have that

$$\overline{\mathscr{M}}_N v_N = \overline{\mathscr{M}}v_N - (\overline{\mathscr{M}}v_N - \overline{\mathscr{M}}_N v_N) = \overline{\mathscr{M}}v_N - \overline{Q}(t),$$

with

$$\begin{aligned} \overline{Q}(t) &= \overline{\mathscr{M}}v_N - \overline{\mathscr{M}}_N v_N = \int_{-1}^{1}(\frac{t+1}{2})\widetilde{a}(t)v_N(s(t,\theta))d\theta \\ &- \sum_{j=0}^{N}(\frac{t+1}{2})\widetilde{a}(t)v_N(s(t,\theta_j))\omega_{-\alpha,-\beta}(\theta_j)\omega_j \\ &= ((\frac{t+1}{2})a(t)\omega_{-\alpha,-\beta}, v_N(s(t,\cdot)))_{\omega_{\alpha,\beta}} \\ &- ((\frac{t+1}{2})\widetilde{a}(t)\omega_{-\alpha,-\beta}, v_N(s(t,\cdot)))_N, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \overline{Q}_1(t) &= (\frac{q(t+1)}{2}\widetilde{b}(t)\omega_{-\alpha,-\beta}, v_N(\lambda(t,\cdot)))_{\omega_{\alpha,\beta}} \\ &- (\frac{q(t+1)}{2}\widetilde{b}(t)\omega_{-\alpha,-\beta}, v_N(\lambda(t,\cdot)))_N. \end{aligned}$$

$$\widetilde{\mathscr{M}}_N u_N = \widetilde{\mathscr{M}}u_N - (\widetilde{\mathscr{M}}u_N - \widetilde{\mathscr{M}}_N u_N) = \widetilde{\mathscr{M}}u_N - \widetilde{Q}(t),$$

with

$$\begin{aligned} \widetilde{Q}(t) &= \widetilde{\mathscr{M}}u_N - \widetilde{\mathscr{M}}_N u_N = \int_{-1}^{1}(\frac{t+1}{2})\widetilde{K}_1(t,s(t,\theta))u_N(\theta)d\theta \\ &- \sum_{j=0}^{N}(\frac{t+1}{2})\widetilde{K}_1(t,\theta_j)\omega_{-\alpha,-\beta}(\theta_j)u_N(s(t,\theta_j))\omega_j \\ &= \{((\frac{t+1}{2})\widetilde{K}_1(t,s(t,\cdot))\omega_{-\alpha,-\beta}(\cdot), u_N(s(t,\cdot)))_{\omega_{\alpha,\beta}} \\ &- ((\frac{t+1}{2})\widetilde{K}_1(t,s(t,\cdot))\omega_{-\alpha,-\beta}(\cdot), u_N(s(t,\cdot)))_N\}, \end{aligned} \tag{4.4}$$

and

$$\widehat{\mathscr{M}}_N w_N = \widehat{\mathscr{M}}w_N - (\widehat{\mathscr{M}}w_N - \widehat{\mathscr{M}}_N w_N) = \widehat{\mathscr{M}}w_N - \widehat{Q}(t),$$

with

$$\widehat{Q}(t)=\widehat{\mathscr{M}}w_N-\widehat{\mathscr{M}}_N w_N=\int_{-1}^{1}(\frac{t+1}{2})\overline{K}_2(t,\theta)w_N(\theta)d\theta$$
$$-\sum_{j=0}^{N}(\frac{t+1}{2})\overline{K}_2(t,s(t,\theta_j))w_N(s(t,\theta_j))\omega_{-\alpha,-\beta}(\theta_j)\omega_j$$
$$=((\frac{t+1}{2})\overline{K}_2(t,s(t,\cdot))\omega_{-\alpha,-\beta},w_N(s(t,\cdot)))_{\omega_{\alpha,\beta}}$$
$$-((\frac{t+1}{2})\overline{K}_2(t,s(t,\cdot))\omega_{-\alpha,-\beta},w_N(s(t,\cdot)))_N, \tag{4.5}$$

The combination of (4.1)-(4.5) yields

$$\begin{cases}(u_N+I_N^{\alpha,\beta}Q(t)-I_N^{\alpha,\beta}\mathscr{M}v_N,v_1)_{\omega_{\alpha,\beta}}=(I_N^{\alpha,\beta}y_0,v_1)_{\omega_{\alpha,\beta}},\\(w_N+I_N^{\alpha,\beta}Q_1(t)-I_N^{\alpha,\beta}\mathscr{M}^1v_N,v_2)_{\omega_{\alpha,\beta}}=(I_N^{\alpha,\beta}y_0,v_2)_{\omega_{\alpha,\beta}},\\(v_N+I_N^{\alpha,\beta}\widetilde{Q}(t)-I_N^{\alpha,\beta}\widetilde{\mathscr{M}}u_N\\+I_N^{\alpha,\beta}\overline{Q}(t)-I_N^{\alpha,\beta}\overline{\mathscr{M}}v_N+I_N^{\alpha,\beta}\widehat{Q}(t)-I_N^{\alpha,\beta}\widehat{\mathscr{M}}w_N\\+I_N^{\alpha,\beta}\overline{Q}_1(t)-I_N^{\alpha,\beta}\overline{\mathscr{M}}^1v_N,v_3)_{\omega_{\alpha,\beta}}=(I_N^{\alpha,\beta}g(t),v_3)_{\omega_{\alpha,\beta}}.\end{cases}$$

which gives rise to

$$\begin{cases}u_N+I_N^{\alpha,\beta}Q(t)-I_N^{\alpha,\beta}\mathscr{M}v_N=I_N^{\alpha,\beta}y_0,\\w_N+I_N^{\alpha,\beta}Q_1(t)-I_N^{\alpha,\beta}\mathscr{M}^1v_N=I_N^{\alpha,\beta}y_0,\\v_N+I_N^{\alpha,\beta}\widetilde{Q}(t)-I_N^{\alpha,\beta}\widetilde{\mathscr{M}}u_N+I_N^{\alpha,\beta}\overline{Q}(t)\\-I_N^{\alpha,\beta}\overline{\mathscr{M}}v_N+I_N^{\alpha,\beta}\widehat{Q}(t)-I_N^{\alpha,\beta}\widehat{\mathscr{M}}w_N\\+I_N^{\alpha,\beta}\overline{Q}_1(t)-I_N^{\alpha,\beta}\overline{\mathscr{M}}^1v_N=I_N^{\alpha,\beta}g(t).\end{cases} \tag{4.6}$$

By the discussion above, (2.18), (4.1) and (4.6) are equivalent.

We first consider an auxiliary problem. We want to find $\widehat{u}_N,\widehat{v}_N,\widehat{w}_N\in\mathscr{P}_N(\Lambda)$ such that

$$\begin{cases}(\widehat{u}_N,v_1)_N-(\mathscr{M}\widehat{v}_N,v_1)_N=(y_0,v_1)_N,\\(\widehat{w}_N,v_2)_N-(\mathscr{M}^1\widehat{v}_N,v_2)_N=(y_0,v_2)_N\\(\widehat{v}_N-\widetilde{\mathscr{M}}\widehat{u}_N-\overline{\mathscr{M}}\widehat{v}_N-\widehat{\mathscr{M}}\widehat{w}_N-\overline{\mathscr{M}}^1\widehat{v}_N,v_3)_N\\=(g(t),v_3)_N,\ \forall v_1,v_2,v_3\in\mathscr{P}_N(\Lambda)\end{cases} \tag{4.7}$$

where $\mathscr{M},\overline{\mathscr{M}},\widetilde{\mathscr{M}}$ and $\widehat{\mathscr{M}}$ are the integral operators defined in Sect 2, and $(\cdot,\cdot)_N$ is still the discrete inner product based on the (N +1)-degree Jacobi-Gauss points. In terms of the definition of $I_N^{\alpha,\beta}$, (4.7) can be written as

$$\begin{cases}\widehat{u}_N-I_N^{\alpha,\beta}\mathscr{M}\widehat{v}_N=y_0,\ \ \widehat{w}_N-I_N^{\alpha,\beta}\mathscr{M}^1\widehat{v}_N=y_0,\\\widehat{v}_N-I_N^{\alpha,\beta}\widetilde{\mathscr{M}}\widehat{u}_N-I_N^{\alpha,\beta}\overline{\mathscr{M}}\widehat{v}_N-I_N^{\alpha,\beta}\widehat{\mathscr{M}}\widehat{w}_N\\-I_N^{\alpha,\beta}\overline{\mathscr{M}}^1\widehat{v}_N=I_N^{\alpha,\beta}g(t).\end{cases} \tag{4.8}$$

When $y_0=g=0$, (4.8) can be written as

$$\begin{cases}\widehat{u}_N-I_N^{\alpha,\beta}\mathscr{M}\widehat{v}_N=0,\ \ \widehat{w}_N-I_N^{\alpha,\beta}\mathscr{M}^1\widehat{v}_N=0,\\\widehat{v}_N-I_N^{\alpha,\beta}\widetilde{\mathscr{M}}\widehat{u}_N-I_N^{\alpha,\beta}\overline{\mathscr{M}}\widehat{v}_N-I_N^{\alpha,\beta}\widehat{\mathscr{M}}\widehat{w}_N-I_N^{\alpha,\beta}\overline{\mathscr{M}}^1\widehat{v}_N=0.\end{cases}$$

In terms of the fact that

$$\begin{cases}\widehat{u}_N-I_N^{\alpha,\beta}\mathscr{M}\widehat{v}_N=\widehat{u}_N-\mathscr{M}\widehat{v}_N+(\mathscr{M}\widehat{v}_N-I_N^{\alpha,\beta}\mathscr{M}\widehat{v}_N),\\\widehat{v}_N-I_N^{\alpha,\beta}\widetilde{\mathscr{M}}\widehat{u}_N-I_N^{\alpha,\beta}\overline{\mathscr{M}}\widehat{v}_N-I_N^{\alpha,\beta}\widehat{\mathscr{M}}\widehat{w}_N-I_N^{\alpha,\beta}\overline{\mathscr{M}}^1\widehat{v}_N\\=\begin{cases}\widehat{v}_N-\widetilde{\mathscr{M}}\widehat{u}_N+(\widetilde{\mathscr{M}}\widehat{u}_N-I_N^{\alpha,\beta}\widetilde{\mathscr{M}}\widehat{u}_N)\\-\overline{\mathscr{M}}\widehat{v}_N+(\overline{\mathscr{M}}\widehat{v}_N-I_N^{\alpha,\beta}\overline{\mathscr{M}}\widehat{v}_N)\\-\widehat{\mathscr{M}}\widehat{w}_N+(\widehat{\mathscr{M}}\widehat{w}_N-I_N^{\alpha,\beta}\widehat{\mathscr{M}}\widehat{w}_N)\\-\overline{\mathscr{M}}^1\widehat{v}_N+(\overline{\mathscr{M}}^1\widehat{v}_N-I_N^{\alpha,\beta}\overline{\mathscr{M}}^1\widehat{v}_N).\end{cases}\end{cases}$$

Suppose that $\max\{|\widetilde{K}_1(t,s)|,|\overline{K}_2(t,s)|,|\widetilde{a}(t)|,|\widetilde{b}(t)|,1\}\le L$. It is clear that from (2.4)-(2.9)

$$\begin{cases}\widehat{u}_N=\int_{-1}^{t}\widehat{v}_N(s)ds+I_N^{\alpha,\beta}\mathscr{M}\widehat{v}_N-\mathscr{M}\widehat{v}_N,\\\widehat{w}_N=\int_{-1}^{qt+q-1}\widehat{v}_N(s)ds+I_N^{\alpha,\beta}\mathscr{M}^1\widehat{v}_N-\mathscr{M}^1\widehat{v}_N,\\\widehat{v}_N=\int_{-1}^{t}(\widetilde{K}_1(t,s)\widehat{u}_N(s)+\widetilde{a}(t)\widehat{v}_N(s)+\overline{K}_2(t,s)\widehat{w}_N(s))ds\\+\int_{-1}^{qt+q-1}\widetilde{b}(t)\widehat{v}_N(s)ds+I_N^{\alpha,\beta}\widetilde{\mathscr{M}}\widehat{u}_N-\widetilde{\mathscr{M}}\widehat{u}_N+I_N^{\alpha,\beta}\overline{\mathscr{M}}\widehat{v}_N\\-\overline{\mathscr{M}}\widehat{v}_N+I_N^{\alpha,\beta}\widehat{\mathscr{M}}\widehat{w}_N-\widehat{\mathscr{M}}\widehat{w}_N+I_N^{\alpha,\beta}\overline{\mathscr{M}}^1\widehat{v}_N-\overline{\mathscr{M}}^1\widehat{v}_N.\end{cases}$$

which yields

$$(|\widehat{u}_N|+|\widehat{v}_N|+|\widehat{w}_N|)\le c_1\int_{-1}^{t}(|\widehat{u}_N(s)|+|\widehat{v}_N|+|\widehat{w}_N(s)|)ds$$
$$+c_2\int_{-1}^{qt+q-1}(|\widehat{u}_N(s)|+|\widehat{v}_N|+|\widehat{w}_N(s)|)ds+\sum_{j=1}^{6}|I_j|.$$

where $I_1=I_N^{\alpha,\beta}\mathscr{M}\widehat{v}_N-\mathscr{M}\widehat{v}_N, I_2=I_N^{\alpha,\beta}\mathscr{M}^1\widehat{v}_N-\mathscr{M}^1\widehat{v}_N, I_3=I_N^{\alpha,\beta}\widetilde{\mathscr{M}}\widehat{u}_N-\widetilde{\mathscr{M}}\widehat{u}_N, I_4=I_N^{\alpha,\beta}\overline{\mathscr{M}}\widehat{v}_N-\overline{\mathscr{M}}\widehat{v}_N, I_5=I_N^{\alpha,\beta}\widehat{\mathscr{M}}\widehat{w}_N-\widehat{\mathscr{M}}\widehat{w}_N, I_6=I_N^{\alpha,\beta}\overline{\mathscr{M}}^1\widehat{v}_N-\overline{\mathscr{M}}^1\widehat{v}_N$. Using Lemma 3.6 leads to

$$(|\widehat{u}_N|+|\widehat{v}_N|+|\widehat{w}_N|)\le c\int_{-1}^{t}(\sum_{j=1}^{6}|I_j|)ds+\sum_{j=1}^{6}|I_j|\le c\sum_{j=1}^{6}\|I_j\|_{L^\infty}.$$

This gives,

$$\||\widehat{u}_N|+|\widehat{v}_N|+|\widehat{w}_N|\|_{L^\infty}\le c\sum_{j=1}^{6}\|I_j\|_{L^\infty}. \tag{4.9}$$

We now estimate $\|I_j\|_{L^\infty}, j=1,2,3,4,5,6$. By virtue of (3.6),(3.7) and Lemma 3.7, we obtain that

$$\|I_N^{\alpha,\beta}\widetilde{\mathscr{M}}\widehat{u}_N-\widetilde{\mathscr{M}}\widehat{u}_N\|_{L^\infty}=\|(I-I_N^{\alpha,\beta})\widetilde{\mathscr{M}}\widehat{u}_N\|_{L^\infty}$$
$$=\|(I-I_N^{\alpha,\beta})(\widetilde{\mathscr{M}}\widehat{u}_N-\tau_N\widetilde{\mathscr{M}}\widehat{u}_N)\|_{L^\infty}$$
$$\le(1+\|I_N^{\alpha,\beta}\|_{L^\infty})\|\widetilde{\mathscr{M}}\widehat{u}_N-\tau_N\widetilde{\mathscr{M}}\widehat{u}_N\|_{L^\infty}$$
$$\le\begin{cases}\le c\log NN^{-\kappa}\|\widehat{u}_N\|_{L^\infty},\quad -1<\alpha,\beta\le-\frac{1}{2}\\\le cN^{\frac{1}{2}-\kappa+\gamma}\|\widehat{u}_N\|_{L^\infty},\quad \gamma=\max(\alpha,\beta),\text{otherwise}.\end{cases}$$

This inequality also holds for $I_j,\quad j=1,2,4,5,6$.

These, together with (4.9), give

$$\||\widehat{u}_N|+|\widehat{v}_N|+|\widehat{w}_N|\|_{L^\infty}\le\begin{cases}c\log NN^{-\kappa}\||\widehat{u}_N|+|\widehat{v}_N|+|\widehat{w}_N|\|_{L^\infty},\\-1<\alpha,\beta\le-\frac{1}{2}\\cN^{\frac{1}{2}-\kappa+\gamma}\||\widehat{u}_N|+|\widehat{v}_N|+|\widehat{w}_N|\|_{L^\infty},\\\gamma=\max(\alpha,\beta),\text{otherwise},\end{cases}$$

which implies, taking $\kappa\in(0,1)$ such that $\kappa>\frac{1}{2}+\gamma$, when N is large enough ,$\widehat{u}_N=\widehat{v}_N=\widehat{w}_N=0$. Hence, the $\widehat{u}_N,\widehat{v}_N$ and $\widehat{w}_N$ are existent and unique as $\mathscr{P}_N(\Lambda)$ is finite-dimensional.

Lemma 4.1. Suppose that $u \in H^m_{\omega_{m-\mu,m-\mu}}(\Lambda)$ and $\max\{|\widetilde{K}_1(t,s)|, |\overline{K}_2(t,s)|, |\widehat{a}(t)|, |\widetilde{b}(t)|, 1\} \leq L$, then we have

$$\||u-\widehat{u}_N| + |v-\widehat{v}_N| + |w-\widehat{w}_N|\|_{L^\infty} \leq \begin{cases} c\log N N^{\frac{3}{4}-m}\|u\|_{m+1,\infty}, & -1<\alpha,\beta \leq -\frac{1}{2} \\ cN^{\frac{5}{4}-m+\gamma}\|u\|_{m+1,\infty}, & \gamma = \max(\alpha,\beta), \text{otherwise}. \end{cases} \tag{4.10}$$

and

$$\||u-\widehat{u}_N| + |v-\widehat{v}_N| + |w-\widehat{w}_N|\|_{L^2_{\omega_{\alpha,\beta}}(\Lambda)} \leq \begin{cases} cN^{-m}(\sum_{k=0}^{1}\|\partial_t^{m+k}u\|_{\omega_{m-\mu,m-\mu}}) \\ +c\log N N^{\frac{3}{4}-m}\|u\|_{m+1,\infty}, \quad -1<\alpha,\beta \leq -\frac{1}{2}, \\ cN^{-m}(\sum_{k=0}^{1}\|\partial_t^{m+k}u\|_{\omega_{m-\mu,m-\mu}}) \\ +cN^{\frac{5}{4}-m+\gamma}\|u\|_{m+1,\infty}, \quad \gamma = \max(\alpha,\beta), \text{otherwise}. \end{cases} \tag{4.11}$$

Proof. Subtracting (4.8) from (2.2) yields

$$\begin{cases} u(t) - \widehat{u}_N + I_N^{\alpha,\beta}\mathscr{M}\widehat{v}_N - \mathscr{M}v(t) = 0, \\ w(t) - \widehat{w}_N + I_N^{\alpha,\beta}\mathscr{M}^1\widehat{v}_N - \mathscr{M}^1 v(t) = 0, \\ v(t) - \widehat{v}_N + I_N^{\alpha,\beta}\widetilde{\mathscr{M}}\widehat{u}_N - \widetilde{\mathscr{M}}u + I_N^{\alpha,\beta}\overline{\mathscr{M}}\widehat{v}_N \\ -\overline{\mathscr{M}}v(t) + I_N^{\alpha,\beta}\widehat{\mathscr{M}}\widehat{w}_N - \widehat{\mathscr{M}}w(t) \\ +I_N^{\alpha,\beta}\overline{\mathscr{M}}^1\widehat{v}_N - \overline{\mathscr{M}}^1 v(t) = g(t) - I_N^{\alpha,\beta}g(t). \end{cases} \tag{4.12}$$

Set $\varepsilon = u(t) - \widehat{u}_N, \overline{\varepsilon} = v(t) - \widehat{v}_N, \widehat{\varepsilon} = w(t) - \widehat{w}_N$. Direct computation shows that

$$\begin{aligned} &\mathscr{M}v(t) - I_N^{\alpha,\beta}\mathscr{M}\widehat{v}_N = \mathscr{M}v - I_N^{\alpha,\beta}\mathscr{M}v + I_N^{\alpha,\beta}\mathscr{M}(v-\widehat{v}_N) \\ &= \mathscr{M}v + \mathscr{M}(v-\widehat{v}_N) - [\mathscr{M}(v-\widehat{v}_N) - I_N^{\alpha,\beta}\mathscr{M}(v-\widehat{v}_N)] \\ &-I_N^{\alpha,\beta}\mathscr{M}v = \mathscr{M}(v-\widehat{v}_N) - [\mathscr{M}(v-\widehat{v}_N) - I_N^{\alpha,\beta}\mathscr{M}(v-\widehat{v}_N)] \\ &+u - y_0 - I_N^{\alpha,\beta}(u-y_0) = u - I_N^{\alpha,\beta}u + \mathscr{M}\overline{\varepsilon} - [\mathscr{M}\overline{\varepsilon} - I_N^{\alpha,\beta}\mathscr{M}\overline{\varepsilon}]. \end{aligned} \tag{4.13}$$

Similarly

$$\begin{aligned} &\widetilde{\mathscr{M}}u - I_N^{\alpha,\beta}\widetilde{\mathscr{M}}\widehat{u}_N + \overline{\mathscr{M}}v - I_N^{\alpha,\beta}\overline{\mathscr{M}}\widehat{v}_N + \widehat{\mathscr{M}}w(t) - I_N^{\alpha,\beta}\widehat{\mathscr{M}}\widehat{w}_N \\ &+\overline{\mathscr{M}}^1 v - I_N^{\alpha,\beta}\overline{\mathscr{M}}^1\widehat{v}_N = v - I_N^{\alpha,\beta}v - g(t) + I_N^{\alpha,\beta}g(t) + \widetilde{\mathscr{M}}\varepsilon \\ &-[\widetilde{\mathscr{M}}\varepsilon - I_N^{\alpha,\beta}\widetilde{\mathscr{M}}\varepsilon] + \overline{\mathscr{M}}\overline{\varepsilon} - [\overline{\mathscr{M}}\overline{\varepsilon} - I_N^{\alpha,\beta}\overline{\mathscr{M}}\overline{\varepsilon}] + \widehat{\mathscr{M}}\widehat{\varepsilon} - [\widehat{\mathscr{M}}\widehat{\varepsilon} \\ &-I_N^{\alpha,\beta}\widehat{\mathscr{M}}\widehat{\varepsilon}] + \overline{\mathscr{M}}^1\overline{\varepsilon} - [\overline{\mathscr{M}}^1\overline{\varepsilon} - I_N^{\alpha,\beta}\overline{\mathscr{M}}^1\overline{\varepsilon}]. \end{aligned} \tag{4.14}$$

The insertion of (4.13)-(4.14) into (4.12) yields

$$\begin{cases} \varepsilon = u - I_N^{\alpha,\beta}u + \mathscr{M}\overline{\varepsilon} - [\mathscr{M}\overline{\varepsilon} - I_N^{\alpha,\beta}\mathscr{M}\overline{\varepsilon}], \\ \widehat{\varepsilon} = w - I_N^{\alpha,\beta}w + \mathscr{M}^1\overline{\varepsilon} - [\mathscr{M}^1\overline{\varepsilon} - I_N^{\alpha,\beta}\mathscr{M}^1\overline{\varepsilon}], \\ \overline{\varepsilon} = v - I_N^{\alpha,\beta}v + \widetilde{\mathscr{M}}\varepsilon - [\widetilde{\mathscr{M}}\varepsilon - I_N^{\alpha,\beta}\widetilde{\mathscr{M}}\varepsilon] + \overline{\mathscr{M}}\overline{\varepsilon} \\ -[\overline{\mathscr{M}}\overline{\varepsilon} - I_N^{\alpha,\beta}\overline{\mathscr{M}}\overline{\varepsilon}] + \widehat{\mathscr{M}}\widehat{\varepsilon} - [\widehat{\mathscr{M}}\widehat{\varepsilon} - I_N^{\alpha,\beta}\widehat{\mathscr{M}}\widehat{\varepsilon}] \\ +\overline{\mathscr{M}}^1\overline{\varepsilon} - [\overline{\mathscr{M}}^1\overline{\varepsilon} - I_N^{\alpha,\beta}\overline{\mathscr{M}}^1\overline{\varepsilon}]. \end{cases}$$

which implies that

$$\begin{aligned} |\varepsilon| + |\overline{\varepsilon}| + |\widehat{\varepsilon}| \leq \sum_{i=1}^{9}|J_i| + c_1\int_{-1}^{t}(|\varepsilon(s)| + |\overline{\varepsilon}(s)| + |\widehat{\varepsilon}(s)|)ds \\ +c_2\int_{-1}^{qt+q-1}(|\varepsilon(s)| + |\overline{\varepsilon}(s)| + |\widehat{\varepsilon}(s)|)ds, \end{aligned} \tag{4.15}$$

where
$J_1 = u - I_N^{\alpha,\beta}u$, $J_2 = v - I_N^{\alpha,\beta}v$, $J_3 = w - I_N^{\alpha,\beta}w$, $J_4 = \widetilde{\mathscr{M}}\varepsilon - I_N^{\alpha,\beta}\widetilde{\mathscr{M}}\varepsilon$, $J_5 = \overline{\mathscr{M}}\overline{\varepsilon} - I_N^{\alpha,\beta}\overline{\mathscr{M}}\overline{\varepsilon}$, $J_6 = \widehat{\mathscr{M}}\widehat{\varepsilon} - I_N^{\alpha,\beta}\widehat{\mathscr{M}}\widehat{\varepsilon}$, $J_7 = \mathscr{M}\overline{\varepsilon} - I_N^{\alpha,\beta}\mathscr{M}\overline{\varepsilon}$, $J_8 = \mathscr{M}^1\overline{\varepsilon} - I_N^{\alpha,\beta}\mathscr{M}^1\overline{\varepsilon}$, $J_9 = \overline{\mathscr{M}}^1\overline{\varepsilon} - I_N^{\alpha,\beta}\overline{\mathscr{M}}^1\overline{\varepsilon}$. Using Lemma 3.6 gives

$$|\varepsilon| + |\overline{\varepsilon}| + |\widehat{\varepsilon}| \leq \sum_{k=1}^{9}|J_k| + c\int_{-1}^{t}(\sum_{k=1}^{9}|J_k|)ds. \tag{4.16}$$

Similar to (4.9), we have that

$$\||\varepsilon| + |\overline{\varepsilon}| + |\widehat{\varepsilon}|\|_{L^\infty} \leq c\sum_{k=1}^{9}\|J_k\|_{L^\infty}. \tag{4.17}$$

By using (3.2), Lemma 3.7, we obtain that

$$\begin{aligned} &\|u - I_N^{\alpha,\beta}u\|_{L^\infty} = \|(I - I_N^{\alpha,\beta})(u - P_N u)\|_{L^\infty} \leq c(1 + \|I_N^{\alpha,\beta}\|_\infty)\|u - P_N u\|_{L^\infty} \\ &\leq \begin{cases} c\log N N^{\frac{3}{4}-m}\|u\|_{m,\infty}, & -1<\alpha,\beta \leq -\frac{1}{2} \\ cN^{\frac{5}{4}-m+\gamma}\|u\|_{m,\infty}, & \gamma = \max(\alpha,\beta), \text{otherwise}. \end{cases} \end{aligned} \tag{4.18}$$

$$\|v - I_N^{\alpha,\beta}v\|_{L^\infty} \leq \begin{cases} c\log N N^{\frac{3}{4}-m}\|u'\|_{m,\infty}, & -1<\alpha,\beta \leq -\frac{1}{2} \\ cN^{\frac{5}{4}-m+\gamma}\|u'\|_{m,\infty}, & \gamma = \max(\alpha,\beta), \text{otherwise}. \end{cases}$$

and

$$\|w - I_N^{\alpha,\beta}w\|_{L^\infty} \leq \begin{cases} c\log N N^{\frac{3}{4}-m}\|u\|_{m,\infty}, & -1<\alpha,\beta \leq -\frac{1}{2} \\ cN^{\frac{5}{4}-m+\gamma}\|u\|_{m,\infty}, & \gamma = \max(\alpha,\beta), \text{otherwise}. \end{cases} \tag{4.19}$$

We now estimate J_4. It is clear that $\varepsilon \in C[-1,1]$. Consequently, using (3.6),(3.7) and Lemma 3.7 it follows that

$$\begin{aligned} &\|J_4\|_{L^\infty} = \|(I - I_N^{\alpha,\beta})(\widetilde{M}\varepsilon - \tau_N\widetilde{M}\varepsilon)\|_{L^\infty} \leq (1 + \|I_N^{\alpha,\beta}\|_{L^\infty}) \\ &\times\|\widetilde{M}\varepsilon - \tau_N\widetilde{M}\varepsilon\|_{L^\infty} \leq c(1 + \|I_N^{\alpha,\beta}\|_{L^\infty})N^{-\kappa}\|\widetilde{M}\varepsilon\|_{0,\kappa} \\ &\leq \begin{cases} c\log N N^{-\kappa}\||\varepsilon| + |\overline{\varepsilon}| + |\widehat{\varepsilon}|\|_{L^\infty}, & -1<\alpha,\beta \leq -\frac{1}{2} \\ cN^{\frac{1}{2}-\kappa+\gamma}\||\varepsilon| + |\overline{\varepsilon}| + |\widehat{\varepsilon}|\|_{L^\infty}, & \gamma = \max(\alpha,\beta), \text{otherwise}. \end{cases} \end{aligned} \tag{4.20}$$

where $\kappa \in (0,1)$ and $\tau_N\mathscr{M}\varepsilon \in \mathscr{P}_N(\Lambda)$. (4.20) also holds for $\|J_5\|_{L^\infty}, \|J_6\|_{L^\infty}, \|J_7\|_{L^\infty}, \|J_8\|_{L^\infty}$ and $\|J_9\|_{L^\infty}$. Taking $\kappa \in (0,1)$ such that $\kappa > \frac{1}{2} + \gamma$, the estimate (4.10) follows from (4.17)-(4.20), provider that N is large enough.

Next we prove (4.11). For any $-1 < \alpha, \beta < 1$, we have from (4.16) that

$$\||\varepsilon| + |\overline{\varepsilon}| + |\widehat{\varepsilon}|\|^2_{L^2_{\omega_{\alpha,\beta}}} \leq c\sum_{i=1}^{9}\|J_i\|^2_{L^2_{\omega_{\alpha,\beta}}}. \tag{4.21}$$

We obtain that from (3.6) (3.7) and Lemma 3.8

$$\begin{aligned} \|J_4\|_{L^2_{\omega_{\alpha,\beta}}} &= \|(I - I_N^{\alpha,\beta})(\widetilde{\mathscr{M}}\varepsilon - \tau_N\widetilde{\mathscr{M}}\varepsilon)\|_{L^2_{\omega_{\alpha,\beta}}} \\ &\leq c\|\widetilde{\mathscr{M}}\varepsilon - \tau_N\widetilde{\mathscr{M}}\varepsilon\|_{L^\infty} \leq cN^{-\kappa}\||\varepsilon| + |\overline{\varepsilon}| + |\widehat{\varepsilon}|\|_{L^\infty}. \end{aligned}$$

It also holds for $\|J_5\|_{L^2_{\omega_{\alpha,\beta}}} - \|J_9\|_{L^2_{\omega_{\alpha,\beta}}}$. These result, together with the estimates (4.10), (4.21) and (3.3), yields (4.11).

Now subtracting (4.6) from (4.8) leads to

$$\begin{cases} \widehat{u}_N - u_N - I_N^{\alpha,\beta}Q(t) + I_N^{\alpha,\beta}\mathscr{M}v_N - I_N^{\alpha,\beta}\mathscr{M}\widehat{v}_N = 0, \\ \widehat{w}_N - w_N - I_N^{\alpha,\beta}Q_1(t) + I_N^{\alpha,\beta}\mathscr{M}^1 v_N - I_N^{\alpha,\beta}\mathscr{M}^1\widehat{v}_N = 0, \\ \widehat{v}_N - v_N - I_N^{\alpha,\beta}\widetilde{Q}(t) - I_N^{\alpha,\beta}\overline{Q}(t) - I_N^{\alpha,\beta}\widehat{Q}(t) - I_N^{\alpha,\beta}\overline{Q}_1(t) \\ +I_N^{\alpha,\beta}\widetilde{\mathscr{M}}u_N + I_N^{\alpha,\beta}\overline{\mathscr{M}}v_N + I_N^{\alpha,\beta}\widehat{\mathscr{M}}w_N + I_N^{\alpha,\beta}\overline{\mathscr{M}}^1 v_N \\ -I_N^{\alpha,\beta}\widetilde{\mathscr{M}}\widehat{u}_N - I_N^{\alpha,\beta}\overline{\mathscr{M}}\widehat{v}_N - I_N^{\alpha,\beta}\widehat{\mathscr{M}}\widehat{w}_N - I_N^{\alpha,\beta}\overline{\mathscr{M}}^1\widehat{v}_N = 0, \end{cases}$$

which can be simplified as, by setting $E = \widehat{u}_N - u_N, E_1 = \widehat{v}_N - v_N$, $E_2 = \widehat{w}_N - w_N$,

$$\begin{cases} E - I_N^{\alpha,\beta}Q(t) - I_N^{\alpha,\beta}\mathscr{M}E_1 = 0, \\ E_2 - I_N^{\alpha,\beta}Q_1(t) - I_N^{\alpha,\beta}\mathscr{M}^1 E_1 = 0, \\ E_1 - I_N^{\alpha,\beta}\widetilde{Q}(t) - I_N^{\alpha,\beta}\overline{Q}(t) - I_N^{\alpha,\beta}\widehat{Q}(t) - I_N^{\alpha,\beta}\overline{Q}_1(t) \\ -I_N^{\alpha,\beta}\widetilde{\mathscr{M}}E - I_N^{\alpha,\beta}\overline{\mathscr{M}}E_1 - I_N^{\alpha,\beta}\widehat{\mathscr{M}}E_2 - I_N^{\alpha,\beta}\overline{\mathscr{M}}^1 E_1 = 0. \end{cases} \tag{4.22}$$

Let $e_N = u - u_N$, $\overline{e}_N = v - v_N$ and $\widehat{e}_N = w - w_N$ be the error corresponding to the Jacobi pseudo-spectral Galerkin solution u_N, v_N, w_N of (2.18). Now we are prepared to get our global convergence result for problem (2.2).

Theorem 4.1. Suppose that $\max\{|\widetilde{K}_1(t,s)|, |\overline{K}_2(t,s)|, |\widetilde{a}(t)|, |\widetilde{b}(t)|, 1\} \le L$and the solution of (2.2) is sufficiently smooth. For the Jacobi pseudo spectral Galerkin solution defined in (2.18), we have the following estimates

(1) L^∞ norm of $|e_N| + |\overline{e}_N| + |\widehat{e}_N|$ satisfies,

$$\begin{aligned} &\|(|e_N| + |\overline{e}_N| + |\widehat{e}_N|)\|_{L^\infty} \\ &\le \begin{cases} c\log N N^{\frac{3}{4}-m}\|u\|_{m+1,\infty} \\ +c\log N N^{-m}\|u\|_{1,\infty}, \ -1 < \alpha,\beta \le -\frac{1}{2}, \\ cN^{\frac{5}{4}-m+\gamma}\|u\|_{m+1,\infty} + cN^{\frac{1}{2}-m+\gamma}\|u\|_{1,\infty}, \\ \gamma = \max(\alpha,\beta), \text{otherwise.} \end{cases} \end{aligned} \tag{4.23}$$

(2) The Jacobi spectral error $|e_N| + |\overline{e}_N| + |\widehat{e}_N|$ satisfies,

$$\begin{aligned} &\||e_N| + |\overline{e}_N| + |\widehat{e}_N|\|_{L^2_{\omega_{\alpha,\beta}}} \\ &\le \begin{cases} c\log N N^{\frac{3}{4}-m}\|u\|_{m+1,\infty} + c\log N N^{-m}\|u\|_{1,\infty} \\ +N^{-m}\sum_{k=0}^{1}\|\partial_t^{m+k}u\|_{\omega_{m+\alpha,m+\beta}}, \quad -1 < \alpha,\beta \le -\dfrac{1}{2}, \\ +cN^{\frac{5}{4}-m+\gamma}\|u\|_{m+1,\infty} + cN^{\frac{1}{2}-m+\gamma}\|u\|_{1,\infty} \\ +cN^{-m}\sum_{k=0}^{1}\|\partial_t^{m+k}u\|_{\omega_{m+\alpha,m+\beta}}, \ \gamma = \max(\alpha,\beta), \text{otherwise.} \end{cases} \end{aligned} \tag{4.24}$$

Proof. We first prove the existence and uniqueness of the Jacobi pseudo-spectral Galerkin solution u_N. As the dimension of $\mathscr{P}_N(\Lambda)$ is finite and (2.18) and (4.6) are equivalent, we only need to prove that the solution of (4.6) is $u_N = v_N = w_N = 0$ when $g = y_0 = 0$. For this purpose, we consider equations

$$\begin{cases} u_N + I_N^{\alpha,\beta}Q(t) - I_N^{\alpha,\beta}\mathscr{M}v_N = 0, \\ w_N + I_N^{\alpha,\beta}Q_1(t) - I_N^{\alpha,\beta}\mathscr{M}^1 v_N = 0, \\ v_N + I_N^{\alpha,\beta}\widetilde{Q}(t) + I_N^{\alpha,\beta}\widehat{Q}(t) + I_N^{\alpha,\beta}\overline{Q}(t) + I_N^{\alpha,\beta}\overline{Q}_1(t) \\ -I_N^{\alpha,\beta}\widetilde{\mathscr{M}}u_N - I_N^{\alpha,\beta}\overline{\mathscr{M}}v_N - I_N^{\alpha,\beta}\widehat{\mathscr{M}}w_N - I_N^{\alpha,\beta}\overline{\mathscr{M}}^1 v_N = 0. \end{cases} \tag{4.25}$$

Obviously (4.25) can be written as

$$\begin{cases} u_N - \mathscr{M}v_N = I_N^{\alpha,\beta}\mathscr{M}v_N - I_N^{\alpha,\beta}Q(t) - \mathscr{M}v_N = R_1 + R_2, \\ w_N - \mathscr{M}^1 v_N = I_N^{\alpha,\beta}\mathscr{M}^1 v_N - I_N^{\alpha,\beta}Q_1(t) - \mathscr{M}^1 v_N = R_3 + R_4, \\ v_N - \widetilde{\mathscr{M}}u_N - \overline{\mathscr{M}}v_N - \widehat{\mathscr{M}}w_N - \overline{\mathscr{M}}^1 v_N = I_N^{\alpha,\beta}\widetilde{\mathscr{M}}u_N + I_N^{\alpha,\beta}\overline{\mathscr{M}}v_N \\ +I_N^{\alpha,\beta}\widehat{\mathscr{M}}w_N + I_N^{\alpha,\beta}\overline{\mathscr{M}}^1 v_N - I_N^{\alpha,\beta}\widetilde{Q}(t) - I_N^{\alpha,\beta}\overline{Q}(t) - I_N^{\alpha,\beta}\widehat{Q}(t) \\ -I_N^{\alpha,\beta}\overline{Q}_1(t) - \widetilde{\mathscr{M}}u_N - \overline{\mathscr{M}}v_N - \widehat{\mathscr{M}}w_N - \overline{\mathscr{M}}^1 v_N \\ = R_5 + R_6 + R_7 + R_8 + R_9 + R_{10} + R_{11} + R_{12}. \end{cases}$$

namely,

$$\begin{cases} u_N = \int_{-1}^{t} v_N(s)ds + R_1 + R_2, \ w_N = \int_{-1}^{qt+q-1} v_N(s)ds + R_3 + R_4, \\ w_N = \int_{-1}^{t}(\widetilde{K}_1(t,s)u_N(s) + \widetilde{a}(t)v_N(s) + \overline{K}_2(t,s)w_N(s))ds \\ +\int_{-1}^{qt+q-1}\widetilde{b}(t)w_N(s)ds + \sum_{i=5}^{12} R_i. \end{cases} \tag{4.26}$$

with $R_1 = I_N^{\alpha,\beta}\mathscr{M}v_N - \mathscr{M}v_N$, $R_2 = -I_N^{\alpha,\beta}Q(t), R_3 = I_N^{\alpha,\beta}\mathscr{M}^1 v_N - \mathscr{M}^1 v_N$, $R_4 = -I_N^{\alpha,\beta}Q_1(t)$, $R_5 = I_N^{\alpha,\beta}\widetilde{\mathscr{M}}u_N - \widetilde{\mathscr{M}}u_N, R_6 = I_N^{\alpha,\beta}\overline{\mathscr{M}}v_N - \overline{\mathscr{M}}v_N$, $R_7 = I_N^{\alpha,\beta}\widehat{\mathscr{M}}w_N - \widehat{\mathscr{M}}w_N, R_8 = -I_N^{\alpha,\beta}\widetilde{Q}(t), R_9 = -I_N^{\alpha,\beta}\overline{Q}(t), R_{10} = -I_N^{\alpha,\beta}\widehat{Q}(t), R_{11} = -I_N^{\alpha,\beta}\overline{Q}_1(t), R_{12} = I_N^{\alpha,\beta}\overline{\mathscr{M}}^1 v_N - \overline{\mathscr{M}}^1 v_N$. Using (4.26) gives

$$\begin{aligned} &(|u_N| + |v_N| + |w_N|) \le c_1\int_{-1}^{t}(|u_N(s)| + |v_N(s)| + |w_N(s)|)ds \\ &+c_2\int_{-1}^{qt+q-1}(|u_N(s)| + |v_N(s)| + |w_N(s)|)ds + \sum_{j=1}^{12}|R_j|. \end{aligned} \tag{4.27}$$

Using Lemma 3.6 yields

$$\|(|u_N| + |v_N| + |w_N|)\|_{L^\infty} \le c\sum_{j=1}^{12}\|R_j\|_{L^\infty}. \tag{4.28}$$

On the other hand, according to Lemma 3.7,

$$\begin{aligned} &\|R_8\|^2_{L^\infty} = \|I_N^{\alpha,\beta}\widetilde{Q}(t)\|^2_{L^\infty} \\ &\le \begin{cases} c(\log N)^2\|\widetilde{Q}(t)\|^2_{L^\infty}, \ \ -1 < \alpha,\beta \le -\frac{1}{2} \\ cN^{1+2\gamma}\|\widetilde{Q}(t)\|^2_{L^\infty}, \ \ \gamma = \max(\alpha,\beta), \text{otherwise.} \end{cases} \end{aligned} \tag{4.29}$$

By the expression of $\widetilde{Q}(t)$ in (4.4), we have from Lemma 3.2

$$\begin{aligned} |\widetilde{Q}(t)| &\le cN^{-m}\|\partial_\theta^m((\frac{t+1}{2})\widetilde{K}_1(\tau(t,\theta))\omega_{-\alpha,-\beta}(\theta)\|_{L^2_{\omega_{m+\alpha,m+\beta}}} \\ &\times\|u_N\|_{L^2_{\omega_{\alpha,\beta}}} \le cN^{-m}\|u_N\|_{L^2_{\omega_{\alpha,\beta}}}, \end{aligned}$$

which, together with (4.29), gives

$$\|R_8\|_{L^\infty} \le \begin{cases} c\log NN^{-m}\|(|u_N|+|v_N|+|w_N|)\|_{L^\infty}, \\ -1<\alpha,\beta\le -\frac{1}{2}, \\ \le cN^{\frac{1}{2}-m+\gamma}\|(|u_N|+|v_N|+|w_N|)\|_{L^\infty}, \\ \gamma=\max(\alpha,\beta), \text{otherwise}. \end{cases} \tag{4.30}$$

Similarly, (4.30) holds for $\|R_2\|_{L^\infty}, \|R_4\|_{L^\infty}, \|R_9\|_{L^\infty}, \|R_{10}\|_{L^\infty}$ and $\|R_{11}\|_{L^\infty}$. The combination of (4.20) and (4.30) yields

$$\begin{aligned} &\|(|u_N|+|v_N|+|w_N|)\|_{L^\infty} \\ &\le \begin{cases} c(\log NN^{-m}+\log NN^{-\kappa})\|(|u_N|+|v_N|+|w_N|)\|_{L^\infty}, \\ -1<\alpha,\beta\le -\frac{1}{2}, \\ c(N^{\frac{1}{2}-m+\gamma}+N^{\frac{1}{2}-\kappa+\gamma})\|(|u_N|+|v_N|+|w_N|)\|_{L^\infty}, \\ \gamma=\max(\alpha,\beta), \text{otherwise}. \end{cases} \end{aligned} \tag{4.31}$$

Based on (4.31) and Lemma 3.4 with $\kappa > \frac{1}{2}+\gamma$, when N is large enough,$u_N = v_N = w_N = 0$. As a result, the existence and uniqueness of the Jacobi pseudo-spectral Galerkin solutions u_N, v_N, w_N are proved.

Now we turn to the L^∞ error estimate. Actually (4.22) can be transformed into

$$\begin{cases} E=\int_{-1}^{t}E_1(s)ds+I_N^{\alpha,\beta}\mathscr{M}E_1-\mathscr{M}E_1+I_N^{\alpha,\beta}Q(t), \\ E_2=\int_{-1}^{qt+q-1}E_1(s)ds+I_N^{\alpha,\beta}\mathscr{M}^1E_1-\mathscr{M}^1E_1+I_N^{\alpha,\beta}Q_1(t), \\ E_1=\int_{-1}^{t}(\widetilde{K}_1(t,s)E(s)+\widetilde{a}(t)E_1(s)+\overline{K}_2(t,s)E_2(s))ds \\ +\int_{-1}^{qt+q-1}\widetilde{b}(t)E_1(s)ds-\widetilde{\mathscr{M}}E-\overline{\mathscr{M}}E_1-\widehat{\mathscr{M}}E_2 \\ -\overline{\mathscr{M}}^1E_1+I_N^{\alpha,\beta}\widetilde{\mathscr{M}}E+I_N^{\alpha,\beta}\overline{\mathscr{M}}E_1+I_N^{\alpha,\beta}\widehat{\mathscr{M}}E_2+I_N^{\alpha,\beta}\overline{\mathscr{M}}^1E_1 \\ +I_N^{\alpha,\beta}\widetilde{Q}(t)+I_N^{\alpha,\beta}\overline{Q}(t)+I_N^{\alpha,\beta}\widehat{Q}(t)+I_N^{\alpha,\beta}\overline{Q}_1(t). \end{cases} \tag{4.32}$$

which yields

$$\begin{aligned} &|E|+|E_1|+|E_2| \le |R_2|+|R_4|+\sum_{i=8}^{11}|R_i|+\sum_{i=13}^{18}|R_i| \\ &+c_1\int_{-1}^{t}(|E(s)|+|E_1(s)|+|E_2(s)|)ds \\ &+c_2\int_{-1}^{qt+q-1}(|E(s)|+|E_1(s)|+|E_2(s)|)ds. \end{aligned} \tag{4.33}$$

with $R_{13} = I_N^{\alpha,\beta}\mathscr{M}E_1 - \mathscr{M}E_1$, $R_{14} = I_N^{\alpha,\beta}\mathscr{M}^1E_1 - \mathscr{M}^1E_1, R_{15} = I_N^{\alpha,\beta}\widetilde{\mathscr{M}}E - \widetilde{\mathscr{M}}E, R_{16} = I_N^{\alpha,\beta}\overline{\mathscr{M}}E_1 - \overline{\mathscr{M}}E_1, R_{17} = I_N^{\alpha,\beta}\widehat{\mathscr{M}}E_2 - \widehat{\mathscr{M}}E_2, R_{18} = I_N^{\alpha,\beta}\overline{\mathscr{M}}^1E_1 - \overline{\mathscr{M}}^1E_1$. Similar to (4.11),it follows from (4.33) and Lemma 3.6 that

$$\begin{aligned} \||E|+|E_1|+|E_2|\|_{L^\infty} &\le c(\|R_2\|_{L^\infty}+\|R_4\|_{L^\infty} \\ &+\sum_{i=8}^{11}\|R_i\|_{L^\infty}+\sum_{i=13}^{18}\|R_i\|_{L^\infty}). \end{aligned} \tag{4.34}$$

Similar to the estimate of (4.20), we obtain

$$\|R_{13}\|_{L^\infty} \le \begin{cases} c\log NN^{-\kappa}\|E_1\|_{L^\infty}, & -1<\alpha,\beta\le -\frac{1}{2}, \\ cN^{\frac{1}{2}-\kappa+\gamma}\|E_1\|_{L^\infty}, & \gamma=\max(\alpha,\beta), \text{otherwise}. \end{cases} \tag{4.35}$$

It also holds for $R_{14}-R_{18}$. In terms of (4.30), (4.34) and (4.35), when N is large enough, we obtain

$$\begin{aligned} &\||E|+|E_1|+|E_2|\|_{L^\infty} \\ &\le \begin{cases} c\log NN^{-m}\|(|u_N|+|v_N|+|w_N|)\|_{L^\infty} \\ \le c\log NN^{-m}(\|(|u|+|v|+|w|)\|_{L^\infty} \\ +\|(|u-u_N|+|v-v_N|+|w-w_N|)\|_{L^\infty}), \quad -1<\alpha,\beta\le -\frac{1}{2}, \\ cN^{\frac{1}{2}-m+\gamma}\|(|u_N|+|v_N|+|w_N|)\|_{L^\infty} \\ \le cN^{\frac{1}{2}-m+\gamma}(\|(|u|+|v|+|w|)\|_{L^\infty} \\ +\|(|u-u_N|+|v-v_N|+|w-w_N|)\|_{L^\infty}), \\ \gamma=\max(\alpha,\beta), \text{otherwise}. \end{cases} \end{aligned} \tag{4.36}$$

By the triangular inequality,

$$\begin{aligned} &\||u-u_N|+|v-v_N|+|w-w_N|\|_{L^\infty} \le \||E|+|E_1|+|E_2|\|_{L^\infty} \\ &+\||u-\widehat{u}_N|+|v-\widehat{v}_N|+|w-\widehat{w}_N|\|_{L^\infty}. \end{aligned} \tag{4.37}$$

as well as (4.36),(4.37) and Lemma 4.1, we can obtain (4.23) provided N is sufficiently large.

Next we prove (4.24). Using Lemma 3.6, one obtains that from (4.33)

$$\begin{aligned} &\||E|+|E_1|+|E_2|\|^2_{L^2_{\omega_{\alpha,\beta}}} \le c(\|R_2\|^2_{L^2_{\omega_{\alpha,\beta}}}+\|R_4\|^2_{L^2_{\omega_{\alpha,\beta}}} \\ &+\sum_{i=8}^{11}\|R_i\|^2_{L^2_{\omega_{\alpha,\beta}}}+\sum_{i=13}^{18}\|R_i\|^2_{L^2_{\omega_{\alpha,\beta}}}) \end{aligned} \tag{4.38}$$

The combination of (4.30), (4.35) and (4.36) yields

$$\||E|+|E_1|+|E_2|\|_{L^2_{\omega_{\alpha,\beta}}} \le \begin{cases} c\log NN^{-m}(\|(|u|+|v|+|w|)\|_{L^\infty} \\ +\||e_N|+|\overline{e}_N|+|\widehat{e}_N|)\|_{L^\infty}), \\ -1<\alpha,\beta\le -\frac{1}{2}, \\ cN^{\frac{1}{2}-m+\gamma}(\|(|u|+|v|+|w|)\|_{L^\infty} \\ +\||e_N|+|\overline{e}_N|+|\widehat{e}_N|\|_{L^\infty}), \\ \gamma=\max(\alpha,\beta), \text{otherwise}. \end{cases} \tag{4.39}$$

By the triangular inequality again,

$$\begin{aligned} &\||e_N|+|\overline{e}_N|+|\widehat{e}_N|\|_{L^2_{\omega_{\alpha,\beta}}} \le \||E|+|E_1|+|E_2|\|_{L^2_{\omega_{\alpha,\beta}}} \\ &+\||u-\widehat{u}_N|+|v-\widehat{v}_N|+|w-\widehat{w}_N|\|_{L^2_{\omega_{\alpha,\beta}}}. \end{aligned} \tag{4.40}$$

In terms of (4.23)(4.39)(4.40) and Lemma 4.1, we obtain the desired result.

5 Numerical results

We give two numerical example to confirm our analysis.

Example 1 Consider the Volterra delay integro-differential equation

$$\begin{aligned} u'(t) &= -2tu(t)+u(qt+q-1)+g(t)+\int_{-1}^{t}\tau^2u(\tau)d\tau \\ &+\int_{-1}^{qt+q-1}u(\eta)d\eta. \end{aligned}$$

with $g(t)$ chosen so that $u(t)=\cos(0.5t)$. By calculation

$$\begin{aligned} g(t) &= (\frac{31}{2}-2t^2)\sin(0.5t)+12\sin(0.5)-6t\cos(0.5t) \\ &-8\cos(0.5)-\cos(\frac{qt+q-1}{2})-2\sin(\frac{qt+q-1}{2}). \end{aligned}$$

Fig. 1 shows the errors $u-u_N$ of approximate solution in L^∞ and weighted $L^2_{\omega_{\alpha,\beta}}$ norms. Fig. 2 shows the errors $u'-u'_N$ in L^∞ and weighted $L^2_{\omega_{\alpha,\beta}}$ norms obtained by using the pseudo-spectral methods described above. It is observed that the desired exponential rate of convergence is obtained.

Example 2 Consider the Volterra delay integro-differential equation

$$u'(t)=u(qt)+\frac{3}{4}+\frac{t}{2}+\frac{3}{2}e^{2t}-(\frac{5}{4}+\frac{t(1-q)}{2})e^{2qt}+\int_0^t u(\tau)d\tau$$
$$+\int_0^{qt}(t-\tau)u(\tau)d\tau.$$

The corresponding exact solution is given by $u(t)=e^{2t}$.

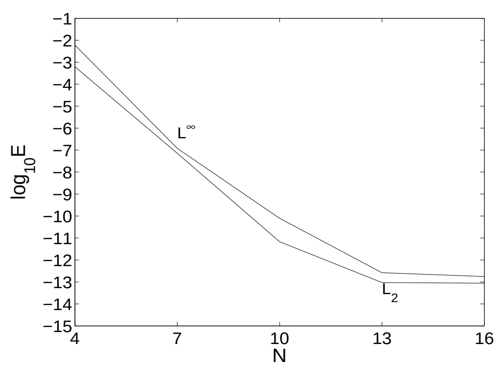

Figure 1. error of u for example 6.1

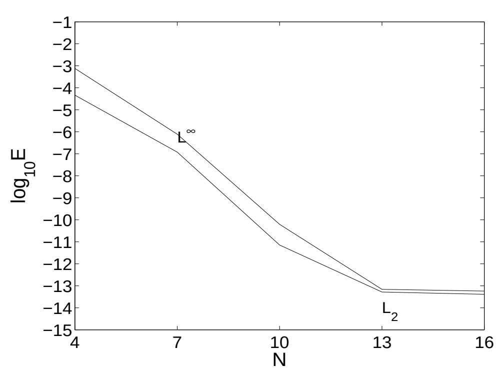

Figure 2. error of u' for example 6.1

Fig.3 and Fig.4 plot the errors $u-u_N$ and $u'-u'_N$ for $4\leq N\leq 16$ in L^∞ and $L^2_{\omega_{\alpha,\beta}}$ norms. Once again the desired spectral accuracy is obtained.

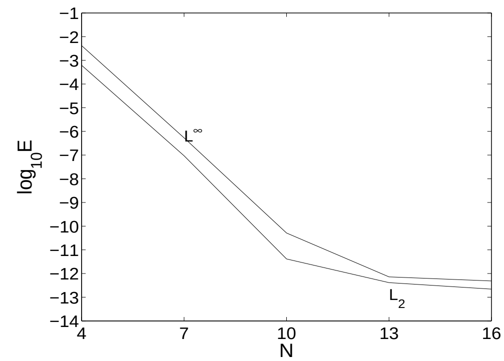

Figure 3. error of u for example 6.2

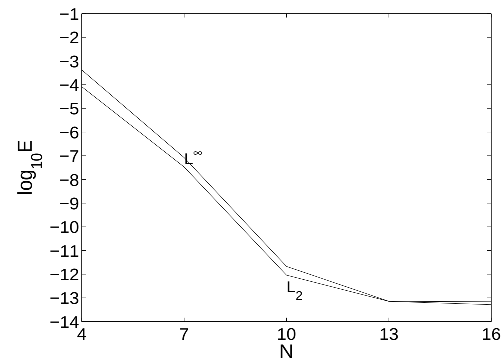

Figure 4. error of u' for example 6.2

6 Conclusion

This paper proposes a numerical method for Volterra delay integro-differential equations based on a Jacobi pseudospectral approach. To facilitate the use of the method, we first restate the original Volterra delay integro-differential equation as three simple integral equations of the second kind. The most important contribution of this work is that we are able to demonstrate rigorously that the errors of approximations decay exponentially in $L^\infty$and$L^2_{\omega_{\alpha,\beta}}$, which is a desired feature for a spectral method.

References

[1] Bock I, Lovisek J. On a reliable solution of a Volterra integral equation in a Hilbert space. Appl Math, 2003, **48**, 469-486.

[2] Brunner H. The numerical solutions of weakly singular Volterra integral equations by collocation on graded meshes, Math.Comp., **45** (1985), pp.417-437. MR804933 (87b:65223).

[3] Brunner H. Polynomial spline collocation methods for Volterra integro-differential equations with weakly singular kernels, IMA J.Numer.Anal., **6** (1986), pp.221-239. MR967664 (89h:65217).

[4] Brunner H. Collocation Methods for Volterra Integral and Related Functional Equations. Cambridge:Cambridge University Press, 2004.
[5] Brunner H.AND D.SCHÖTZAU, hp-discontinuous Galerkin time-stepping for Volterra Integrodifferential equations, SIAM J.Numer.Anal.,**44** (2006),pp.224-245.
[6] Canuto C, Hussaini M Y, Quarteroni A, Zang T A. Spectral Methods Fundamentals in Single Domains. Berlin: Springer-Verlag, 2006.
[7] D. L. Ragozin, Polynomial approximation on compact manifolds and homogeneous spaces, Trans.Amer.Math.Soc., **150** (1970), pp. 41-53. MR0410210 (53:13960).
[8] D.L.Ragozin, Constructive polynomial approximation on spheres and projective spaces, Trans.Amer.Math.Soc., **162** (1971), pp. 157-170. MR0288468 (44:5666).
[9] D.Willett, A linear generalization of Gronwall's inequality, Proceedings of the American Mathematical Society, **16**, (Aug., 1965), pp.774-778. MR0181726 (31:5953).
[10] L.E.GAREY AND R.E.SHAW, Algorithms for the solution of second order Volterra integrodifferential equations, Comput.Math.Appl., **22** (1991), pp. 27-34.
[11] M.AGUILAR AND H.BRUNNER, Collocation methods for second-order volterra integro-differential equations, Appl.Numer.Math., **4** (1988), pp. 455-470.
[12] M.BOLOGNA, Asymptotic solution for first and second order linear Volterra integro-differential equations with convolution kernels, J. Phys. A: Math. Theor., **43** (2010), pp.1-13.
[13] M.ZAREBNIA AND Z.NIKPOUR, Solution of linear Volterra integro-differential equations via Sine functions. Int.J.Appl.Math.Comput., **2** (2010), pp. 1-10.
[14] Shen J, Tang T. Spectral and High-Order Methods with Applications. Beijing: Science Press, 2006.
[15] Shen,J., Tang,T., Wang,L.-L.,Spectral Methods: Algorithms, Analysis and Applications. Springer Series in Computational Mathematics.Springer, New York (2011).
[16] Tang T. A note on collocation methods for Volterra integro-differential equations with weakly singular kernels, IMA J.Numer.Anal., **13** (1993), pp.93-99. MR1199031 (93k:65111).
[17] Y.Chen and T.Tang,Convergence analysis of the Jacobi Spectral-Collocation Methods for Volterra integral equations with a weakly singular kernel,Mathemacics of Computation, **79** (2010), Pages 147-167.
[18] Y.Chen and T.Tang, Spectral methods for weakly singular Volterra integral equations with smooth solutions, Journal of Computational and Applied Mathematics, **233** (2009) 938-950.

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