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Solution of Master Equations for the Anharmonic Oscillator Interacting with a Heat Bath and for Parametric Down Conversion Process

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We solve analytically master equations that describe a cavity filled with a kerr medium, taking into account the dissipation induced by the environment, and parametric down conversion processes. We use superoperator techniques.

Keywords: Superoperator techniques, master equations, harmonic oscillator.

1 Introduction

The study of the quantum anharmonic oscillator has attracted some attention because of their significant role in nonlinear quantum optics, such as the prediction of Yurker and Stoler that with this system Schrödinger cat state can be produced [1]. The study of the anharmonic oscillator, in the isolated case, was carried out by Milburn [2]. The study of the anharmonic oscillator interacting with a heat bath was made by many authors [3–7]. In particular, Milburn and Holmes [4] studied this system when the oscillators in the reservoir are at zero temperature. Daniel and Milburn [3] and Peřinová and Lukš [5] studied this model with the reservoir at temperature different from zero. All these authors found that the quantum coherence effects are destroyed because of the interaction of the system with the environment. This phenomenon was called decoherence [8]. A common feature of the majority of these studies is that they were realized using the usual technique of finding a Fokker-Planck type equation and the result found was that the time of environment-induced decoherence is very short. Recently there have been some studies of the behaviour of this system [9, 10] that shows that there can be quantum effects at times greater than the decoherence time.

It is well known that one way to produce optical bistability is for instance by injecting a field inside a high Q cavity filled with a Kerr medium [11]. Solving the master equation for stationary conditions shows that optical bistability may be produced in such a system.

In this contribution we show solutions of master equations using superoperator techniques for both parametric down conversion and for the field interacting with a Kerr medium. This enable us to find the evolution of the system for an arbitrary initial state.

2 Kerr Medium and Dissipation

Milburn and Holmes [4] have studied the damped annharmonic oscillator by mapping the master equation in quantum state space into an evolution equation in classical state space. They managed to solve the master equation for an initial coherent state by using Fokker-Planck equations. Here we develop a solution that may be applied to any initial state. We write the master equation (in a frame rotating at field frequency ω) for a lossy cavity filled with a Kerr medium

$$\dot{\rho} = (S + J_- + L)\rho, \tag{2.1}$$

where the superoperators S, J and L are defined as

$$S\rho = -i\chi[a^{\dagger 2}a^2, \rho] \tag{2.2}$$

and

$$J_-\rho = 2\gamma_- a\rho a^\dagger, \qquad L\rho = -\gamma_-(a^\dagger a\rho + \rho a^\dagger a). \tag{2.3}$$

It is not difficult to show that the solution to (2.1) is

$$\rho(t) = e^{St}e^{Lt}\exp\left(\frac{1 - e^{-2t(\gamma_- + i\chi R)}}{2(\gamma_- + i\chi R)}J_-\right)\rho(0), \tag{2.4}$$

where

$$R\rho = a^\dagger a\rho - \rho a^\dagger a. \tag{2.5}$$

To check that the above density matrix is the solution of Equation (2.1) we need to use the commutation relations

$$[L, J_-]\rho = 2\gamma_- J_-\rho, \qquad [S, J_-]\rho = i2\chi R J_-\rho, \qquad [R, J_-]\rho = 0. \tag{2.6}$$

3 Non-Zero Temperature

The master equation in the non-zero temperature case is slightly different from (2.1)

$$\dot{\tilde{\rho}} = (S + J_- + J_+ + \mathcal{L} + C_\gamma)\tilde{\rho}. \tag{3.1}$$

superoperators that appear are defined as

$$J_+\rho = 2\gamma_+ a^\dagger \rho a, \qquad \mathcal{L}\rho = -\gamma_0(a^\dagger a\rho + \rho a^\dagger a). \tag{3.2}$$

C_γ is a constant. We will now perform a set of transformations to arrive to a master equation that has a known solution. First we get rid off the constant term via $\tilde{\rho} = \exp(C_\gamma t)\rho$ to obtain

$$\rho = e^{C_\gamma t}\rho, \tag{3.3}$$

$$\dot{\rho} = (S + J_- + J_+ + \mathcal{L})\rho. \tag{3.4}$$

Next we do $\rho = \exp(St)\rho_1$ so we obtain

$$\dot{\rho}_1 = (J_- e^{-2i\chi Rt} + J_+ e^{2i\chi Rt} + \mathcal{L})\rho_1. \tag{3.5}$$

The next transformation allows us to get rid off the superoperator J_+, $\rho_1 = \exp(\beta e^{2i\chi Rt} J_+)\rho_2$, with

$$\beta_{1,2} = \frac{(i\chi R + \gamma_0 \pm \sqrt{(\gamma_0 + i\chi R)^2 - 4\gamma_-\gamma_+})}{4\gamma_-\gamma_+}, \tag{3.6}$$

so we get

$$\dot{\rho}_2 = (J_- e^{-2i\chi Rt} + \alpha\mathcal{L} + F)\rho_2, \tag{3.7}$$

where $\alpha = 1 - 4\beta\gamma_-\gamma_+/\gamma_0$ and $F = 4\gamma_-\gamma_+\beta$. We have almost succeeded in taking the master equation to a known one, as the superoperators involved in the above equation are the ones involved in the master equation at zero temperature (because they commute with R and functions of it). Therefore we do $\rho_2 = \exp[F(R)t]\rho_3$

$$\dot{\rho}_3 = (J_- e^{-2i\chi Rt} + \alpha\mathcal{L})\rho_3, \tag{3.8}$$

and by doing $\rho_3 = \exp[\delta J_- e^{-\chi 2iRt}]\rho_4$ we arrive to the equation

$$\dot{\rho}_4 = \alpha\mathcal{L}\rho_4, \tag{3.9}$$

where we have set

$$\delta = -\frac{1}{2(\gamma_0\alpha + i\chi R)} \tag{3.10}$$

with the solution

$$\dot{\rho}_4(t) = \exp\left[\alpha(R)t\mathcal{L}\right]\rho_4(0). \tag{3.11}$$

We now write the solution for $\tilde{\rho}$ with the initial condition $\tilde{\rho}(0)$

$$\tilde{\rho}(t) = e^{C_\gamma t} e^{St} e^{\beta e^{2i\chi Rt} J_+} e^{F(R)t} e^{\delta J_- e^{-2i\chi Rt}} e^{\alpha(R)t\mathcal{L}} e^{-\delta J_-} e^{-\beta J_+}\tilde{\rho}(0). \tag{3.12}$$

4 Parametric Down Conversion

Parametric down conversion processes consists in the conversion of an incident photon of frequency ω into two photons of frequency $\omega/2$, called *idler* and *signal photons*. This process occurs in the interaction of an electromagnetic field with a nonlinear crystal. In the case where the incident wave is very intense, the Hamiltonian of this interaction is given by [12]:

$$H = \hbar\omega a^\dagger a + \hbar(\epsilon a^{\dagger 2} + \epsilon^* a^2). \tag{4.1}$$

We consider the interaction of this system with a reservoir. In this case, the master equation at nonzero temperature in the so called diffusive limit i.e, when the damping constant goes to zero $\kappa \to 0$ and the number of thermal photon goes to infinity, but keeping the product $\gamma = \kappa\bar{n}$ finite, may be written as (in the interaction picture)

$$\frac{d\rho}{dt} = (S + J + K + L)\rho, \tag{4.2}$$

where

$$S\rho = -i[\epsilon a^{\dagger 2} + \epsilon^* a^2, \rho], \tag{4.3}$$

and

$$J\rho = 2\gamma a\rho a^{\dagger 2}, \quad K\rho = 2\gamma a^\dagger \rho a, \quad L\rho = -\gamma(a^\dagger a\rho + \rho a^\dagger a). \tag{4.4}$$

The terms S and L suggest a phase sensitive reservoir as they may be expressed as only one superoperator via a squeeze transformation. Because of this, we introduce some operators used in the phase sensitive reservoir we previously studied [13] via the transformations

$$\begin{aligned} \rho_1 &= e^{\alpha_+ J_+}\rho, \quad J_+\rho = a^\dagger \rho a^\dagger, \\ \tilde{\rho} &= e^{\alpha_- J_-}\rho_1, \quad J_- = a\rho a. \end{aligned} \tag{4.5}$$

The commutation relations between J_-, J_+ and the relevant superoperator in (4.2) are

$$[J_+, J]\rho = -2\gamma\rho a^{\dagger 2}, \qquad [J_+, K]\rho = 2\gamma a^{\dagger 2}\rho, \tag{4.6}$$

$$[J_+, S]\rho = i\frac{\epsilon^*}{\gamma}(J + K)\rho, \qquad [J_-, J]\rho = -2\gamma a^2\rho, \tag{4.7}$$

$$[J_-, K]\rho = 2\gamma\rho a^2, \qquad [J_-, S]\rho = -i\frac{\epsilon}{\gamma}(J + K)\rho, \tag{4.8}$$

and

$$[J_-, L]\rho = [J_+, L]\rho = 0, \qquad [J_-, R_+]\rho = \frac{\beta}{\gamma}(K + J). \tag{4.9}$$

We still do the definition

$$X_-\rho = i\epsilon\rho a^{\dagger 2}, \qquad X_+\rho = -i\epsilon^* a^2\rho, \tag{4.10}$$
$$Y_-\rho = -i\epsilon a^{\dagger 2}\rho, \qquad Y_+\rho = i\epsilon^*\rho a^2, \tag{4.11}$$

so that we write the superoperator S as

$$S\rho = (X_- + X_+ + Y_- + Y_+)\rho. \tag{4.12}$$

After transformation we obtain

$$\begin{aligned}\frac{d\tilde{\rho}}{dt} &= \left[S + (J+K)\left(1 + i\epsilon^*\frac{\alpha_+}{\gamma} - i\epsilon\frac{\alpha_-}{\gamma} - i\epsilon\frac{\alpha_-}{\gamma}\beta\right) + L\right]\tilde{\rho} \\ &+ \left[(X_+ + Y_+)\left[2i\gamma\frac{\alpha_-}{\epsilon^*}\left(1 + i\epsilon^*\frac{\alpha_+}{\gamma}\right) + \beta\alpha_-^2\frac{\epsilon}{\epsilon^*}\right] + \beta(X_- + Y_-)\right]\tilde{\rho}\end{aligned} \tag{4.13}$$

with $\beta = 2i\gamma\alpha_+ - \epsilon^*\alpha_+^2$. In order to cancel the term $X_- + Y_-$ with the same term in S, see equation (4.12), we need $\beta = -1$, which produces the value for α_+

$$\alpha_+ = \frac{-\gamma \pm \sqrt{\gamma^2 - |\epsilon|^2}}{i\epsilon^*}. \tag{4.14}$$

Finally, to cancel the term $X_+ + Y_+$ we have to find the value for α_-, which is given by

$$\alpha_- = \mp\frac{i\epsilon^*}{2\sqrt{\gamma^2 - |\epsilon|^2}}. \tag{4.15}$$

In [15] there was a mistake in finding such parameters, which however does not change the conclusion of it. With the choices given above, (4.13) is finally written in the form

$$\frac{d\tilde{\rho}}{dt} = \left[(J+K)\sqrt{1 - \frac{|\epsilon|^2}{\gamma^2}} + L\right]\tilde{\rho}, \tag{4.16}$$

i.e. a master equation that has only the terms present in the case of losses at nonzero temperature and which has been solved in [14].

The parameter α_- has a singularity at $\gamma^2 = |\epsilon|^2$, which indicates that it was not necessary to make the (second) transformation to ρ_1, see (4.5). In that case it would be only needed to do the first transformation, which yields master equation

$$\rho_1 = (X_+ + Y_+ + L)\rho_1,$$

and because

$$[X_+ + Y_+), L]\rho = -2\gamma(X_+ + Y_+)\rho, \tag{4.17}$$

the superoperators involved close an algebra and the solution may be easily found [14].

Finally, it is worth to note that, if we consider the following definitions

$$N = J_+ + J_-, \tag{4.18}$$

$$M = \frac{iX_-}{\epsilon} - \frac{iX_+}{\epsilon^*}, \tag{4.19}$$

$$Q = \frac{iY_-}{\epsilon} - \frac{iY_+}{\epsilon^*}, \tag{4.20}$$

$$J' = \frac{2J}{\gamma}, \qquad K' = \frac{2K}{\gamma}, \tag{4.21}$$

$[,] =$	M	J'	N	Q	K'
M	0	0	$-J'$	$4\left(\frac{L}{\gamma}-1\right)$	$-2N$
J'	0	0	$-4M$	$8N$	$-16\left(\frac{L}{\gamma}-1\right)$
N	J'	$4M$	0	K'	$4Q$
Q	$-4\left(\frac{L}{\gamma}-1\right)$	$-8N$	$-K'$	0	0
K'	$2N$	$16\left(\frac{L}{\gamma}-1\right)$	$-4Q$	0	0

Table 4.1: Table of commutators of relevant superoperators.

we can identify two subalgebras in the table $\{J', M, N\}, \{K', Q, N\}$.

5 Conclusions

We have shown how a solution to master equations describing the interaction of an anharmonic oscillator and a heat bath can be obtained. Also, we have given a solution to the master equation describing parametric down conversion.

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