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Concomitants of Dual Generalized Order Statistics from Farlie Gumbel Morgenstern Type Bivariate Inverse Rayleigh Distribution

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Abstract: Dual generalized order statistics constitute a unified model for descending order random variables, like reverse order statistics and lower record values. In this paper, we have considered concomitants of dual generalized order statistics for the Farlie-Gumbel-Morgenstern type bivariate inverse Rayleigh distribution and single and joint distribution of concomitants of dual generalized order statistics are obtained. Further, Single and product moments are derived and recurrence relations between moments are established. Also results are deduced for order statistics and lower record values and some computation works are carried out.

Keywords: Dual generalized order statistics; lower record values; concomitants; single and product moments

AMS Subject Classification: 62E15, 62G30

1 Introduction

The most important use of concomitants arises in selection procedures when $k < (n)$ individuals are chosen on the basis of their X-values. Then the corresponding Y-values represent performance on an associated characteristic. For example, X might be the score of a candidate on a screening test and Y the score on a later test.

Statements are made such as ”if a student is good in mathematics then he will be poor in languages”. If the average score in language of students good in mathematics is lower than average score in language of All students, then the statement may be justified. To test hypotheses of this kind we need the distribution of the concomitant of order statistics. Thus to study a variable associated with another, distribution of concomitants of order statistics are usually crucial.

The Farlie Gumbel Morgenstern (FGM) family of bivariate distributions has found extensive use in practice. This family is characterized by the specified marginal distribution functions $F_X(x)$ and $F_Y(y)$ of random variables X and Y respectively and a parameter α, resulting in the bivariate distribution function (df) is given by

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)[1+\alpha(1-F_X(x))(1-F_Y(y))], \tag{1.1}$$

with the corresponding probability density function (pdf)

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)[1+\alpha(2F_X(x)-1)(1-2F_Y(y))]. \tag{1.2}$$

Here, $f_X(x)$ and $f_Y(y)$ are the marginal $pdfs$ of $f_{X,Y}(x,y)$. The parameter α is known as the association parameter; the two random variables X and Y are independent when α is zero. Such a model was originally introduced by Morgenstern (1956) and investigated by Gumbel (1960) for exponential marginal. The general form in (1.1) is due to Farlie (1960) and

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Johnson and Kotz (1975). The admissible range of association parameter α is $-1 \leq \alpha \leq 1$ and the Pearson correlation coefficient ρ between X and Y can never exceed 1/3. The conditional df and pdf of Y given X, are given by

$$F_{Y|X}(y|x) = F_Y(y)[1+\alpha(1-F_X(x))(1-F_Y(y))] \tag{1.3}$$

and

$$f_{Y|X}(y|x) = f_Y(y)[1+\alpha(1-2F_X(x))(1-2F_Y(y))]. \tag{1.4}$$

($c.f.$ Beg and Ahsanullah, 2007).

In this paper, we consider FGM type bivariate inverse Rayleigh distribution with pdf

$$f(x,y) = \frac{4\theta_1}{x^3}\frac{\theta_2}{y^3}e^{-\frac{\theta_1}{x^2}}e^{-\frac{\theta_2}{y^2}}[1+\alpha(1-2e^{-\frac{\theta_1}{x^2}})(1-2e^{-\frac{\theta_2}{y^2}})],\ 0<x,y<\infty, -1\leq\alpha\leq 1 \tag{1.5}$$

and corresponding df

$$F(x,y) = e^{-\frac{\theta_1}{x^2}}e^{-\frac{\theta_2}{y^2}}[1+\alpha(1-e^{-\frac{\theta_1}{x^2}})(1-e^{-\frac{\theta_2}{y^2}})],\ \ 0<x,y<\infty, -1\leq\alpha\leq 1. \tag{1.6}$$

The conditional pdf of Y given X is

$$f(y|x) = \frac{2\theta_2}{y^3}e^{-\frac{\theta_2}{y^2}}[1+\alpha(1-2e^{-\frac{\theta_1}{x^2}})(1-2e^{-\frac{\theta_2}{y^2}})],\ \ 0<x,y<\infty, -1\leq\alpha\leq 1. \tag{1.7}$$

The marginal pdf and df of X are

$$f(x) = \frac{2\theta_1}{x^3}e^{-\frac{\theta_1}{x^2}},\ \ 0<x<\infty,\ \ \theta_1>0, \tag{1.8}$$

$$F(x) = e^{-\frac{\theta_1}{x^2}},\ \ 0<x<\infty,\ \ \theta_1>0, \tag{1.9}$$

respectively.

Let X be a continuous random variables with df $F(x)$ and the pdf $f(x)$; $x \in (\alpha,\beta)$. Further, Let $n \in N, k \geq 1$, $\tilde{m} = (m_1,m_2,...,m_n) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$ such that $\gamma_r = k+(n-r)+M_r \geq 1$ for all $r \in 1,2,...,n-1$. Then $X_d(r,n,\tilde{m},k)$, $r=1,2,...,n$ are called dual generalized order statistics ($dgos$) if their pdf is given by [Burkschat *et al.* (2003)]

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left[F(x_i)\right]^{m_i}f(x_i)\right)\left[F(x_n)\right]^{k-1}f(x_n) \tag{1.10}$$

on the cone $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$.

If $m_i = 0$, $i=1,2,...,n$, $k=1$, then $X_d(r,n,m,k)$ reduces to the $(n-r+1)-th$ order statistic $X_{n-r+1:n}$ from sample $X_1,X_2,...,X_n$ and when $m=-1$, then $X_d(r,n,m,k)$ reduces to $r-th$, $k-$ lower record values.

The pdf of $X_d(r,n,m,k)$ is

$$f_{X_d(r,n,m,k)} = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x)), \tag{1.11}$$

and joint pdf of $X_d(r,n,m,k)$ and $X_d(s,n,m,k)$, is $1 \leq r < s \leq n$ is

$$f_{X_d(r,s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\ [F(x)]^m f(x)\ g_m^{r-1}\big(F(x)\big)$$

$$\times[h_m\big(F(y)\big)-h_m\big(F(x)\big)]^{s-r-1}[F(y)]^{\gamma_s-1}f(y),\ \alpha\leq x<y\leq\beta, \tag{1.12}$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i,\quad \gamma_i = k+(n-i)(m+1)$$

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$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}x^{m+1} & , \; m \neq -1 \\ -\log x & , \; m = -1 \end{cases}$$

and $\quad g_m(x) = h_m(x) - h_m(1), \; x \in (0,1).$

Let $(X_i,Y_i), i = 1,2,...,n$, be n pairs of independent random variables from some bivariate population with $df\ F(x,y)$. If we arrange the $X-$ variates in descending order as $X_d(1,n,m,k) \geq X_d(2,n,m,k) \geq ... \geq X_d(n,n,m,k)$, then $Y-$ variates paired (not necessarily in descending order) with these *dgos* are called the concomitants of *dgos* and are denoted by $Y_{[1,n,m,k]}, Y_{[2,n,m,k]}, ..., Y_{[n,n,m,k]}$. The pdf of $Y_{[r,n,m,k]}$, the $r-th$ concomitant of *dgos*, is given as

$$g_{[r,n,m,k]}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x)\; f_{X_d(r,n,m,k)}(x)\; dx. \tag{1.13}$$

The joint *pdf* of $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ $\; 1 \leq r < s \leq n$ is

$$g_{[r,s,n,m,k]}(y_1,y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_1} f_{Y|X}(y_1|x_1)\; f_{Y|X}(y_2|x_2)\; f_{X_d(r,s,n,m,k)}(x_1,x_2)\; dx_1\; dx_2, \tag{1.14}$$

where $f_{X_d(r,s,n,m,k)}(x)$ is the joint pdf of $X_d(r,n,m,k)$ and $X_d(s,n,m,k)$, $1 \leq r < s \leq n$.

An excellent review on concomitants of order statistics is given by Bhattacharya (1984) and David and Nagaraja (1998). Balasubramanian and Beg (1996, 1997, 1998) studied the concomitants for bivariate exponential distribution of Marshall-Olkin, Morgesnstern type bivariate exponential distribution and Gumbel's bivariate exponential distribution and gave the recurrence relation between single and product moment of order statistics. Begum and Khan (1997, 1998, 2000) studied the concomitants of order statistics for Gumbel's bivariate Weibull distribution, bivariate Burr distribution, Marshall and Olkin bivariate Weibull distribution and established expression for single and product moments. Ahsanullah and Beg (2006) studied the concomitants for Gumbel's bivariate exponential distribution and derived the recurrence relation between single and product moment of generalized order statistics.

2 Probability Density Function of $Y_{[r,n,m,k]}$

For the FGM type bivariate inverse Rayleigh distribution as given in (1.5), using (1.7) and (1.11) in (1.13), the pdf of $r-th$ concomitant of *dgos* $Y_{[r,n,m,k]}$ for $m \neq -1$ is given as:

$$g_{[r,n,m,k]}(y) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \frac{4\theta_1\theta_2}{y^3} e^{-\frac{\theta_2}{y^2}} \sum_{i=0}^{r-1}(-1)^i \binom{r-1}{i}$$

$$\times \int_0^{\infty} \frac{1}{x^3} e^{-\frac{\theta_1\gamma_{r-i}}{x^2}} [1+\alpha(1-2e^{-\frac{\theta_1}{x^2}})(1-2e^{-\frac{\theta_2}{y^2}})]dx. \tag{2.1}$$

Setting $x^{-2} = t$ in (2.1), we get

$$= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \frac{2\theta_2}{y^3}\; e^{-\frac{\theta_2}{y^2}} \sum_{i=0}^{r-1}(-1)^i \binom{r-1}{i} \left[\frac{1}{\gamma_{r-i}} + \alpha\left(\frac{1}{\gamma_{r-i}} - \frac{2}{\gamma_{r-i}+1}\right)(1-2e^{-\frac{\theta_2}{y^2}})\right]. \tag{2.2}$$

$$= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \frac{2\theta_2}{y^3}\; e^{-\frac{\theta_2}{y^2}} \sum_{i=0}^{r-1}(-1)^i \binom{r-1}{i}$$

$$\times \left[\frac{1}{\frac{k}{m+1}+n-r+i} + \alpha\left(\frac{1}{\frac{k}{m+1}+n-r+i} - \frac{2}{\frac{k+1}{m+1}+n-r+i}\right)(1-2e^{-\frac{\theta_2}{y^2}})\right]. \tag{2.3}$$

$$= \frac{C_{r-1}}{(r-1)!(m+1)^{r}} \frac{2\theta_2}{y^3}\; e^{-\frac{\theta_2}{y^2}} \sum_{i=0}^{r-1}(-1)^i \binom{r-1}{i} \left[B\left(\frac{k}{m+1}+n-r+i,1\right)\right.$$

$$\left. +\alpha\left\{B\left(\frac{k}{m+1}+n-r+i,1\right) - 2B\left(\frac{k+1}{m+1}+n-r+i,1\right)\right\}(1-2e^{-\frac{\theta_2}{y^2}})\right]. \tag{2.4}$$

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For real positive p, c and a positive integer b, we have

$$\sum_{a=0}^{b}(-1)^a\binom{b}{a}B(a+p,c)=B(p,c+b). \tag{2.5}$$

Thus using (2.5) in (2.4), we get

$$g_{[r,n,m,k]}(y)=\frac{C_{r-1}}{(r-1)!(m+1)^r}\frac{2\theta_2}{y^3}\,e^{-\frac{\theta_2}{y^2}}\Big[B\Big(\frac{k}{m+1}+n-r,r\Big)$$

$$+\alpha\Big\{B\Big(\frac{k}{m+1}+n-r,r\Big)-2B\Big(\frac{k+1}{m+1}+n-r,r\Big)\Big\}(1-2e^{-\frac{\theta_2}{y^2}})\Big], \tag{2.6}$$

which after simplification yields

$$g_{[r,n,m,k]}(y)=\frac{2\theta_2}{y^3}\,e^{-\frac{\theta_2}{y^2}}\Big[1+\alpha(1-2e^{-\frac{\theta_2}{y^2}})\Big\{1-2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}\Big\}\Big]. \tag{2.7}$$

The expression (2.7) may also be represented as

$$g_{[r,n,m,k]}(y)=f(y)+\alpha(1-2F(y))f(y)\Big\{1-2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}\Big\}. \tag{2.8}$$

The above expression for $g_{[r,n,m,k]}(y)$ does not depend on $F(x)$ at all. Observing that $2F(y)f(y)$ is the pdf of $Y_{2,2}$ the second order statistic of a random sample of size two of the Y variate. we find that the distribution of the $r-th$ concomitant depends only on the marginal distribution of Y and the distribution of $Y_{2:2}$.

Now from (2.8), we have

$$g_{[r,n,m,k]}(y)=f_{1:1}(y)-\alpha\Big[1-2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}\Big]\Big[f_{2:2}(y)-f_{1:1}(y)\Big]. \tag{2.9}$$

It may be verified that $\int_0^\infty g_{[r,n,m,k]}(y)dy=1$.

Remark 2.1: Set $m=0,k=1$ in (2.7), to get the pdf of $r-th$ concomitant of order statistics from FGM type bivariate inverse Rayleigh distribution as

$$g_{[n-r+1:n]}(y)=\frac{2\theta_2}{y^3}e^{-\frac{\theta_2}{y^2}}\Big[1+\alpha(1-2e^{-\frac{\theta_2}{y^2}})\Big\{1-\frac{2(n-r+1)}{n+1}\Big\}\Big].$$

Replace $n-r+1$ by r, we get

$$g_{[r:n]}(y)=\frac{2\theta_2}{y^3}e^{-\frac{\theta_2}{y^2}}\Big[1+\alpha(1-2e^{-\frac{\theta_2}{y^2}})\Big\{1-\frac{2r}{n+1}\Big\}\Big].$$

Remark 2.2: At $m=-1$ in (2.7), we get the pdf of $r-th$ concomitant of $k-th$ lower record values from FGM type bivariate inverse Rayleigh distribution as

$$g_{[r,n,-1,k]}(y)=\frac{2\theta_2}{y^3}e^{-\frac{\theta_2}{y^2}}\Big[1+\alpha(1-2e^{-\frac{\theta_2}{y^2}})\Big\{1-2\Big(1+\frac{k}{k+1}\Big)^r\Big\}\Big].$$

3 Moments of $Y_{[r,n,m,k]}$

In this section,we derive the moments of $Y_{[r,n,m,k]}$ for FGM type bivariate inverse Rayleigh distribution by using the results of the previous section.

In view of (2.7), the moments of $Y_{[r,n,m,k]}$ is given as

$$E\Big[Y^{(a)}_{[r,n,m,k]}\Big]=2\theta_2\int_0^\infty y^a\frac{1}{y^3}e^{-\frac{\theta_2}{y^2}}\Big[1+\alpha(1-2e^{-\frac{\theta_2}{y^2}})\Big\{1-2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}\Big\}\Big]dy. \tag{3.1}$$

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If we put $y^{-2} = z$ in (3.1), then we have

$$= \theta_2 \int_0^\infty z^{(1-\frac{a}{2})-1} e^{-\theta_2 z} \left[1 + \alpha(1 - 2e^{-\theta_2 z})\left\{1 - 2\prod_{i=1}^{r}\left(1+\frac{1}{\gamma_i}\right)^{-1}\right\}\right] dz. \tag{3.2}$$

Note that

$$\int_0^\infty x^{\alpha-1} e^{-x} dx = \Gamma(\alpha). \tag{3.3}$$

Using (3.3) in (3.2), we get after simplification

$$E\left[Y^{(a)}_{[r,n,m,k]}\right] = (\theta_2)^{\frac{a}{2}} \Gamma(1-\frac{a}{2})\left[1 + \alpha(1 - (2)^{\frac{a}{2}})\left\{1 - 2\prod_{i=1}^{r}\left(1+\frac{1}{\gamma_i}\right)^{-1}\right\}\right]. \tag{3.4}$$

Remark 3.1: At $m = 0, k = 1$ in (3.4), we get the moments of concomitants of order statistics from FGM type bivariate inverse Rayleigh distribution as

$$E\left[Y^{(a)}_{[n-r+1:n]}\right] = (\theta_2)^{\frac{a}{2}} \Gamma(1-\frac{a}{2})\left[1 + \alpha(1 - (2)^{\frac{a}{2}})\left\{1 - \frac{2(n-r+1)}{n+1}\right\}\right].$$

Replace $n-r+1$ by r, we get

$$E\left[Y^{(a)}_{[r:n]}\right] = (\theta_2)^{\frac{a}{2}} \Gamma(1-\frac{a}{2})\left[1 + \alpha(1 - (2)^{\frac{a}{2}})\left\{1 - \frac{2r}{n+1}\right\}\right].$$

Remark 3.2: Setting $m = -1$ in (3.4), we get the moments of concomitants of $k-th$ lower record statistics from FGM type bivariate inverse Rayleigh distribution as

$$E\left[Y^{a}_{[r,n,-1,k]}\right] = (\theta_2)^{\frac{a}{2}} \Gamma(1-\frac{a}{2})\left[1 + \alpha(1 - (2)^{\frac{a}{2}})\left\{1 - 2\left(\frac{k}{k+1}\right)^{r}\right\}\right].$$

4 Relations Between Single Moments of Concomitants

In this section we shall present several recurrence relations between $pdfs, moments$ and $mgfs$ of concomitants.

The following relations between $pdfs$ can be seen in view of expression (2.9).

$$g_{[r,n,m,k]}(y) - g_{[r-1,n,m,k]}(y) = 2\alpha\left[\prod_{i=1}^{r}\left(1+\frac{1}{\gamma_i}\right)^{-1} - \prod_{i=1}^{r-1}\left(1+\frac{1}{\gamma_i}\right)^{-1}\right]\left[f_{2:2}(y) - f_{1:1}(y)\right]. \tag{4.1}$$

$$g_{[r,n,m,k]}(y) - g_{[r-1,n-1,m,k]}(y) = 2\alpha\left[\prod_{i=1}^{r}\left(1+\frac{1}{\gamma_i}\right)^{-1} - \prod_{i=1}^{r-1}\left(1+\frac{1}{\gamma_{i+1}}\right)^{-1}\right]\left[f_{2:2}(y) - f_{1:1}(y)\right]. \tag{4.2}$$

$$g_{[r-1,n,m,k]}(y) - g_{[r-1,n-1,m,k]}(y) = 2\alpha\left[\prod_{i=1}^{r-1}\left(1+\frac{1}{\gamma_i}\right)^{-1} - \prod_{i=1}^{r-1}\left(1+\frac{1}{\gamma_{i+1}}\right)^{-1}\right]\left[f_{2:2}(y) - f_{1:1}(y)\right]. \tag{4.3}$$

Now on applications of (4.1) to (4.3), we have the following Theorems:

Theorem 4.1: Let $n \in N,\ m \in \Re,\ \ 2 \le r \le n$. For a bivariate random variable (X,Y) having pdf (1.5), the following recurrence relations between moments of concomitants are valid:

$$\mu^{(a)}_{[r,n,m,k]} - \mu^{(a)}_{[r-1,n,m,k]} = 2\alpha\left[\prod_{i=1}^{r}\left(1+\frac{1}{\gamma_i}\right)^{-1} - \prod_{i=1}^{r-1}\left(1+\frac{1}{\gamma_i}\right)^{-1}\right]\left[\mu^{(a)}_{[2:2]} - \mu^{(a)}_{[1:1]}\right]. \tag{4.4}$$

$$\mu^{(a)}_{[r,n,m,k]} - \mu^{(a)}_{[r-1,n-1,m,k]} = 2\alpha\left[\prod_{i=1}^{r}\left(1+\frac{1}{\gamma_i}\right)^{-1} - \prod_{i=1}^{r-1}\left(1+\frac{1}{\gamma_{i+1}}\right)^{-1}\right]\left[\mu^{(a)}_{[2:2]} - \mu^{(a)}_{[1:1]}\right]. \tag{4.5}$$

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$$\mu^{(a)}_{[r-1,n,m,k]} - \mu^{(a)}_{[r-1,n-1,m,k]} = 2\alpha\left[\prod_{i=1}^{r-1}\left(1+\frac{1}{\gamma_i}\right)^{-1} - \prod_{i=1}^{r-1}\left(1+\frac{1}{\gamma_{i+1}}\right)^{-1}\right]\left[\mu^{(a)}_{[2:2]} - \mu^{(a)}_{[1:1]}\right]. \tag{4.6}$$

Remark 4.1: At $m=0, k=1$ in Theorem 4.1, we get recurrence relations between the moments of concomitants of order statistics from FGM type bivariate inverse Rayleigh distribution as

$$\mu^{(a)}_{[n-r+1:\,n]} - \mu^{(a)}_{[n-r+2:\,n]} = -\frac{2\alpha}{n+1}\left[\mu^{(a)}_{[2:2]} - \mu^{(a)}_{[1:1]}\right]. \tag{4.7}$$

$$\mu^{(a)}_{[n-r+1:\,n]} - \mu^{(a)}_{[n-r+2:\,n-1]} = -\frac{2\alpha(n-r+1)}{n(n+1)}\left[\mu^{(a)}_{[2:2]} - \mu^{(a)}_{[1:1]}\right]. \tag{4.8}$$

$$\mu^{(a)}_{[n-r+2:\,n]} - \mu^{(a)}_{[n-r+2:\,n-1]} = -\frac{2\alpha(r-1)}{n(n+1)}\left[\mu^{(a)}_{[2:2]} - \mu^{(a)}_{[1:1]}\right]. \tag{4.9}$$

Now after replacing $n-r+1$ by r, we get

$$\mu^{(a)}_{[r:\,n]} - \mu^{(a)}_{[r-1:\,n]} = -\frac{2\alpha}{n+1}\left[\mu^{(a)}_{[2:2]} - \mu^{(a)}_{[1:1]}\right]. \tag{4.10}$$

$$\mu^{(a)}_{[r:\,n]} - \mu^{(a)}_{[r-1:\,n-1]} = -\frac{2r\alpha}{n(n+1)}\left[\mu^{(a)}_{[2:2]} - \mu^{(a)}_{[1:1]}\right]. \tag{4.11}$$

$$\mu^{(a)}_{[r-1:\,n]} - \mu^{(a)}_{[r-1:\,n-1]} = -\frac{2\alpha(n-r)}{n(n+1)}\left[\mu^{(a)}_{[2:2]} - \mu^{(a)}_{[1:1]}\right]. \tag{4.12}$$

Remark 4.2: At $m=-1$ in Theorem 4.1, we get recurrence relations between the moments of concomitants of record statistics from FGM type bivariate inverse Rayleigh distribution as

$$\mu^{(a)}_{[r,n,-1,k]} - \mu^{(a)}_{[r-1,n,-1,k]} = 2\alpha\left[\left(\frac{k}{k+1}\right)^{r} - \left(\frac{k}{k+1}\right)^{r-1}\right]\left[\mu^{(a)}_{[2:2]} - \mu^{(a)}_{[1:1]}\right]. \tag{4.13}$$

Theorem 4.2: Let $n \in N,\ m \in \Re,\ \ 2 \le r \le n$. For a bivariate random variable (X,Y) having pdf (1.5), the following recurrence relations between mgf of concomitants are valid

$$M_{[r,n,m,k]}(t) - M_{[r-1,n,m,k]}(t) = 2\alpha\left[\prod_{i=1}^{r}\left(1+\frac{1}{\gamma_i}\right)^{-1} - \prod_{i=1}^{r-1}\left(1+\frac{1}{\gamma_i}\right)^{-1}\right]\left[M_{[2:2]}(t) - M_{[1:1]}(t)\right]. \tag{4.14}$$

$$\begin{aligned} M_{[r,n,m,k]}(t) - M_{[r-1,n-1,m,k]}(t) &= 2\alpha\left[\prod_{i=1}^{r}\left(1+\frac{1}{\gamma_i}\right)^{-1} - \prod_{i=1}^{r-1}\left(1+\frac{1}{\gamma_{i+1}}\right)^{-1}\right] \\ &\quad \times\left[M_{[2:2]}(t) - M_{[1:1]}(t)\right]. \end{aligned} \tag{4.15}$$

$$\begin{aligned} M_{[r-1,n,m,k]}(t) - M_{[r-1,n-1,m,k]}(t) &= 2\alpha\left[\prod_{i=1}^{r-1}\left(1+\frac{1}{\gamma_i}\right)^{-1} - \prod_{i=1}^{r-1}\left(1+\frac{1}{\gamma_{i+1}}\right)^{-1}\right] \\ &\quad \times\left[M_{[2:2]}(t) - M_{[1:1]}(t)\right]. \end{aligned} \tag{4.16}$$

Remark 4.3: At $m=0, k=1$ in Theorem 4.2, we get recurrence relations between the mgf of concomitants of order statistics from FGM type bivariate inverse Rayleigh distribution as

$$M_{[n-r+1:\,n]}(t) - M_{[n-r+2:\,n]}(t) = -\frac{2\alpha}{n+1}\left[M_{[2:2]}(t) - M_{[1:1]}(t)\right]. \tag{4.17}$$

$$M_{[n-r+1:\,n]}(t) - M_{[n-r+2:\,n-1]}(t) = -\frac{2\alpha(n-r+1)}{n(n+1)}\left[M_{[2:2]}(t) - M_{[1:1]}(t)\right]. \tag{4.18}$$

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$$M_{[n-r+2:\,n]}(t) - M_{[n-r+2:\,n-1]}(t) = -\frac{2\alpha(r-1)}{n(n+1)}\Big[M_{[2:2]}(t) - M_{[1:1]}(t)\Big]. \tag{4.19}$$

If we replace $n-r+1$ by r, we get

$$M_{[r:\,n]}(t) - M_{[r-1:\,n]}(t) = -\frac{2\alpha}{n+1}\Big[M_{[2:2]}(t) - M_{[1:1]}(t)\Big]. \tag{4.20}$$

$$M_{[r:\,n]}(t) - M_{[r-1:\,n-1]}(t) = -\frac{2r\alpha}{n(n+1)}\Big[M_{[2:2]}(t) - M_{[1:1]}(t)\Big]. \tag{4.21}$$

$$M_{[r-1:\,n]}(t) - M_{[r-1:\,n-1]}(t) = -\frac{2\alpha(n-r)}{n(n+1)}\Big[M_{[2:2]}(t) - M_{[1:1]}(t)\Big]. \tag{4.22}$$

Remark 4.4: At $m=-1$ in Theorem 4.2, we get recurrence relations between the mgf of concomitants of record statistics from FGM type bivariate inverse Rayleigh distribution as

$$M_{[r,n,-1,k]} - M_{[r-1,n,-1,k]} = 2\alpha\left[\left(\frac{k}{k+1}\right)^{r} - \left(\frac{k}{k+1}\right)^{r-1}\right]\Big[M_{[2:2]} - M_{[1:1]}\Big]. \tag{4.23}$$

Table 4.1 : Means of the concomitants of order statistics (Remark 3.1)

n	r	$\alpha=-1.0000,\ \theta_2=1.0000$	$\alpha=-0.5000,\ \theta_2=0.5000$	$\alpha=0.5000,\ \theta_2=0.5000$
1	1	1.7725	1.2533	1.2533
2	1	1.5277	1.3398	1.1668
	2	2.0172	1.1668	1.3398
3	1	1.4054	1.3831	1.1235
	2	1.7725	1.2533	1.2533
	3	2.1395	1.1235	1.3831
4	1	1.3319	1.4090	1.0976
	2	1.6256	1.3052	1.2014
	3	1.9193	1.2014	1.3052
	4	2.2130	1.0976	1.4090
5	1	1.2830	1.4263	1.0803
	2	1.5277	1.3398	1.1668
	3	1.7725	1.2533	1.2533
	4	2.0172	1.1668	1.3398
	5	2.2619	1.0803	1.4263
6	1	1.2480	1.4387	1.0679
	2	1.4578	1.3645	1.1421
	3	1.6676	1.2904	1.2162
	4	1.8773	1.2162	1.2904
	5	2.0871	1.1421	1.3645
	6	2.2969	1.0679	1.4387

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Table 4.2 : Means of the concomitants of lower record values(Remark 3.2)

r	$\alpha=-0.5000,\ \theta_2=0.500$	$\alpha=-1.0000,\ \theta_2=1.000$	$\alpha=0.5000,\ \theta_2=0.500$
1	1.3831	2.1395	1.1235
2	1.3182	1.9560	1.1884
3	1.2857	1.8642	1.2209
4	1.2695	1.8183	1.2371
5	1.2614	1.7954	1.2452
6	1.2614	1.7839	1.2492
7	1.2553	1.7782	1.2513
8	1.2543	1.7753	1.2523
9	1.2538	1.7739	1.2528
10	1.2536	1.7732	1.2530

5 Joint Probability Density Function of $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$

For the FGM type bivariate inverse Rayleigh distribution as given in (1.5), using (1.7) and (1.12) in (1.14), the joint pdf of $r-th$ and $s-th$ concomitants of $dgos$ $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ for $m\neq -1$ is

$$g_{[r,s,n,m,k]}(y_1,y_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\frac{2\theta_2}{y_1^3}\frac{2\theta_2}{y_2^3}e^{-\frac{\theta_2}{y_1^2}}e^{-\frac{\theta_2}{y_2^2}}$$
$$\times\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\int_0^\infty\frac{2\theta_1}{x_1^3}e^{-\frac{\theta_1(s-r+i-j)(m+1)}{x_1^2}}$$
$$\times[1+\alpha(1-2e^{-\frac{\theta_1}{x_1^2}})(1-2e^{-\frac{\theta_2}{y_1^2}})]I(x_1,y_2)dx_1 \qquad (5.1)$$

where,

$$I(x_1,y_2)=\int_0^{x_1}\frac{2\theta_1}{x_2^3}e^{-\frac{\theta_1\gamma_{s-j}}{x_2^2}}[1+\alpha(1-2e^{-\frac{\theta_1}{x_2^2}})(1-2e^{-\frac{\theta_2}{y_2^2}})]dx_2. \qquad (5.2)$$

By setting $x^{-2}=t$ in (5.2), we get after simplification

$$I(x_1,y_2)=\Big[\frac{e^{-\frac{\theta_1\gamma_{s-j}}{x_1^2}}}{\gamma_{s-j}}+\alpha(1-2e^{-\frac{\theta_2}{y_2^2}})\Big(\frac{e^{-\frac{\theta_1\gamma_{s-j}}{x_1^2}}}{\gamma_{s-j}}-2\frac{e^{-\frac{\theta_1(\gamma_{s-j}+1)}{x_1^2}}}{\gamma_{s-j}+1}\Big)\Big]. \qquad (5.3)$$

Substituting the value of $I(x_1,y_2)$ in (5.1),we get

$$g_{[r,s,n,m,k]}(y_1,y_2)=\frac{C_{s-1}\frac{2\theta_2}{y_1^3}\frac{2\theta_2}{y_2^3}e^{-\frac{\theta_2}{y_1^2}}e^{-\frac{\theta_2}{y_2^2}}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}$$
$$\times\int_0^\infty\frac{2\theta_1}{x_1^3}e^{-\frac{\theta_1\gamma_{r-i}}{x_1^2}}\Big[\frac{1}{\gamma_{s-j}}+\alpha(1-2e^{-\frac{\theta_2}{y_2^2}})\Big(\frac{1}{\gamma_{s-j}}-2\frac{e^{-\frac{\theta_1}{x_1^2}}}{\gamma_{s-j}+1}\Big)\Big]$$
$$\times\Big[1+\alpha\Big(1-2e^{-\frac{\theta_2}{y_1^2}}\Big)\Big(1-2e^{-\frac{\theta_1}{x_1^2}}\Big)\Big]dx_1. \qquad (5.4)$$

Let $x_1^{-2}=t_2$ in (5.4), we get after simplification

$$g_{[r,s,n,m,k]}(y_1,y_2)=\frac{2\theta_1}{y_1^3}\frac{2\theta_2}{y_2^3}e^{-\frac{\theta_2}{y_1^2}}e^{-\frac{\theta_2}{y_2^2}}\Big[1+\alpha^2\Big(1-2e^{-\frac{\theta_2}{y_1^2}}\Big)\Big(1-2e^{-\frac{\theta_2}{y_2^2}}\Big)$$

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$$\times\left\{1-2\prod_{i=1}^{r}\left(1+\frac{1}{\gamma_i}\right)^{-1}-2\prod_{i=1}^{s}\left(1+\frac{1}{\gamma_i}\right)^{-1}+4\prod_{i=1}^{r}\left(1+\frac{2}{\gamma_i}\right)^{-1}\right\}$$

$$+\alpha\left\{\left(1-2e^{-\frac{\theta_2}{y_1^2}}\right)+\left(1-2e^{-\frac{\theta_2}{y_2^2}}\right)\right\}\left\{1-2\prod_{i=1}^{s}\left(1+\frac{1}{\gamma_i}\right)^{-1}\right\}\Bigg]. \tag{5.5}$$

It may be verified that $\int_0^\infty\int_0^\infty g_{[r,s,n,m,k]}(y_1,y_2)dy_1dy_2=1$.

Remark 5.1: By setting $m=0,\ k=1$ in (5.5), we get the joint pdf of two concomitants of order statistics and at $m=-1$, we have the joint pdf of two concomitants of $k-th$ lower record values for FGM type bivariate inverse Rayleigh distribution.

6 Product Moments of Two Concomitants $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$

Product moments of two concomitant $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ is given as

$$E\left(Y_{[r,n,m,k]}^{(a)}Y_{[s,n,m,k]}^{(b)}\right)=\int_0^\infty\int_0^\infty y_1^a\,y_2^b\ g_{[r,s,n,m,k]}(y_1,y_2)\,dy_1\,dy_2. \tag{6.1}$$

In view of (5.5) and (6.1), we have

$$E\left(Y_{[r,n,m,k]}^{(a)}Y_{[s,n,m,k]}^{(b)}\right)=\int_0^\infty y_2^{b-3}e^{-\frac{\theta_2}{y_2^2}}\left\{\int_0^\infty y_1^{a-3}\,e^{-\frac{\theta_2}{y_1^2}}\left[1+\alpha^2\left(1-2e^{-\frac{\theta_2}{y_1^2}}\right)\left(1-2e^{-\frac{\theta_2}{y_2^2}}\right)\right.\right.$$

$$\times\left\{1-2\prod_{i=1}^{r}\left(1+\frac{1}{\gamma_i}\right)^{-1}-2\prod_{i=1}^{s}\left(1+\frac{1}{\gamma_i}\right)^{-1}+4\prod_{i=1}^{r}\left(1+\frac{2}{\gamma_i}\right)^{-1}\right\}$$

$$+\alpha\left\{\left(1-2e^{-\frac{\theta_2}{y_1^2}}\right)+\left(1-2e^{-\frac{\theta_2}{y_2^2}}\right)\right\}\left\{1-2\prod_{i=1}^{s}\left(1+\frac{1}{\gamma_i}\right)^{-1}\right\}\Bigg]dy_1\Bigg\}dy_2. \tag{6.2}$$

Let $y_1^{-2}=z_1$ in (6.2)and simplify, we have

$$=\Gamma(1-\frac{a}{2})(\theta_2)^{\frac{a}{2}}\int_0^\infty y_2^{b-3}e^{-\frac{\theta_2}{y_2^2}}\left[1+\alpha^2\,(1-(2)^{\frac{a}{2}})\left(1-2e^{-\frac{\theta_2}{y_2^2}}\right)\right.$$

$$\times\left\{1-2\prod_{i=1}^{r}\left(1+\frac{1}{\gamma_i}\right)^{-1}-2\prod_{i=1}^{s}\left(1+\frac{1}{\gamma_i}\right)^{-1}+4\prod_{i=1}^{r}\left(1+\frac{2}{\gamma_i}\right)^{-1}\right\}$$

$$+\alpha\left\{(1-(2)^{\frac{a}{2}})+\left(1-2e^{-\frac{\theta_2}{y_2^2}}\right)\right\}\left\{1-2\prod_{i=1}^{s}\left(1+\frac{1}{\gamma_i}\right)^{-1}\right\}\Bigg]dy_1. \tag{6.3}$$

Set $y_2^{-2}=z_2$ in (6.3), to get

$$E\left(Y_{[r,n,m,k]}^{(a)}Y_{[s,n,m,k]}^{(b)}\right)=\Gamma(1-\frac{a}{2})\Gamma(1-\frac{b}{2})(\theta_2)^{\frac{a+b}{2}}\left[1+\alpha^2(1-(2)^{\frac{a}{2}})(1-(2)^{\frac{b}{2}})\right.$$

$$\times\left\{1+4\prod_{i=1}^{r}\left(1+\frac{2}{\gamma_i}\right)^{-1}-2\prod_{i=1}^{r}\left(1+\frac{1}{\gamma_i}\right)^{-1}-2\prod_{i=1}^{s}\left(1+\frac{1}{\gamma_i}\right)^{-1}\right\}\Bigg]$$

$$+\alpha\left\{(1-(2)^{\frac{a}{2}})+(1-(2)^{\frac{b}{2}})\right\}\left\{1-2\prod_{i=1}^{s}\left(1+\frac{1}{\gamma_i}\right)^{-1}\right\}. \tag{6.4}$$

Remark 6.1: Set $m=0,\ k=1$ in (6.4), to get product moments of concomitants order statistics from FGM type bivariate inverse Rayleigh distribution as

$$E\left(Y_{[n-s+1:n]}^{(a)}Y_{[n-r+1:n]}^{(b)}\right)=\Gamma(1-\frac{a}{2})\Gamma(1-\frac{b}{2})(\theta_2)^{\frac{a+b}{2}}\left[1+\alpha^2(1-(2)^{\frac{a}{2}})(1-(2)^{\frac{b}{2}})\right.$$

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$$\times\left\{1+4\left(\frac{(n-s+1)(n-s+2)}{(n+1)(n+2)}\right)-2\left(\frac{n-s+1}{n+1}\right)-2\left(\frac{n-r+1}{n+1}\right)\right\}$$

$$+\alpha\left\{(1-(2)^{\frac{a}{2}})+(1-(2)^{\frac{b}{2}})\right\}\left\{1-2\left(\frac{n-r+1}{n+1}\right)\right\}\Big].$$

By replacing $n-s+1$ by r and $n-r+1$ by s, we have

$$E\left(Y_{[r:n]}^{(a)}Y_{[s:n]}^{(b)}\right)=\Gamma(1-\frac{a}{2})\Gamma(1-\frac{b}{2})(\theta_2)^{\frac{a+b}{2}}\Big[1+\alpha^2(1-(2)^{\frac{a}{2}})(1-(2)^{\frac{b}{2}})$$

$$\times\left\{1+4\left(\frac{r(r+1)}{(n+1)(n+2)}\right)-2\left(\frac{r}{n+1}\right)-2\left(\frac{s}{n+1}\right)\right\}$$

$$+\alpha\left\{(1-(2)^{\frac{a}{2}})+(1-(2)^{\frac{b}{2}})\right\}\left\{1-2\left(\frac{s}{n+1}\right)\right\}\Big].$$

Remark 6.2: At $m=-1$ in (6.4), we get the product moments of concomitants of $k-th$ lower record statistics from FGM type bivariate inverse Rayleigh distribution as

$$E\left(Y_{[r,n,-1,k]}^{(a)}Y_{[s,n,-1,k]}^{(b)}\right)=\Gamma(1-\frac{a}{2})\Gamma(1-\frac{b}{2})(\theta_2)^{\frac{a+b}{2}}\Big[1+\alpha^2(1-(2)^{\frac{a}{2}})(1-(2)^{\frac{b}{2}})$$

$$\times\left\{1+4\left(\frac{k}{k+2}\right)^r-2\left(\frac{k}{k+1}\right)^r-2\left(\frac{k}{k+1}\right)^s\right\}$$

$$+\alpha\left\{1-(2)^{\frac{a}{2}})+(1-(2)^{\frac{b}{2}})\right\}\left\{1-2\left(\frac{k}{k+1}\right)^s\right\}\Big].$$

7 Relations Between Product Moments of Two Concomitants

In this section we shall present recurrence relations between joint $pdfs$, product moments and $mgfs$ of two concomitants.

From (5.5), we have

$$g_{[r,s,n,m,k]}(y_1,y_2)=f(y_1)f(y_2)-\alpha[f_{2:2}(y_1)-f_{1:1}(y_1)]f(y_2)\left[1-2\prod_{i=1}^{s}\left(1+\frac{1}{\gamma_i}\right)^{-1}\right]$$

$$-\alpha[f_{2:2}(y_2)-f_{1:1}(y_2)]f(y_1)\left[1-2\prod_{i=1}^{s}\left(1+\frac{1}{\gamma_i}\right)^{-1}\right]$$

$$+\alpha^2\,[f_{2:2}(y_1)-f_{1:1}(y_1)][f_{2:2}(y_2)-f_{1:1}(y_2)]$$

$$\times\left\{1-2\prod_{i=1}^{r}\left(1+\frac{1}{\gamma_i}\right)^{-1}-2\prod_{i=1}^{s}\left(1+\frac{1}{\gamma_i}\right)^{-1}+4\prod_{i=1}^{r}\left(1+\frac{2}{\gamma_i}\right)^{-1}\right\}. \quad (7.1)$$

Now using $1-2F(y)=1-F(y)-F(y)$ in (7.1), we get

$$g_{[r,s,n,m,k]}(y_1,y_2)=f(y_1)f(y_2)+\frac{\alpha}{2}[f_{1:2}(y_1)-f_{2:2}(y_1)]f(y_2)\left[1-2\prod_{i=1}^{r}\left(1+\frac{1}{\gamma_i}\right)^{-1}\right]$$

$$-\frac{\alpha}{2}\,[f_{1:2}(y_2)-f_{2:2}(y_2)]f(y_1)\left[1-2\prod_{i=1}^{r}\left(1+\frac{1}{\gamma_i}\right)^{-1}\right]$$

$$+\frac{\alpha^2}{4}\,[f_{1:2}(y_1)-f_{2:2}(y_1)][f_{1:2}(y_2)-f_{2:2}(y_2)]$$

$$\times\left\{1-2\prod_{i=1}^{r}\left(1+\frac{1}{\gamma_i}\right)^{-1}-2\prod_{i=1}^{s}\left(1+\frac{1}{\gamma_i}\right)^{-1}+4\prod_{i=1}^{r}\left(1+\frac{2}{\gamma_i}\right)^{-1}\right\}. \quad (7.2)$$

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$$g_{[r,s,n,m,k]}(y_1,y_2)-g_{[r,s-1,n,m,k]}(y_1,y_2)$$

$$=-\frac{\alpha^2}{2}\left\{\prod_{i=1}^{s}\left(1+\frac{1}{\gamma_i}\right)^{-1}-\prod_{i=1}^{s-1}\left(1+\frac{1}{\gamma_i}\right)^{-1}\right\}[f_{1:2}(y_1)-f_{2:2}(y_1)][f_{1:2}(y_2)-f_{2:2}(y_2)]. \tag{7.3}$$

Using (7.3), the recurrence relation for joint moment generating function of $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ is given as

$$M_{[r,s,n,m,k]}(t_1,t_2)-M_{[r,s-1,n,m,k]}(t_1,t_2)$$

$$=-\frac{\alpha^2}{2}\left\{\prod_{i=1}^{s}\left(1+\frac{1}{\gamma_i}\right)^{-1}-\prod_{i=1}^{s-1}\left(1+\frac{1}{\gamma_i}\right)^{-1}\right\}[M_{1:2}(t_1)-M_{2:2}(t_1)][M_{1:2}(t_2)-M_{2:2}(t_2)]. \tag{7.4}$$

Remark 7.1: Set $m=0,k=1$ in (7.4), to get the recurrence relation for joint mgf of two concomitants of order statistics from FGM type bivariate inverse Rayleigh distribution

$$M_{[n-s+1,n-r+1:n]}(t_1,t_2)-M_{[n-s+1,n-r+2:n]}(t_1,t_2)$$
$$=-\frac{\alpha^2}{2}\frac{1}{n+1}[M_{1:2}(t_1)-M_{2:2}(t_1)][M_{1:2}(t_2)-M_{2:2}(t_2)].$$

Replacing $n-r+1$ by s and $n-s+1$ by r , we get

$$M_{[r,s:n]}(t_1,t_2)-M_{[r,s-1:]}(t_1,t_2)$$
$$=-\frac{\alpha^2}{2}\frac{1}{n+1}[M_{1:2}(t_1)-M_{2:2}(t_1)][M_{1:2}(t_2)-M_{2:2}(t_2)].$$

Remark 7.2: By setting $m=-1$ in (7.4), we get the recurrence relation for joint mgf of two concomitants of $k-th$ lower record statistics from FGM type bivariate inverse Rayleigh distribution as

$$M_{[r,s,n,-1,k]}(t_1,t_2)-M_{[r,s-1,n,-1,k]}(t_1,t_2)$$
$$=-\frac{\alpha^2}{2}\left\{\left(\frac{k}{k+1}\right)^{s}-\left(\frac{k}{k+1}\right)^{s-1}\right\}[M_{1:2}(t_1)-M_{2:2}(t_1)][M_{1:2}(t_2)-M_{2:2}(t_2)].$$

Now using (7.3), the recurrence relation for product moment of $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ is given as

$$\mu^{(a,b)}_{[r,s,n,m,k]}-\mu^{(a,b)}_{[r,s-1,n,m,k]}$$

$$=-\frac{\alpha^2}{2}\left\{\prod_{i=1}^{s}\left(1+\frac{1}{\gamma_i}\right)^{-1}-\prod_{i=1}^{s-1}\left(1+\frac{1}{\gamma_i}\right)^{-1}\right\}[\mu^{(a)}_{[1:2]}-\mu^{(a)}_{[2:2]}][\mu^{(b)}_{[1:2]}-\mu^{(b)}_{[2:2]}]. \tag{7.5}$$

Remark 7.3: Set $m=0,k=1$ in (7.5), to get the recurrence relation for product moments of two concomitants of order statistics from FGM type bivariate inverse Rayleigh distribution as

$$\mu^{(a,b)}_{[n-s+1,n-r+1:n]}-\mu^{(a,b)}_{[n-s+1,n-r+2:n]}=-\frac{\alpha^2}{2}\frac{1}{n+1}[\mu^{(a)}_{[1:2]}-\mu^{(a)}_{[2:2]}][\mu^{(b)}_{[1:2]}-\mu^{(b)}_{[2:2]}].$$

Replace $n-r+1$ by s and $n-s+1$ by r , we have

$$\mu^{(a,b)}_{[r,s:n]}-\mu^{(a,b)}_{[r,s-1:n]}=-\frac{\alpha^2}{2}\frac{1}{n+1}[\mu^{(a)}_{[1:2]}-\mu^{(a)}_{[2:2]}][\mu^{(b)}_{[1:2]}-\mu^{(b)}_{[2:2]}].$$

Remark 7.4: Set $m=-1$ in (7.5), to get the recurrence relation for product moments of two concomitants of $k-th$ lower record statistics from FGM type bivariate inverse Rayleigh distribution as

$$\mu^{(a,b)}_{[r,s,n,-1,k]}-\mu^{(a,b)}_{[r,s-1,n,-1,k]}$$
$$=-\frac{\alpha^2}{2}\left\{\left(\frac{k}{k+1}\right)^{s}-\left(\frac{k}{k+1}\right)^{s-1}\right\}[\mu^{(a)}_{[1:2]}-\mu^{(a)}_{[2:2]}][\mu^{(b)}_{[1:2]}-\mu^{(b)}_{[2:2]}].$$

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Table 7.1: Product moments between the concomitants of order statistics (Remark 6.1)

			$\alpha = 0.5000$		$\theta_2 = 0.5000$				
n	$s \setminus r$	1	2	3	4	5	6	7	8
1	1	3.1410							
2	1	2.5802							
	2	3.1635	3.1859						
3	1	2.3902							
	2	2.7791	2.7881						
	3	3.1680	3.1769	3.2129					
4	1	2.2953							
	2	2.5869	2.5892						
	3	2.8786	2.8808	2.9010					
	4	3.1702	3.1724	3.1927	3.2308				
5	1	2.2386							
	2	2.4719	2.4706						
	3	2.7052	2.7039	2.7154					
	4	2.9385	2.9372	2.9488	2.9731				
	5	3.1718	3.1705	3.1821	3.2065	3.2437			
6	1	2.2010							
	2	2.3954	2.3922						
	3	2.5898	2.5866	2.5930					
	4	2.7842	2.7810	2.7874	2.8035				
	5	2.9787	2.9755	2.9819	2.9979	3.0236			
	6	3.1731	3.1699	3.1763	3.1923	3.2180	3.2533		
7	1	2.1742							
	2	2.3409	2.3366						
	3	2.5076	2.5033	2.5065					
	4	2.6742	2.6699	2.6731	2.6838				
	5	2.8409	2.8366	2.8398	2.8505	2.8687			
	6	3.0075	3.0032	3.0064	3.0171	3.0353	3.0610		
	7	3.1742	3.1699	3.1731	3.1838	3.2020	3.2276	3.2608	
8	1	2.1543							
	2	2.3001	2.2953						
	3	2.4460	2.4411	2.4422					
	4	2.5918	2.5869	2.5880	2.5952				
	5	2.7376	2.7327	2.7339	2.7410	2.7541			
	6	2.8834	2.8786	2.8797	2.8868	2.8999	2.9190		
	7	3.0292	3.0244	3.0255	3.0326	3.0457	3.0648	3.0899	
	8	3.1751	3.1702	3.1713	3.1784	3.1915	3.2106	3.2357	3.2668

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Table 7.2: Product moments between the concomitants of lower record statistics (Remark 6.2)

n	r	$\alpha = -1.000$, $\theta_2 = 1.000$	$\alpha = -0.500$, $\theta_2 = 0.500$	$\alpha = 0.500$, $\theta_2 = 0.500$
1	1	3.3214	1.5933	1.5933
2	1	4.8922	1.9523	1.3016
	2	4.6826	1.9261	1.2754
3	1	5.6776	2.1318	1.1558
	2	5.4680	2.1056	1.1296
	3	5.4430	2.1025	1.1265
4	1	6.0703	2.2216	1.0829
	2	5.8607	2.1954	1.0567
	3	5.8357	2.1922	1.0536
	4	5.8499	2.1940	1.0554
5	1	6.2667	2.2664	1.0465
	2	6.0570	2.2402	1.0203
	3	6.0321	2.2371	1.0171
	4	6.0462	2.2389	1.0189
	5	6.0622	2.2409	1.0209

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