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Connected Resolving Partitions in Unicyclic Graphs

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Abstract: A k-partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ is resolving if for every two distinct vertices u and v of a connected graph G, there is a set S_i in Π so that the minimum distance between u and a vertex of S_i is different from the minimum distance between v and a vertex of S_i. A resolving partition Π is said to be connected if each subgraph $< S_i >$ induced by S_i $(1 \le i \le k)$ is connected in G. In this paper, we investigate the minimum connected resolving partitions in unicyclic graphs. Also, modified sharp lower and upper bounds for the connected partition dimension of unicyclic graphs are provided.

Keywords: partition dimension, connected partition dimension, unicyclic graphs
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1 Introduction

Partition dimension was firstly studied by Chartrand, Salehi and Zhang in [5,6] perhaps as a variation of metric dimension. Resolving sets and resolving partitions have since been widely investigated [2,4,6,9,10,12,13,16,17,18,19,20] and arise in many diverse areas including network discovery and verification [1], strategies for the Mastermind game [7,8], robot navigation [14] and connected joins in graphs [19].

For the vertices u and v in a connected graph G, the *distance* $d(u,v)$ is the length of a shortest path between u and v in G. For an ordered set $W = \{v_1, v_2, \ldots, v_k\}$ of vertices in a connected graph G and a vertex v of G, the k-vector $c_W(v) = (d(v,v_1), d(v,v_2), \ldots, d(v,v_k))$ is referred to as the *code* of v with respect to W. The set W is called a *resolving set* for G if all the vertices of $V(G) \setminus W$ have distinct codes. A resolving set containing a minimum number of vertices is called a *minimum resolving set* or a *metric basis* for G. The number of elements in a metric basis of G is called the *metric dimension* of G, and is denoted by $dim(G)$ [2,3].

For a set S of vertices of G and a vertex v of G, the distance $d(v,S)$ between v and S is defined as $d(v,S) = \min\{d(v,x) : x \in S\}$. For an ordered k-partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ and a vector v of G, the code of v with respect to Π is defined as the k-vector $c_\Pi(v) = (d(v,S_1), d(v,S_2), \ldots, d(v,S_k))$. The partition Π is called a *resolving partition* for G if the distinct vertices of G have distinct codes with respect to Π. The minimum k for which there is a resolving k-partition of $V(G)$ is the *partition dimension* of G, denoted by $pd(G)$ [5,6].

A resolving partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ is said to be a *connected resolving partition* if the subgraph $< S_i >$ induced by each subset S_i $(1 \le i \le k)$ is connected in G. The minimum k for which there is a connected resolving k-partition of $V(G)$ is the *connected partition dimension* of G, denoted by $cpd(G)$. A connected resolving partition of $V(G)$ containing $cpd(G)$ elements is called a *minimum connected resolving partition* (or cr-*partition*) of $V(G)$. If G is a non-trivial connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, then the n-partition $\{S_1, S_2, \ldots, S_n\}$, where $S_i = \{v_i\}$ for $1 \le i \le n$, is a connected resolving partition for G. Thus, $cpd(G)$ is defined for every non-trivial connected graph G. Indeed, every connected resolving partition of a connected graph is a resolving partition. Thus, if G is a connected graph of order $n \ge 2$, then

$$2 \le pd(G) \le cpd(G) \le n.$$

Moreover, $pd(G) = cpd(G)$ if and only if G contains a minimum resolving partition that is connected [16,17].

For any $S \subseteq V(G)$, if $d(x,S) \ne d(y,S)$, then we say that the set S separates two distinct vertices x and y of G. If a class of a partition Π separates two distinct vertices x and y, then we say that Π separates x and y. From these definitions, it can be observed that the property of a given partition Π of the vertices of a graph G to be a resolving

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partition of G can be verified by investigating the pairs of vertices in the same class. Indeed, every vertex $x \in S_i$ $(1 \leq i \leq k)$ is at distance 0 from S_i, but is at a distance different from zero from any other class S_j with $j \neq i$. It follows that $x \in S_i$ and $y \in S_j$ are separated either by S_i, or by S_j for every $i \neq j$.

A connected graph with exactly one cycle is called a *unicyclic graph*. The metric dimension of unicyclic graphs was studied by Poisson and Zhang in [15]. We adopt the terminology, used in [15], to study the connected partition dimension of unicyclic graphs: The graph $G = G[G_1, G_2, u, v]$ obtained from G_1 and G_2 by identifying u and v is called an *identification graph*, where G_1 and G_2 are non-trivial connected graphs with $u \in V(G_1)$ and $v \in V(G_2)$. Therefore $u = v$ in G and we name the vertex $u = v$, the *joint* in G. The *identification is said to be of type-1* if an end vertex of a path is identified with a vertex of degree two of a cycle in a graph, or an end vertex of a path is identified with a vertex of degree 1 of a graph, otherwise *identification is said to be of type-2*. A unicyclic graph can be obtained by the addition of a single edge between two vertices of a tree. Also a unicyclic graph that is not a cycle can be obtained from a cycle and one or more trees by identifying some specified vertices on the cycle and on the trees.

Unicyclic graphs first time, in the context of connected partition dimension, were considered by Javaid in [11]. Together with some basic results, he proved the following major results for the partition dimension of unicyclic graphs:

Lemma 1.[11] *Let $G = G[G_1, G_2, u, v]$ be an identification graph of type-2. Then $cpd(G) \leq cpd(G_1) + cpd(G_2) - 1$.*

Theorem 1.[11] *Let G be a unicyclic graph of type-2 with unique cycle C of order n. Then*

$$4 \leq cpd(G) \leq 3 + \sum_{i=1}^{k} cpd(T_i) - k,$$

where Ti is a subtree of G rooted at the vertex u_i $(1 \leq i \leq k)$ of the cycle C.

Theorem 2.[11] *Let T be a tree which is not a path and e is an edge. Then*

$$cpd(T) - 2 \leq cpd(T + e) \leq cpd(T) + 1.$$

We investigate that the bounds for the connected partition dimension of unicyclic graphs provided by Javaid are not tight. In this paper, we reconsider the unicyclic graphs in the context of connected partition dimension and, together with some basic results, we provide modified sharp bounds for the connected partition dimension of unicyclic graphs.

2 Results

The following result gives the connected partition dimension of an identification graph of type-1.

Lemma 2.*Let $G = G[G_1, G_2, u, v]$ be an identification graph of type-1, where G_1 be any non-trivial connected graph and G_2 is a path on $n \geq 2$ vertices. Then $cpd(G) = cpd(G_1)$.*

*Proof.*Let $cpd(G_1) = k$ with connected resolving partitions $\Pi_1 = \{S_1 S_2, \ldots, S_k\}$ of $V(G_1)$. Since connected partition dimension of a graph is 2 if and only if the graph is a path [17], we have $cpd(G_2) = 2$ with connected resolving partition $\Pi_2 = \{U_1, U_2\}$. Let v be the joint in G such that $v \in S_k$ and $v \in U_1$. Let $\Pi = \{S_1, S_2, \ldots, S'_k = S_k \cup V(G_2)\}$ be a partition of $V(G)$ of cardinality $cpd(G_1) + cpd(G_2) - 2$, then any two distinct vertices v_1 and v_2 of $V(G)$ have different codes with respect to Π as shown in the following three cases:
Case A. If $v_1, v_2 \in V(G_1)$, then since $c_{\Pi_1}(v_1) \neq c_{\Pi_1}(v_2)$ and $d(v_i, S_k) = d(v_i, S'_k)$ for $i = 1, 2$, we have $c_\Pi(v_1) \neq c_\Pi(v_2)$.
Case B. If $v_1, v_2 \in V(G_2)$, then $v, v_1, v_2 \in S'_k$ and $c_\Pi(v_i) = c_{\Pi_1}(v) + (d(v_i, v), d(v_i, v), \ldots, 0)$ for $i = 1, 2$. Since $d(v_1, v) \neq d(v_2, v)$, we have $c_\Pi(v_1) \neq c_\Pi(v_2)$.
Case C. If $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, then $v_2 \in S'_k$ and we have the following two subcases:
Subcase C_1. If v is identified with a vertex of degree two of a cycle in G_1, then since the vertices of a cycle are divided into at least three classes, it is easy to see that v_1 and v_2 are at different distance from a class containing the vertices of the cycle, which implies that $c_\Pi(v_1) \neq c_\Pi(v_2)$.
Subcase C_2. If v is identified with a vertex of degree one of G_1, then $d(v_1 S_i) < d(v_2, S_i)$ $(1 \leq i \leq k-1)$, which yields that $c_\Pi(v_1) \neq c_\Pi(v_2)$.

Thus, it is concluded that Π is a connected resolving partition of $V(G)$ and hence

$$cpd(G) \leq cpd(G_1) + cpd(G_2) - 2.$$

Now, if $cpd(G) \ngeq cpd(G_1) + cpd(G_2) - 2$, then $cpd(G) < cpd(G_1) + cpd(G_2) - 2$. Since $cpd(G_2) = 2$, this implies that $cpd(G) < cpd(G_1) = k$. This suggest that there exists a connected resolving partition of $V(G_1)$ with cardinality less than the cardinality of Π_1, which is a contradiction. Therefore $cpd(G) \geq cpd(G_1) + cpd(G_2) - 2$ and hence $cpd(G) = cpd(G_1)$.

The following result gives the sharp upper and lower bounds for the connected partition dimension of an identification graph of type-2.

Lemma 3.*Let $G = G[G_1, G_2, u, v]$ be an identification graph of type-2. Then*

$$\max\{cpd(G_1), cpd(G_2)\} \leq cpd(G) \leq cpd(G_1) + cpd(G_2).$$

Moreover, both bounds are sharp.

*Proof.*Let $cpd(G_1) = k$ with connected resolving partitions $\Pi_1 = \{S_1 S_2, \ldots, S_k\}$ of $V(G_1)$ and $cpd(G_2) = l$

with connected resolving partition $\Pi_2 = \{U_1, U_2, \dots, U_l\}$ of $V(G_2)$. Then, clearly, $\Pi = \{S_1, S_2, \dots, S_k, U_1, U_2, \dots, U_l\}$ is a connected resolving partition of $V(G)$ of cardinality $cpd(G_1) + cpd(G_2)$.

For the lower bound, there is no loss of generality in assuming that $\max\{cpd(G_1), cpd(G_2)\} = cpd(G_2) = l$. Now, if $cpd(G) \ngeq \max\{cpd(G_1), cpd(G_2)\}$, then $cpd(G) < l$, which implies that a connected partition of $V(G_2)$ with the partite sets fewer than the partite sets of Π_2 resolves all the vertices of G_2, which is a contradiction. Hence

$$cpd(G) \geq \max\{cpd(G_1), cpd(G_2)\}.$$

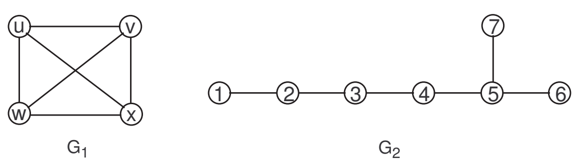

Fig. 1: G_1: The complete graph K_4 and G_2: The tree T.

For sharpness, consider the graphs G_1 and G_2 shown in Figure 1, which are the complete graph K_4 and the tree T, respectively. Also consider a path P_n on $n \geq 2$ vertices with an end vertex y and a star $S_{1,n-1}$ with the vertex z adjacent to $n-1$ vertices $z_1, z_1, \dots, z_{n-1}$ each of degree one. Then, note that, $cpd(G_1) = 4$, with connected resolving partition $\{\{u\}, \{v\}, \{w\}, \{x\}\}$; $cpd(G_2) = 3$ with a connected resolving partition, say $\{\{1,2,3,4\}, \{5,6\}, \{7\}\}$; $cpd(P_n) = 2$ with a connected resolving partition, say $\{\{y\}, V(P_n) \setminus \{y\}\}$ and $cpd(s_{1,n-1}) = n-1$ with a connected resolving partition, say $\{\{z, z_1\}, \{z_2\}, \dots, \{z_{n-1}\}\}$ [17]. Now, if $G = G[G_1, P_n, x, y]$ and $G' = G'[G_2, S_{1,n-1}, 5, z]$ are the identification graphs of type-2, then one can easily see that the connected partitions $\{\{u\}, \{v\}, \{w\}, \{x\} \cup V(P_n)\}$ and $\{\{1,2,3,4\}, \{5 = z, 6\}, \{7\}, \{z_1\}, \{z_2\}, \dots, \{z_{n-1}\}\}$ of $V(G)$ and $V(G')$, respectively, are the minimal connected resolving partitions, which implies that $cpd(G) = 4 = \max\{cpd(G_1), cpd(P_n)\}$ and $cpd(G') = n+2 = cpd(G_2) + cpd(S_{1,n-1})$.

Let G be a unicyclic graph with unique cycle C. Let $u_1, u_2, \dots, u_k$ be the vertices of C at which the subtrees of G are rooted. Let $T_1^i, T_2^i, \dots, T_{\lambda_i}^i$ be the subtrees of G rooted at u_i, where λ_i denotes the number of subtrees rooted at u_i. Then G is said to be unicyclic graph of type-1 if and only if $\lambda_i = 1$ for every i and $T_1^i = T_i$ is a path. Otherwise, G is unicyclic graph of type-2. The following result was proved for the connected partition dimension of unicyclic graphs of type-1 in [11].

Theorem 3.[11] *Let G be a unicyclic graph of type-1, then $cpd(G) = 3$.*

The following definitions, given in [3], will be used in the proof of next results. A vertex of degree at least three in a graph G will be called a *major vertex* of G. An end vertex u of G is said to be a *terminal vertex* of a major vertex v of G if $d(u,v) < d(u,w)$ for every other major vertex w of G. The *terminal degree* of a major vertex v is the number of terminal vertices of v. A major vertex of G is an *exterior major vertex* if it has positive terminal degree. Let $\sigma(G)$ denotes the sum of the terminal degrees of the major vertices of G, and let $ex(G)$ denotes the number of exterior major vertices of G.

Theorem 4. *Let G be a unicyclic graph of type-2 with unique cycle C of order n. Then*

$$4 \leq cpd(G) \leq 3 + \sum_{i=1}^{k} \sum_{j=1}^{\lambda_i} (\sigma(T_j^i) - ex(T_j^i)) + \sum_{i=1}^{k} \lambda_i,$$

where λ_i is the number of subtrees $T_1^i, T_2^i, \dots, T_{\lambda_i}^i$ of G rooted at each vertex u_i $(1 \leq i \leq k \leq n)$ of the cycle C.

Proof. For the upper bound, first we assume that $k = 1$. Then there are λ_1 subtrees $T_1^1, T_2^1, \dots, T_{\lambda_1}^1$ of G rooted at u_1. Let $G_1^1 = G_1^1[C, T_1^1, u_1, v_1^1]$ be an identification graph of type-2, where v_1^1 is a vertex of degree one in T_1^1. Then, since $cpd(C) = 3$ [17], we have $cpd(G_1^1) \leq 3 + cpd(T_1^1)$, by Lemma 3. Let $G_2^1 = G_2^1[G_1^1, T_2^1, u_1, v_2^1]$ be an identification graph of type-2, where v_2^1 is a vertex of degree one in T_2^1. Then, again by Lemma 3, we have $cpd(G_2^1) \leq 3 + cpd(T_1^1) + cpd(T_2^1)$. Now, by continuing this identification process λ_1 times, if $G_{\lambda_1}^1 = G_{\lambda_1}^1[G_{\lambda_1 - 1}^1, T_{\lambda_1}^1, u_1, v_{\lambda_1}^1]$ is an identification graph of type-2, where $v_{\lambda_1}^1$ is a vertex of degree one in $T_{\lambda_1}^1$. Then, $cpd(G_{\lambda_1}^1) \leq 3 + \sum_{j=1}^{\lambda_1} cpd(T_j^1)$, by Lemma 3.

For $k = 2$, there are λ_2 subtrees $T_1^2, T_2^2, \dots, T_{\lambda_2}^2$ of G rooted at u_2. Let $G_1^2 = G_1^2[G_{\lambda_1}^1, T_1^2, u_2, v_1^2]$ be an identification graph of type-2, where v_1^2 is a vertex of degree one in T_1^2. Then, $cpd(G_1^2) \leq 3 + \sum_{j=1}^{\lambda_1} cpd(T_j^1) + cpd(T_1^2)$, by Lemma 3. Let $G_2^2 = G_2^2[G_1^2, T_2^2, u_2, v_2^2]$ be an identification graph of type-2, where v_2^2 is a vertex of degree one in T_2^2. Then, again by Lemma 3, we have $cpd(G_2^2) \leq 3 + \sum_{j=1}^{\lambda_1} cpd(T_j^1) + cpd(T_1^2) + cpd(T_2^2)$. Now, by continuing this identification process λ_2 times, if $G_{\lambda_2}^2 = G_{\lambda_2}^2[G_{\lambda_2 - 1}^2, T_{\lambda_2}^2, u_2, v_{\lambda_2}^2]$ is an identification graph of type-2, where $v_{\lambda_2}^2$ is a vertex of degree one in $T_{\lambda_2}^2$. Then, $cpd(G_{\lambda_2}^2) \leq 3 + \sum_{j=1}^{\lambda_1} cpd(T_j^1) + \sum_{j=1}^{\lambda_2} cpd(T_j^2)$, by Lemma 3.

Similarly, by continuing this process up to the kth stage, let $G = G^k_{\lambda_k} = G^k_{\lambda_k}[G^k_{\lambda_k-1}, T^k_{\lambda_k}, u_k, v^k_{\lambda_k}]$ be an identification graph of type-2, where $v^k_{\lambda_k}$ is a vertex of degree one in $T^k_{\lambda_k}$. Then, $cpd(G) \leq 3 + \sum_{i=1}^{k-1}\sum_{j=1}^{\lambda_i} cpd(T^i_j) + \sum_{j=1}^{\lambda_k-1} cpd(T^k_j) + cpd(T^k_{\lambda_k})$, by Lemma 3. This implies that $cpd(G) \leq 3 + \sum_{i=1}^{k}\sum_{j=1}^{\lambda_i} cpd(T^i_j)$. Since for any tree T which is not a path, $cpd(T) = \sigma(T) - ex(T) + 1$ [17], we have

$$cpd(G) \leq 3 + \sum_{i=1}^{k}\sum_{j=1}^{\lambda_i}(\sigma(T^i_j) - ex(T^i_j)) + \sum_{i=1}^{k}\lambda_i.$$

For the lower bound, it is a routine exercise to see that $cpd(G) < 4$ would be possible only if G is a unicycle graph of type-1 or G is not a unicyclic graph.

We call a path of order $n \geq 2$ rooted at a (an exterior) major vertex, say v, in a tree, *a stem of the tree for* v. The following is a useful proposition, and may be is of independent interest.

Proposition 1. *Let T be a tree which is not a path and e is an edge. Then (1) $\sigma(T+e) \geq \sigma(T) - 2$ and (2) $ex(T+e) \leq ex(T) + 2$.*

Proof. Let u and v be two distinct non-adjacent vertices of T such that $e = uv$ in $T + e$.
(1) One can easily see that $\sigma(T+e) = \sigma(T) - 2$ if and only if u and v are the terminal vertices. Otherwise, $\sigma(T+e) > \sigma(T) - 2$.
(2) It is straightforward to see that $ex(T+e) = ex(T) + 2$ if and only if u and v are the non-terminal (non-major) vertices belonging to two different stems (a) for the same exterior major vertex having at least three stems, or (b) for two distinct exterior major vertices having at least two stems. Otherwise, $ex(T+e) \leq ex(T) + 1$.

In [17], it was shown that $cpd(G) \geq \sigma(G) - ex(G) + 1$ for any non-trivial connected graph G. The next result shows that how the connected partition dimension is changed when a single edge is added to a tree T.

Theorem 5. *Let T be a tree which is not a path and e is an edge. Then*

$$cpd(T) - 4 \leq cpd(T+e) \leq cpd(T) + 1.$$

Proof. Since $cpd(G) \geq \sigma(G) - ex(G) + 1$ and $cpd(T) = \sigma(T) - ex(T) + 1$ [17], so by Proposition 1, we have
$cpd(T+e) \geq \sigma(T+e) - ex(T+e) + 1 \geq cpd(T) - 4$.
For the upper bound, suppose that T contains p exterior major vertices $v_1, v_2, \ldots, v_p$. For each i with $1 \leq i \leq p$, let $u^i_1, u^i_2, \ldots, u^i_{k_i}$ be the terminal vertices of v_i. For each i with $1 \leq i \leq p$, let S^i_j be the stem for v_i in T for all $1 \leq j \leq k_i$ and let x^i_j be a vertex in S^i_j that is adjacent to v_i. Then let P^i_j be the $x^i_j - u^i_j$ path in S^i_j for all $1 \leq i \leq p$ and $1 \leq j \leq k_i$. Let $U = \{v_1, u^1_1, u^2_1, \ldots, u^p_1\}$ and let T_1 be the subtree of T of smallest size such that T_1 contains U. Let $U_0 = V(T_1)$ and $U^i_j = V(P^i_j)$ for all $1 \leq i \leq p$ and $2 \leq j \leq k_i$. Define a partition Π of $V(T)$ by $\Pi = \{U_0, U^i_j\ ;\ 1 \leq i \leq p \text{ and } 2 \leq j \leq k_i\}$. Then Π is connected and resolving as was shown in [17]. It is noted that the vertices in one class are separated by more than one class. Let C denotes the unique cycle in $T+e$ and let $e = uv$ in $T+e$, where u and v are two distinct vertices of T. We consider the following two cases:
Case 1. If C contains at least two major vertices, then the connected resolving partition Π for T is also a connected resolving partition for $T+e$. So $cpd(T+e) \leq cpd(T)$.
Case 2. If C contains only one major vertex, say x, then there are two subcases.
Subcase 1a. If u and v belong to two different stems for x, then the connected resolving partition Π for T is also a connected resolving partition for $T+e$. So $cpd(T+e) \leq cpd(T)$.
Subcase 1b. If u and v are the non-terminal vertices belonging to the same stem for x having at least three vertices, then we define a new partition by putting any vertex of that stem other than the major vertex in a new class. This will be a connected resolving partitions for $T+e$. So $cpd(T+e) \leq |\Pi| + 1 \leq cpd(T) + 1$. Hence, by summarizing all the above discission, we have

$$cpd(T) - 4 \leq cpd(T+e) \leq cpd(T) + 1.$$

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