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Dynamic Response of a Solid Bar of Cardioidal Cross-Sections Immersed in an Inviscid Fluid

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Abstract: The dynamic response of a homogeneous transversely isotropic solid bar of cardioidal cross-section immersed in a fluid is studied using the Fourier expansion collocation method (FECM), within the framework of the linearized three-dimensional theory of elasticity. The equations of motion of solid and fluid are respectively formulated using the constitutive equations of a transversely isotropic cylinder and the constitutive equations of an inviscid fluid. Three wave potential functions are introduced to uncouple the equations of motion. The frequency equations of longitudinal and flexural (symmetric and antisymmetric) modes are analyzed numerically for a cardioidal cross-sectional transversely isotropic solid bar immersed in a fluid. The computed non-dimensional frequencies are presented in the form of dispersion curves for the material Zinc.

Keywords: Wave propagation; Transversely isotropic bars; Solid-fluid interaction; Cardioidal cross-sections; Elliptical cross-sections; Ultrasonic transducers.

1 Introduction

Knowledge of various wave propagation characteristics, as a function of material and geometrical parameters is necessary for a wide range of applications, from geophysical prospecting in cased holes, non-destructive evaluation of oil and gas pipelines, to the insulated fiber optic cables for data transmission, ultrasonic transducers and resonators. We have shown that the frequencies depend strongly on the cross-sections of the bar and deviate from the circular one. The propagation of waves in cardioidal bar immersed in fluid has many applications in various fields of science and technology, namely, atomic physics, industrial engineering, submarine structures, pressure vessel, aerospace and metallurgy. The most general form of harmonic waves in a hollow cylinder of circular cross section of infinite length has been analyzed by Gazis[1] . Mirsky[2] investigated analyzed the wave propagation in transversely isotropic circular cylinders of infinite length and presented the frequency equation in Part I and numerical results in Part II. A method, for solving wave propagation in arbitrary cross-sectional cylinders and plates and to find out the phase velocities in different modes of vibrations namely longitudinal, torsional and flexural, by constructing frequency equations was devised by Nagaya [3,4,5,6]. He formulated the Fourier expansion collocation method for this purpose. Following Nagaya, Paul and Venkatesan [7] studied the wave propagation in an infinite piezoelectric solid cylinder of arbitrary cross section using Fourier expansion collocation method.

The longitudinal waves in homogeneous anisotropic cylindrical bars immersed in a fluid is studied by Dayal[8]. Rahman and Ahmad[9] presented the representation of the displacement in terms of scalar functions for use in transversely isotropic materials, later, Ahmad and Rahman[10] has discussed the acoustic scattering by transversely isotropic cylinders. Guided waves in a transversely isotropic cylinder immersed in a fluid is analyzed by Ahmad[11]. Following Ahmad, Nagy[12] have studied the longitudinal guided wave propagation in a transversely isotropic rod immersed in fluid, later, Nagy with Nayfeh[13] discussed the viscosity-induced attenuation of longitudinal guided waves in fluid-loaded rods. The free modes of propagation of an infinite fluid loaded thin cylindrical shell is discussed by Scott[14].

Easwaran and Munjal[15] reported a note on the effect of wall compliance on lowest-order mode propagation in

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fluid-filled/submerged impedance tubes. Sinha et. al.[16] have discussed the axisymmetric wave propagation in circular cylindrical shell immersed in fluid, in two parts. In Part I, the theoretical analysis of the propagating modes are discussed and in Part II, the axisymmetric modes excluding torsional modes are obtained theoretically and experimentally and are compared. Berlinear and Solecki[17] have studied the wave propagation in fluid loaded transversely isotropic cylinder. In that paper, Part I consists of the analytical formulation of the frequency equation of the coupled system consisting of the cylinder with inner and outer fluid and Part II gives the numerical results.

Venkatesan and Ponnusamy[18,19] have obtained the frequency equation of the free vibration of a solid cylinder of arbitrary cross section immersed in a fluid using Fourier expansion collocation method. The frequency equations are obtained for longitudinal and flexural vibrations. Recently, Ponnusamy and Selvamani[20] have studied the wave propagation in a magneto-thermo elastic waves in a transversely isotropic cylindrical panel using the wave propagation approach.

In this paper, the dynamic response of a transversely isotropic solid bar cardioidal cross section immersed in a fluid is studied. Using the Fourier expansion collocation method the boundary conditions on the surface of the cardioidal cross sectional solid bar are satisfied and the frequency equations are obtained. The frequency equations of longitudinal and flexural modes are studied numerically and the computed non dimensional frequencies are presented in the form of dispersion curves.

2 Basic equations and formulation of the problem

We consider a transversely isotropic cylindrical bar of cardioidal cross-section immersed in inviscid fluid. The system is assumed to be linear so that the linearized three-dimensional stress equations of motion are used for both the cylinder and the fluid. The system displacements and stresses are defined by the cylindrical coordinates r, θ and z. In cylindrical coordinates, the three-dimensional stress equations of motion in the absence of body are given by Berliner and Solecki (1996)

$$\begin{aligned}\sigma_{rr,r}+r^{-1}\sigma_{r\theta,\theta}+\sigma_{rz,z}+r^{-1}\left(\sigma_{rr}-\sigma_{\theta\theta}\right)&=\rho u_{r,tt}\\ \sigma_{r\theta,r}+r^{-1}\sigma_{\theta\theta,\theta}+\sigma_{\theta z,z}+2r^{-1}\sigma_{r\theta}&=\rho u_{\theta,tt}\\ \sigma_{rz,r}+r^{-1}\sigma_{\theta z,\theta}+\sigma_{zz,z}+r^{-1}\sigma_{rz}&=\rho u_{z,tt}\end{aligned}\tag{1}$$

The stress strain relation for a transversely isotropic material is given by

$$\begin{aligned}\sigma_{rr}&=c_{11}e_{rr}+c_{12}e_{\theta\theta}+c_{13}e_{zz}\\ \sigma_{\theta\theta}&=c_{12}e_{rr}+c_{11}e_{\theta\theta}+c_{13}e_{zz}\\ \sigma_{zz}&=c_{13}e_{rr}+c_{13}e_{\theta\theta}+c_{33}e_{zz}\\ \sigma_{r\theta}&=2c_{66}e_{r\theta},\ \sigma_{\theta z}=2c_{44}e_{\theta z},\ \sigma_{rz}=2c_{44}e_{rz}\end{aligned}\tag{2}$$

where σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, $\sigma_{r\theta}$, $\sigma_{\theta z}$, σ_{rz} are the stress components, e_{rr}, $e_{\theta\theta}$, e_{zz}, $e_{r\theta}$, $e_{\theta z}$, e_{rz} are the strain components, c_{11}, c_{12}, c_{13}, c_{33}, c_{44} and $c_{66}=(c_{11}-c_{12})/2$ are the five independent elastic constants, ρ is the mass density of the material.

The strain e_{ij} are related to the displacements are given by

$$\begin{aligned}&e_{rr}=u_{r,r},\quad e_{\theta,\theta}=r^{-1}\left(u_r+u_{\theta,\theta}\right),\quad e_{zz}=u_{z,z},\\ &2e_{r\theta}=u_{\theta,r}-r^{-1}\left(u_\theta-u_{r,\theta}\right),\quad 2e_{zr}=\left(u_{r,z}+u_{z,r}\right),\\ &2e_{\theta z}=\left(u_{\theta,z}+r^{-1}u_{z,\theta}\right)\end{aligned}\tag{3}$$

in which u_r, u_θ and u_z are the displacement components along radial, circumferential and axial directions respectively. The comma in the subscripts denotes the partial differentiation with respect to the variables.

Substituting eqn's. (3) and (2) in the eqn. (1), results in the following three-dimensional displacement equations of motion:

$$\begin{aligned}&c_{11}\left(u_{r,rr}+r^{-1}u_{r,r}-r^{-2}u_r\right)-r^{-2}\left(c_{11}+c_{66}\right)u_{\theta,\theta}\\ &\quad+r^{-2}c_{66}u_{r,\theta\theta}+c_{44}u_{r,zz}+\left(c_{44}+c_{13}\right)u_{z,rz}\\ &\quad+r^{-1}\left(c_{66}+c_{12}\right)u_{\theta,r\theta}=\rho u_{r,tt}\end{aligned}\tag{4a}$$

$$\begin{aligned}&r^{-1}\left(c_{12}+c_{66}\right)u_{r,r\theta}+r^{-2}\left(c_{66}+c_{11}\right)u_{r,\theta}\\ &\quad+c_{66}\left(u_{\theta,rr}+r^{-1}u_{\theta,r}-r^{-2}u_\theta\right)+r^{-2}c_{11}u_{\theta,\theta\theta}\\ &\quad+c_{44}u_{\theta,zz}+r^{-1}\left(c_{44}+c_{13}\right)u_{z,\theta z}=\rho u_{\theta,tt}\end{aligned}\tag{4b}$$

$$\begin{aligned}&c_{44}\left(u_{z,rr}+r^{-1}u_{z,r}+r^{-2}u_{z,\theta\theta}\right)\\ &\quad+r^{-1}\left(c_{44}+c_{13}\right)\left(u_{r,z}+u_{\theta,\theta z}\right)\\ &\quad+\left(c_{44}+c_{13}\right)u_{r,rz}+c_{33}u_{z,zz}=\rho u_{z,tt}\end{aligned}\tag{4c}$$

3 Method of solution to the equation of motion

The eqn's. (4) are coupled partial differential equations of the three displacement components. This system of equations can be uncoupled by eliminating two of the three displacement components through two of the three equations, but this results in a partial differential equations of fourth order. To uncouple the eqn's. (4), we follow Mirsky (1964) and assuming the solution of eqn's. (4) as follows:

$$\begin{aligned}u_r\left(r,\theta,z,t\right)=\sum_{n=0}^{\infty}\varepsilon_n\big[&\left(\phi_{n,r}+r^{-1}\psi_{n,\theta}\right)\\ &+\left(\overline{\phi}_{n,r}+r^{-1}\overline{\psi}_{n,\theta}\right)\big]e^{i(kz+\omega t)}\end{aligned}\tag{5a}$$

$$\begin{aligned}u_\theta\left(r,\theta,z,t\right)=\sum_{n=0}^{\infty}\varepsilon_n\big[&\left(r^{-1}\phi_{n,\theta}-\psi_{n,r}\right)\\ &\left(r^{-1}\overline{\phi}_{n,\theta}-\overline{\psi}_{n,r}\right)\big]e^{i(kz+\omega t)}\end{aligned}\tag{5b}$$

$$u_z(r,\theta,z,t) = \frac{i}{a}\sum_{n=0}^{\infty}\varepsilon_n\left[W_n+\overline{W}_n\right]e^{i(kz+\omega t)} \tag{5c}$$

where $\varepsilon_n = \frac{1}{2}$ for $n=0$, $\varepsilon_n = 1$ for $n \geq 1$, $i=\sqrt{-1}$, k is the wave number, ω is the angular frequency, $\phi_n(r,\theta)$, $W_n(r,\theta)$, $\psi_n(r,\theta)$, $\overline{\phi}_n(r,\theta)$, $\overline{W}_n(r,\theta)$, $\overline{\psi}_n(r,\theta)$ are the displacement potentials and a is the geometrical parameter of the cylinder.

By introducing the dimensionless quantities such as $\varsigma = ka$, $\Omega^2 = \frac{\rho\omega^2a^2}{c_44}$, $\bar{c}_{11} = \frac{c_{11}}{c_{44}}$, $\bar{c}_{13} = \frac{c_{13}}{c_{44}}$, $\bar{c}_{33} = \frac{c_{33}}{c_{44}}$, $T = t\sqrt{\frac{c_{44}}{\frac{\rho}{a}}}$ and $x = \frac{r}{a}$ and substituting eqn's (5) in eqn's. (4), we obtain

$$\begin{aligned}\left(\bar{c}_{11}\nabla^2+\left(\Omega^2-\varsigma^2\right)\right)\phi_n-\left(1+\bar{c}_{13}\right)W_n=0\\ \varsigma\left(1+c_{13}\right)\phi_n+\left(\nabla^2+\left(\Omega^2-\bar{c}_{33}\varsigma^2\right)\right)W_n=0\end{aligned} \tag{6}$$

and

$$\left(\nabla^2+\frac{\Omega^2-\varsigma^2}{\bar{c}_{66}}\right)\psi_n=0 \tag{7}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+x^{-1}\frac{\partial}{\partial x}+x^{-2}\frac{\partial^2}{\partial\theta^2}$.

Eliminating W_n from the eqn's. (6), we obtain

$$\left(A\nabla^4+B\nabla^2+C\right)\phi_n=0 \tag{8}$$

where

$$\begin{aligned}A&=\bar{c}_{11},\\ B&=-\left[\left(1+\bar{c}_{11}\right)\Omega^2+\varsigma^2\left(\bar{c}_{13}^2+2\bar{c}_{13}+\bar{c}_{11}\bar{c}_{33}\right)\right],\\ C&=\left(\Omega^2-\varsigma^2\right)\left(\Omega^2-\bar{c}_{33}\varsigma^2\right).\end{aligned} \tag{9}$$

Solving the eqn.(8), the solutions for the symmetric modes are obtained as

$$\phi_n = \sum_{i=1}^{2}\left[A_{in}J_n(\alpha_i ax)+B_{in}Y_n(\alpha_i ax)\right]\cos n\theta \tag{10a}$$

$$W_n = \sum_{i=1}^{2} d_i\left[A_{in}J_n(\alpha_i ax)+B_{in}Y_n(\alpha_i ax)\right]\cos n\theta \tag{10b}$$

where J_n and Y_n are Bessel functions of the first and second kind of order n. The solution for the antisymmetric modes $\overline{\phi}_n$ and $\overline{W}_n$ are obtained by replacing $\cos n\theta$ by $\sin n\theta$ in eqn's. (10).

Here $(\alpha_i a)^2 > 0$, $(i=1,2)$ are the roots of the algebraic equation

$$A(\alpha a)^4 - B(\alpha a)^2 + C = 0. \tag{11}$$

The Bessel functions J_n and Y_n is used when the roots $(\alpha_i a)^2$, $(i=1,2)$ are real or complex and the modified Bessel function I_n and K_n is used when the roots are imaginary.

The constants d_i defined in the eqn. (10b) can be calculated from the equation

$$d_i = \frac{\left(1+\bar{c}_{13}\right)(\alpha_i a)^2}{(\alpha_i a)^2+\Omega^2-\bar{c}_{33}^2\varsigma^2},\ i=1,2. \tag{12}$$

Solving the eqn. (7), the solution to the symmetric mode is obtained as

$$\psi_n = \left[A_{3n}J_n(\alpha_3 ax)+B_{3n}Y_n(\alpha_3 ax)\right]\sin n\theta \tag{13}$$

where $(\alpha_3 a)^2 = \frac{\Omega^2-\varsigma^2}{\bar{c}_{66}}$. If $(\alpha_3 a)^2 < 0$, the Bessel function J_n and Y_n is replaced by the modified Bessel function I_n and K_n. The solution for the antisymmetric mode $\overline{\psi}_n$ is obtained from eqn.(13) by replacing $\sin n\theta$ by $\cos n\theta$.

4 Equations of motion of the fluid

In cylindrical polar coordinates r, θ and z, the acoustic pressure and radial displacement equations of motion for an inviscid fluid are of the form [16]

$$p^f = -B^f\left(u^f_{r,r}+r^{-1}(u^f_r+u^f_{\theta,\theta})+u^f_{z,z}\right) \tag{14}$$

and

$$c_f^{-2}u^f_{r,tt} = \Delta_{,r} \tag{15}$$

respectively where B^f is the adiabatic bulk modulus, ρ^f is the density, $c^f = \sqrt{\frac{B^f}{\rho^f}}$ is the acoustic phase velocity in the fluid, and $(u^f_r, u^f_\theta, u^f_z)$ is the displacement vector.

$$\Delta = (u^f_{r,r}+r^{-1}(u^f_r+u_{\theta,\theta})+u^f_{z,z}). \tag{16}$$

Substituting

$$u^f_r = \phi^f_{,r},\ u^f_\theta = r^{-1}\phi^f_{,\theta} \text{ and } u^f_z = \phi^f_{,z} \tag{17}$$

and seeking the solution of (15) in the form

$$\phi^f(r,\theta,z,t) = \sum_{n=0}^{\infty}\varepsilon_n\left[\phi^f_n(r)\cos n\theta+\overline{\phi}^f_n(r)\sin n\theta\right]e^{i(kz+\omega t)} \tag{18}$$

the oscillating waves propagating in the inner fluid located in the annulus is given by

$$\phi^f_n = B_{4n}H^{(1)}_n(\alpha_4 ax) \tag{19}$$

where $(\alpha_4 a)^2 = \frac{\Omega^2}{\overline{\rho}^f\overline{B}^f}-\varsigma^2$, in which $\overline{\rho}^f = \frac{\rho}{\rho^f}$, $\overline{B}^f = \frac{B^f}{c_{44}}$, $H^{(1)}_n$ is the Hankel function of the second kind. If $(\alpha_4 a)^2 < 0$, then the Hankel function of second kind is to be replaced by K_n, where K_n is the modified Bessel function of the second kind. By substituting eqn (17) in (14) along with (19), the acoustic pressure for the fluid can be expressed as

$$p^f = B_{4n}\Omega^2\overline{\rho}H^{(1)}_n(\alpha_4 ax)\cos n\theta e^{i(\varsigma\bar{z}+\Omega T_a)} \tag{20}$$

In the case of antisymmetric, the solutions for fluid are obtained by replacing $\cos n\theta$ by $\sin n\theta$ in the eq. (20).

5 Boundary conditions and frequency equations

The objective of this problem is to study the dynamic response of a cylindrical bar of cardioid cross sectional cylinder immersed in a fluid. The boundary conditions of an infinite cylindrical solid bar are obtained as follows:

$$\left(\sigma_{xx}+p^{f}\right)_{i}=\left(\sigma_{xy}\right)_{i}=\left(\sigma_{zx}\right)_{i}=\left(u_{r}-u_{r}^{f}\right)_{i}=0 \tag{21}$$

σ_{xx} is the normal stress, σ_{xy} and σ_{zx} are the shearing stresses and $\left(\right)_{i}$ is the value at the i-th segment of the boundary. The first and last conditions in eqn. (21) are due to the continuity of the stresses and displacements of the cylindrical bar and fluid on the curved surfaces. Since the boundary of the cross section is irregular in shape, it is difficult to satisfy the boundary conditions along surfaces of the cylinder directly. Hence, to satisfy the boundary conditions, the Fourier expansion collocation method due to Nagaya [3,4,5,6] is applied. If γ_i is the angle between normal to the segment and the reference axis is assumed to be constant, then the transformed expressions for the stresses are

$$\begin{aligned}\sigma_{xx}=c_{66}&\left(r^{-1}\left(u_{\theta}-u_{r,\theta}\right)-u_{\theta,r}\right)\sin 2\left(\theta-\gamma_{i}\right)\\&+\left(c_{11}\cos^{2}\left(\theta-\gamma_{i}\right)+c_{12}\sin^{2}\left(\theta-\gamma_{i}\right)\right)u_{r,r}\\&+r^{-1}\left(c_{11}\sin^{2}\left(\theta-\gamma_{i}\right)+c_{13}u_{z,z}\right.\\&\left.+c_{12}\cos^{2}\left(\theta-\gamma_{i}\right)\right)\left(u_{r}+u_{\theta,\theta}\right)\end{aligned} \tag{22a}$$

$$\begin{aligned}\sigma_{xy}=c_{66}&\left(\left(u_{r,r}-r^{-1}\left(u_{\theta,\theta}+u_{r}\right)\right)\sin 2\left(\theta-\gamma_{i}\right)\right.\\&\left.+\left(r^{-1}\left(u_{r,\theta}-u_{\theta}\right)+u_{\theta,r}\right)\cos 2\left(\theta-\gamma_{i}\right)\right)\end{aligned} \tag{22b}$$

$$\begin{aligned}\sigma_{zx}=c_{44}&\left(\left(u_{r,z}+u_{z,r}\right)\cos\left(\theta-\gamma_{i}\right)\right.\\&\left.-\left(u_{\theta,z}+r^{-1}u_{z,\theta}\right)\sin\left(\theta-\gamma_{i}\right)\right)\end{aligned} \tag{22c}$$

Substituting equations (5), (10) and (13), in the boundary condition (21) the boundary conditions are transformed by applying the Fourier expansion collocation method along the curved surface of the boundary as follows:

$$\begin{aligned}\left[\left(S_{xx}\right)_{i}+\left(\bar{S}_{xx}\right)_{i}\right]e^{i\left(\varsigma\bar{z}+\Omega T_{a}\right)}&=0,\\\left[\left(S_{xy}\right)_{i}+\left(\bar{S}_{xy}\right)_{i}\right]e^{i\left(\varsigma\bar{z}+\Omega T_{a}\right)}&=0,\\\left[\left(S_{zx}\right)_{i}+\left(\bar{S}_{zx}\right)_{i}\right]e^{i\left(\varsigma\bar{z}+\Omega T_{a}\right)}&=0,\\\left[\left(S_{r}\right)_{i}+\left(\bar{S}_{r}\right)_{i}\right]e^{i\left(\varsigma\bar{z}+\Omega T_{a}\right)}&=0\end{aligned} \tag{23}$$

where

$$\begin{aligned}S_{xx}=0.5&\left(e_{0}^{1}A_{10}+e_{0}^{2}B_{10}+e_{0}^{3}A_{20}+e_{0}^{4}B_{20}+e_{0}^{7}A_{40}\right)\\&+\sum_{n=1}^{\infty}\left(e_{n}^{1}A_{1n}+e_{n}^{2}B_{1n}+e_{n}^{3}A_{2n}+e_{n}^{4}B_{2n}\right.\\&\left.+e_{n}^{5}A_{3n}+e_{n}^{6}B_{3n}+e_{n}^{7}A_{4n}\right)\end{aligned} \tag{24a}$$

$$\begin{aligned}S_{xy}=0.5&\left(f_{0}^{1}A_{10}+f_{0}^{2}B_{10}+f_{0}^{3}A_{20}+f_{0}^{4}B_{20}\right)\\&+\sum_{n=1}^{\infty}\left(f_{n}^{1}A_{1n}+f_{n}^{2}B_{1n}+f_{n}^{3}A_{2n}+f_{n}^{4}B_{2n}\right.\\&\left.+f_{n}^{5}A_{3n}+f_{n}^{6}B_{3n}\right)\end{aligned} \tag{24b}$$

$$\begin{aligned}S_{zx}=0.5&\left(g_{0}^{1}A_{10}+g_{0}^{2}B_{10}+g_{0}^{3}A_{20}+g_{0}^{4}B_{20}\right)\\&+\sum_{n=1}^{\infty}\left(g_{n}^{1}A_{1n}+g_{n}^{2}B_{1n}+g_{n}^{3}A_{2n}+g_{n}^{4}B_{2n}\right.\\&\left.+g_{n}^{5}A_{3n}+g_{n}^{6}B_{3n}\right)\end{aligned} \tag{24c}$$

$$\begin{aligned}S_{r}=0.5&\left(h_{0}^{1}A_{10}+h_{0}^{2}B_{10}+h_{0}^{3}A_{20}+h_{0}^{4}B_{20}+h_{0}^{7}A_{40}\right)\\&+\sum_{n=1}^{\infty}\left(h_{n}^{1}A_{1n}+h_{n}^{2}B_{1n}+h_{n}^{3}A_{2n}+h_{n}^{4}B_{2n}\right.\\&\left.+h_{n}^{5}A_{3n}+h_{n}^{6}B_{3n}+h_{n}^{7}A_{4n}\right)\end{aligned} \tag{24d}$$

$$\begin{aligned}\bar{S}_{xx}=0.5&\left(\bar{e}_{0}^{1}\bar{A}_{10}+\bar{e}_{0}^{2}\bar{B}_{10}+\bar{e}_{0}^{3}\bar{A}_{20}+\bar{e}_{0}^{4}\bar{B}_{20}+\bar{e}_{0}^{7}\bar{A}_{40}\right)\\&+\sum_{n=1}^{\infty}\left(\bar{e}_{n}^{1}\bar{A}_{1n}+\bar{e}_{n}^{2}\bar{B}_{1n}+\bar{e}_{n}^{3}\bar{A}_{2n}+\bar{e}_{n}^{4}\bar{B}_{2n}\right.\\&\left.+\bar{e}_{n}^{5}\bar{A}_{3n}+\bar{e}_{n}^{6}\bar{B}_{3n}+\bar{e}_{n}^{7}\bar{A}_{4n}\right)\end{aligned} \tag{25a}$$

$$\begin{aligned}\bar{S}_{xy}=0.5&\left(\bar{f}_{0}^{1}\bar{A}_{10}+\bar{f}_{0}^{2}\bar{B}_{10}+\bar{f}_{0}^{3}\bar{A}_{20}+\bar{f}_{0}^{4}\bar{B}_{20}\right)\\&+\sum_{n=1}^{\infty}\left(\bar{f}_{n}^{1}\bar{A}_{1n}+\bar{f}_{n}^{2}\bar{B}_{1n}+\bar{f}_{n}^{3}\bar{A}_{2n}+\bar{f}_{n}^{4}\bar{B}_{2n}\right.\\&\left.+\bar{f}_{n}^{5}\bar{A}_{3n}+\bar{f}_{n}^{6}\bar{B}_{3n}\right)\end{aligned} \tag{25b}$$

$$\begin{aligned}\bar{S}_{zx}=0.5&\left(\bar{g}_{0}^{1}\bar{A}_{10}+\bar{g}_{0}^{2}\bar{B}_{10}+\bar{g}_{0}^{3}\bar{A}_{20}+\bar{g}_{0}^{4}\bar{B}_{20}\right)\\&+\sum_{n=1}^{\infty}\left(\bar{g}_{n}^{1}\bar{A}_{1n}+\bar{g}_{n}^{2}\bar{B}_{1n}+\bar{g}_{n}^{3}\bar{A}_{2n}+\bar{g}_{n}^{4}\bar{B}_{2n}\right.\\&\left.+\bar{g}_{n}^{5}\bar{A}_{3n}+\bar{g}_{n}^{6}\bar{B}_{3n}\right)\end{aligned} \tag{25c}$$

$$\begin{aligned}\bar{S}_{r}=0.5&\left(\bar{h}_{0}^{1}\bar{A}_{10}+\bar{h}_{0}^{2}\bar{B}_{10}+\bar{h}_{0}^{3}\bar{A}_{20}+\bar{h}_{0}^{4}\bar{B}_{20}+\bar{h}_{0}^{7}\bar{A}_{40}\right)\\&+\sum_{n=1}^{\infty}\left(\bar{h}_{n}^{1}\bar{A}_{1n}+\bar{h}_{n}^{2}\bar{B}_{1n}+\bar{h}_{n}^{3}\bar{A}_{2n}+\bar{h}_{n}^{4}\bar{B}_{2n}\right.\\&\left.+\bar{h}_{n}^{5}\bar{A}_{3n}+\bar{h}_{n}^{6}\bar{B}_{3n}+\bar{h}_{n}^{7}\bar{A}_{4n}\right)\end{aligned} \tag{25d}$$

The equations for $e_n^1 \sim \bar{h}_n^8$ are given in Appendix A. The boundary conditions along both the inner and outer arbitrary surface cannot be satisfied directly. Therefore, performing the Fourier series expansion to (21) along the

boundary, the boundary conditions are expanded in the form of double Fourier series. In the symmetric mode, the necessary boundary conditions for the inner surface are obtained as

$$\sum_{m=0}^{\infty}\varepsilon_m\Big[E^1_{m0}A_{10}+E^2_{m0}B_{10}+E^3_{m0}A_{20}+E^4_{m0}B_{20}+E^7_{m0}A_{50}+\sum_{n=1}^{\infty}\Big(E^1_{mn}A_{1n}+E^2_{mn}B_{1n}+E^3_{mn}A_{2n}+E^4_{mn}B_{2n}+E^5_{mn}A_{3n}+E^6_{mn}B_{3n}+E^7_{mn}A_{5n}\Big)\Big]\cos m\theta=0 \quad (26a)$$

$$\sum_{m=1}^{\infty}\Big[F^1_{m0}A_{10}+F^2_{m0}B_{10}+F^3_{m0}A_{20}+F^4_{m0}B_{20}+\sum_{n=1}^{\infty}\Big(F^1_{mn}A_{1n}+F^2_{mn}B_{1n}+F^3_{mn}A_{2n}+F^4_{mn}B_{2n}+F^5_{mn}A_{3n}+F^6_{mn}B_{3n}\Big)\Big]\sin m\theta=0 \quad (26b)$$

$$\sum_{m=0}^{\infty}\varepsilon_m\Big[G^1_{m0}A_{10}+G^2_{m0}B_{10}+G^3_{m0}A_{20}+G^4_{m0}B_{20}+\sum_{n=1}^{\infty}\Big(G^1_{mn}A_{1n}+G^2_{mn}B_{1n}+G^3_{mn}A_{2n}+G^4_{mn}B_{2n}+G^5_{mn}A_{3n}+G^6_{mn}B_{3n}\Big)\Big]\cos m\theta=0 \quad (26c)$$

$$\sum_{m=0}^{\infty}\varepsilon_m\Big[H^1_{m0}A_{10}+H^2_{m0}B_{10}+H^3_{m0}A_{20}+H^4_{m0}B_{20}+H^7_{m0}A_{50}+\sum_{n=1}^{\infty}\Big(H^1_{mn}A_{1n}+H^2_{mn}B_{1n}+H^3_{mn}A_{2n}+H^4_{mn}B_{2n}+H^5_{mn}A_{3n}+H^6_{mn}B_{3n}+H^7_{mn}A_{5n}\Big)\Big]\cos m\theta=0 \quad (26d)$$

Similarly, for the anti symmetric mode, the boundary conditions for the inner surface are

$$\sum_{m=1}^{\infty}\Big[\overline{E}^5_{m0}\overline{A}_{30}+\overline{E}^6_{m0}\overline{B}_{30}+\sum_{n=1}^{\infty}\Big(\overline{E}^1_{mn}\overline{A}_{1n}+\overline{E}^2_{mn}\overline{B}_{1n}+\overline{E}^3_{mn}\overline{A}_{2n}+\overline{E}^4_{mn}\overline{B}_{2n}+\overline{E}^5_{mn}\overline{A}_{3n}+\overline{E}^6_{mn}\overline{B}_{3n}+\overline{E}^7_{mn}\overline{A}_{5n}\Big)\Big]\sin m\theta=0 \quad (27a)$$

$$\sum_{m=0}^{\infty}\Big[\overline{F}^5_{m0}\overline{A}_{30}+\overline{F}^6_{m0}\overline{B}_{30}+\sum_{n=1}^{\infty}\Big(\overline{F}^1_{mn}\overline{A}_{1n}+\overline{F}^2_{mn}\overline{B}_{1n}+\overline{F}^3_{mn}\overline{A}_{2n}+\overline{F}^4_{mn}\overline{B}_{2n}+\overline{F}^5_{mn}\overline{A}_{3n}+\overline{F}^6_{mn}\overline{B}_{3n}\Big)\Big]\cos m\theta=0 \quad (27b)$$

$$\sum_{m=1}^{\infty}\Big[\overline{G}^5_{m0}\overline{A}_{30}+\overline{G}^6_{m0}\overline{B}_{30}+\sum_{n=1}^{\infty}\Big(\overline{G}^1_{mn}\overline{A}_{1n}+\overline{G}^2_{mn}\overline{B}_{1n}+\overline{G}^3_{mn}\overline{A}_{2n}+\overline{G}^4_{mn}\overline{B}_{2n}+\overline{G}^5_{mn}\overline{A}_{3n}+\overline{G}^6_{mn}\overline{B}_{3n}\Big)\Big]\sin m\theta=0 \quad (27c)$$

$$\sum_{m=1}^{\infty}\Big[\overline{H}^5_{m0}\overline{A}_{30}+\overline{H}^6_{m0}\overline{B}_{30}+\sum_{n=1}^{\infty}\Big(\overline{H}^1_{mn}\overline{A}_{1n}+\overline{H}^2_{mn}\overline{B}_{1n}+\overline{H}^3_{mn}\overline{A}_{2n}+\overline{H}^4_{mn}\overline{B}_{2n}+\overline{H}^5_{mn}\overline{A}_{3n}+\overline{H}^6_{mn}\overline{B}_{3n}+\overline{H}^7_{mn}\overline{A}_{5n}\Big)\Big]\sin m\theta=0 \quad (27d)$$

The frequency equations are obtained from the inner and outer boundary conditions of the equations (26), for the symmetric mode, and for the antisymmetric mode, the frequency equations are obtained from the equations (27) by truncating the series to $N+1$ terms, and equating the determinant of the coefficients of the amplitudes A_{in}, B_{in}, $\overline{A}_{in}$ and $\overline{B}_{in}$ $(i=1,2,3,4)$ to zero. Thus the frequency equation for the symmetric mode is obtained as

$$\begin{vmatrix}
E^1_{00} & E^2_{00} & E^3_{00} & E^4_{00} & E^7_{00} & 0 & E^1_{01} & \cdots & E^1_{0N} & E^6_{01} & \cdots & E^6_{0N} & E^7_{01} & \cdots & E^7_{0N} & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
E^1_{N0} & E^2_{N0} & E^3_{N0} & E^4_{N0} & E^7_{N0} & 0 & E^1_{N1} & \cdots & E^1_{NN} & E^6_{N1} & \cdots & E^6_{NN} & E^7_{N1} & \cdots & E^7_{NN} & 0 & \cdots & 0\\
F^1_{10} & F^2_{10} & F^3_{10} & F^4_{10} & 0 & 0 & F^1_{11} & \cdots & F^1_{1N} & F^6_{11} & \cdots & F^6_{1N} & 0 & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
F^1_{N0} & F^2_{N0} & F^3_{N0} & F^4_{N0} & 0 & 0 & F^1_{N1} & \cdots & F^1_{NN} & F^6_{N1} & \cdots & F^6_{NN} & 0 & \cdots & 0 & 0 & \cdots & 0\\
G^1_{00} & G^2_{00} & G^3_{00} & G^4_{00} & 0 & 0 & G^1_{01} & \cdots & G^1_{0N} & G^6_{01} & \cdots & G^6_{0N} & 0 & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
G^1_{N0} & G^2_{N0} & G^3_{N0} & G^4_{N0} & 0 & 0 & G^1_{N1} & \cdots & G^1_{NN} & G^6_{N1} & \cdots & G^6_{NN} & 0 & \cdots & 0 & 0 & \cdots & 0\\
H^1_{00} & H^2_{00} & H^3_{00} & H^4_{00} & H^7_{00} & 0 & H^1_{01} & \cdots & H^1_{0N} & H^6_{01} & \cdots & H^6_{0N} & H^7_{01} & \cdots & H^7_{0N} & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
H^1_{N0} & H^2_{N0} & H^3_{N0} & H^4_{N0} & H^7_{N0} & 0 & H^1_{N1} & \cdots & H^1_{NN} & H^6_{N1} & \cdots & H^6_{NN} & H^7_{N1} & \cdots & H^7_{NN} & 0 & \cdots & 0\\
E^1_{00} & E^2_{00} & E^3_{00} & E^4_{00} & 0 & E^8_{00} & E^1_{01} & \cdots & E^1_{0N} & E^6_{01} & \cdots & E^6_{0N} & 0 & \cdots & 0 & E^8_{01} & \cdots & E^8_{0N}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
E^1_{N0} & E^2_{N0} & E^3_{N0} & E^4_{N0} & 0 & E^8_{N0} & E^1_{N1} & \cdots & E^1_{NN} & E^6_{N1} & \cdots & E^6_{NN} & 0 & \cdots & 0 & E^8_{N1} & \cdots & E^8_{NN}\\
F^1_{10} & F^2_{10} & F^3_{10} & F^4_{10} & 0 & 0 & F^1_{11} & \cdots & F^1_{1N} & F^6_{11} & \cdots & F^6_{1N} & 0 & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
F^1_{N0} & F^2_{N0} & F^3_{N0} & F^4_{N0} & 0 & 0 & F^1_{N1} & \cdots & F^1_{NN} & F^6_{N1} & \cdots & F^6_{NN} & 0 & \cdots & 0 & 0 & \cdots & 0\\
G^1_{00} & G^2_{00} & G^3_{00} & G^4_{00} & 0 & 0 & G^1_{01} & \cdots & G^1_{0N} & G^6_{01} & \cdots & G^6_{0N} & 0 & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
G^1_{N0} & G^2_{N0} & G^3_{N0} & G^4_{N0} & 0 & 0 & G^1_{N1} & \cdots & G^1_{NN} & G^6_{N1} & \cdots & G^6_{NN} & 0 & \cdots & 0 & 0 & \cdots & 0\\
H^1_{00} & H^2_{00} & H^3_{00} & H^4_{00} & 0 & H^8_{00} & H^1_{01} & \cdots & H^1_{0N} & H^6_{00} & \cdots & H^6_{0N} & 0 & \cdots & 0 & H^8_{00} & \cdots & H^8_{0N}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
H^1_{N0} & H^2_{N0} & H^3_{N0} & H^4_{N0} & 0 & H^8_{N0} & H^1_{N1} & \cdots & H^1_{NN} & H^6_{N1} & \cdots & H^6_{NN} & 0 & \cdots & 0 & H^8_{N1} & \cdots & H^8_{NN}
\end{vmatrix}=0 \quad (28)$$

Similarly, the frequency equation for antisymmetric mode of vibration is given by

$$\begin{vmatrix}
\overline{E}^5_{10} & \overline{E}^6_{10} & \overline{E}^1_{11} & \cdots & \overline{E}^1_{1N} & \overline{E}^6_{11} & \cdots & \overline{E}^6_{1N} & \overline{E}^7_{11} & \cdots & \overline{E}^7_{1N} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\overline{E}^5_{N0} & \overline{E}^6_{N0} & \overline{E}^1_{N1} & \cdots & \overline{E}^1_{NN} & \overline{E}^6_{N1} & \cdots & \overline{E}^6_{NN} & \overline{E}^7_{N1} & \cdots & \overline{E}^7_{NN} & 0 & \cdots & 0 \\
\overline{F}^5_{00} & \overline{F}^6_{00} & \overline{F}^1_{01} & \cdots & \overline{F}^1_{0N} & \overline{F}^6_{01} & \cdots & \overline{F}^6_{0N} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\overline{F}^5_{N0} & \overline{F}^6_{N0} & \overline{F}^1_{N1} & \cdots & \overline{F}^1_{NN} & \overline{F}^6_{N1} & \cdots & \overline{F}^6_{NN} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\overline{G}^5_{10} & \overline{G}^6_{10} & \overline{G}^1_{11} & \cdots & \overline{G}^1_{1N} & \overline{G}^6_{11} & \cdots & \overline{G}^6_{1N} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\overline{G}^5_{N0} & \overline{G}^6_{N0} & \overline{G}^1_{N1} & \cdots & \overline{G}^1_{NN} & \overline{G}^6_{N1} & \cdots & \overline{G}^6_{NN} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\overline{H}^5_{10} & \overline{H}^6_{10} & \overline{H}^1_{11} & \cdots & \overline{H}^1_{1N} & \overline{H}^6_{11} & \cdots & \overline{H}^6_{1N} & \overline{H}^7_{11} & \cdots & \overline{H}^7_{1N} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\overline{H}^5_{N0} & \overline{H}^6_{N0} & \overline{H}^1_{N1} & \cdots & \overline{H}^1_{NN} & \overline{H}^6_{N1} & \cdots & \overline{H}^6_{NN} & \overline{H}^7_{N1} & \cdots & \overline{H}^7_{NN} & 0 & \cdots & 0 \\
\overline{E}^5_{10} & \overline{E}^6_{10} & \overline{E}^1_{11} & \cdots & \overline{E}^1_{1N} & \overline{E}^6_{11} & \cdots & \overline{E}^6_{1N} & 0 & \cdots & 0 & \overline{E}^8_{11} & \cdots & \overline{E}^8_{1N} \\
\vdots & \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\overline{E}^5_{N0} & \overline{E}^6_{N0} & \overline{E}^1_{N1} & \cdots & \overline{E}^1_{NN} & \overline{E}^6_{N1} & \cdots & \overline{E}^6_{NN} & 0 & \cdots & 0 & \overline{E}^8_{N1} & \cdots & \overline{E}^8_{NN} \\
\overline{F}^5_{00} & \overline{F}^6_{00} & \overline{F}^1_{01} & \cdots & \overline{F}^1_{0N} & \overline{F}^6_{01} & \cdots & \overline{F}^6_{0N} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\overline{F}^5_{N0} & \overline{F}^6_{N0} & \overline{F}^1_{N1} & \cdots & \overline{F}^1_{NN} & \overline{F}^6_{N1} & \cdots & \overline{F}^6_{NN} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\overline{G}^5_{10} & \overline{G}^6_{10} & \overline{G}^1_{11} & \cdots & \overline{G}^1_{1N} & \overline{G}^6_{11} & \cdots & \overline{G}^6_{1N} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\overline{G}^5_{N0} & \overline{G}^6_{N0} & \overline{G}^1_{N1} & \cdots & \overline{G}^1_{NN} & \overline{G}^6_{N1} & \cdots & \overline{G}^6_{NN} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\overline{H}^5_{10} & \overline{H}^6_{10} & \overline{H}^1_{11} & \cdots & \overline{H}^1_{1N} & \overline{H}^6_{11} & \cdots & \overline{H}^6_{1N} & 0 & \cdots & 0 & \overline{H}^8_{11} & \cdots & \overline{H}^8_{1N} \\
\vdots & \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\overline{H}^5_{N0} & \overline{H}^6_{N0} & \overline{H}^1_{N1} & \cdots & \overline{H}^1_{NN} & \overline{H}^6_{N1} & \cdots & \overline{H}^6_{NN} & 0 & \cdots & 0 & \overline{H}^8_{N1} & \cdots & \overline{H}^8_{NN}
\end{vmatrix} = 0 \tag{29}$$

where

$$\begin{aligned}
\overline{E}^i_{mn} &= \frac{2\varepsilon_n}{\pi}\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} e^i_n(R_i,\theta)\cos m\theta\, d\theta, \\
\overline{F}^i_{mn} &= \frac{2\varepsilon_n}{\pi}\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} f^i_n(R_i,\theta)\sin m\theta\, d\theta, \\
\overline{G}^i_{mn} &= \frac{2\varepsilon_n}{\pi}\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} g^i_n(R_i,\theta)\cos m\theta\, d\theta, \\
\overline{H}^i_{mn} &= \frac{2\varepsilon_n}{\pi}\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} h^i_n(R_i,\theta)\cos m\theta\, d\theta
\end{aligned} \tag{30}$$

$$\begin{aligned}
E^i_{mn} &= \frac{2\varepsilon_n}{\pi}\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} e^i_n(R_i,\theta)\cos m\theta\, d\theta, \\
F^i_{mn} &= \frac{2\varepsilon_n}{\pi}\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} f^i_n(R_i,\theta)\sin m\theta\, d\theta, \\
G^i_{mn} &= \frac{2\varepsilon_n}{\pi}\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} g^i_n(R_i,\theta)\cos m\theta\, d\theta, \\
H^i_{mn} &= \frac{2\varepsilon_n}{\pi}\sum_{i=1}^{I}\int_{\theta_{i-1}}^{\theta_i} h^i_n(R_i,\theta)\cos m\theta\, d\theta
\end{aligned} \tag{31}$$

where $i = 1, 2, 3, 4, 5, 6, 7$ and 8, $\varepsilon_m = \frac{1}{2}$ for $m = 0$ and $\varepsilon_m = \frac{1}{2}$ for $m \geq 0$, I is the number of segments, $\overline{R}_i$ is the coordinate r at the inner boundary, and R_i is the coordinate r at the outer boundary. The equations for $\overline{E}^i_{mn} \sim \overline{H}^i_{mn}$ can be obtained by replacing $\cos n\theta$ by $\sin n\theta$ and $\sin n\theta$ by $\cos n\theta$ in eqn's.(28) and (29).

6 Numerical results and discussion

The resulting frequency equations of the symmetric and antisymmetric cases of the cylinder of general cross section immersed in a fluid is given in (28) and (29) are transcendental in nature with respect to the dimensionless frequency Ω and dimensionless wave number ς. The analysis is carried out for cardioid cross sections by fixing the dimensionless wave number ς and the dimensionless frequency Ω are obtained. The computation of cylindrical Bessel functions of complex arguments are performed using the method provided by Zhang and Jin[21]. The computation of Fourier coefficients given in (30) is carried out using the five point Gaussian quadrature. To obtain the roots of the frequency equation, the secant method applicable for the complex roots (Antia[22]) is employed. The material chosen for the numerical calculation is zinc, its properties are as follows: for the solid the elastic constants are

$$\begin{aligned}
&c_{11} = 1.628\times10^{11}\,Nm^{-2}, \quad c_{12} = 0.362\times10^{11}\,Nm^{-2}, \\
&c_{13} = 0.508\times10^{11}\,Nm^{-2}, \quad c_{33} = 0.627\times10^{11}\,Nm^{-2}, \\
&c_{44} = 0.385\times10^{11}\,Nm^{-2}
\end{aligned}$$

and for the density $\rho = 7.14\times10^3\,kgm^{-3}$, the fluid density $\rho^f = 1000\,kgm^{-3}$ and the phase velocity $c = 1500\,ms^{-1}$.

In the present problem, three kinds of basic independent modes of wave propagation have been considered, namely, the longitudinal and two flexural (symmetric and antisymmetric) modes for geometries having more than one symmetry. For geometries having only one symmetry, two modes of wave propagations are studied since the two flexural modes are coupled in this case.

6.1 Cardioid cross-section

The relation used for the numerical calculations of cardioid cross sectional solid bar are from equations (22) and (24) of Nagaya(1983a) as follows

$$\frac{R_i}{a} = \frac{1+s^2+2s\cos\theta_1}{1+s}$$
$$\theta = \cos^{-1}\frac{\cos\theta_1 + s\cos 2\theta_1}{(1+s^2+2s\cos\theta_1)^{\frac{1}{2}}}$$

where a is the radius of the circumscribing circle and

$$G(\theta_1) = \frac{\cos\theta_1 + 2s\cos 2\theta_1}{-\sin\theta_1 - 2s\sin 2\theta_1}$$
$$\gamma_i = \begin{cases} \frac{\pi}{2}, & \text{for } G(\theta_i^*) = 0 \\ \frac{\pi}{2} - tan^{-1}(-G(\theta_i^*)), & \text{for } G(\theta_i^*) < 0 \\ \frac{\pi}{2} + tan^{-1}(-G(\theta_i^*)), & \text{for } G(\theta_i^*) > 0 \end{cases}$$

where $\theta_i^* = (\theta_i - \theta_{i-1})/2$, is the mean angle of the segment I, and R_i is the coordinate r at the boundary, γ_i is the angle between the normal to the segment and the reference axis at the i^{th} boundary. This parameter s represents a circle when $s = 0$ and represents a cardioid when $s = 0.5$. In the case of cardioid cross section, the vibration and displacements are symmetrical about only one axis. Hence, the frequency equation for longitudinal case may be obtained from (28) by choosing $n, m = 0, 1, 2, 3, \ldots$. In the case of flexural mode, the vibration and displacements are antisymmetrical about the minor axis. Hence, the frequency equations may be obtained from (29) by choosing $n, m = 1, 2, 3, \ldots$.

6.2 Dispersion curves

The results of longitudinal and flexural (antisymmetric) modes are plotted in figures. The notations LM and FSAM represents the longitudinal mode and flexural antisymmetric modes respectively. The 1, 2 refers to the first and second modes of vibration respectively.
The Figs.1 and 2, shows that the non-dimensional wave number $|\varsigma|$ versus dimensionless frequency Ω of transversely isotropic free and immersed cardioid cross sectional cylindrical bar for longitudinal and flexural (antisymmetric) modes of vibrations with respect to the parameter $s = 0.05$. It is observed that as the wave number increases, the non-dimensional frequency Ω also increases linearly. Beyond $\Omega = 0.4$, there is a small oscillation in the vibrational modes in Fig.2 due to the leakage of waves from solid in to the fluid.

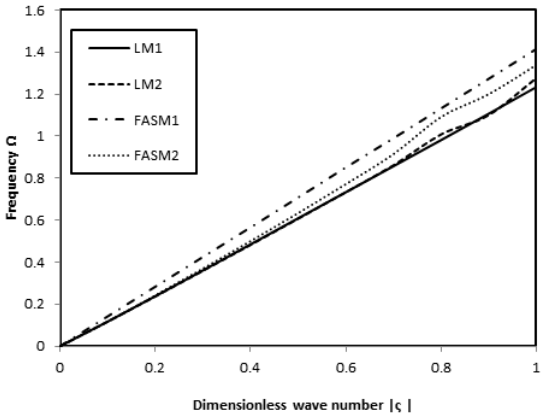

Fig. 1: Non-dimensional wave number $|\varsigma|$ versus dimensionless frequency Ω for free solid bar with $s = 0.05$

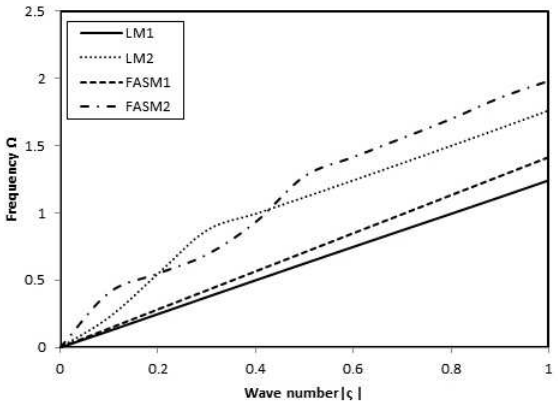

Fig. 2: Non-dimensional wave number $|\varsigma|$ versus dimensionless frequency Ω for immersed solid bar with $s = 0.05$

A graph is drawn between the non-dimensional wave number $|\varsigma|$ versus dimensionless frequency Ω of transversely isotropic cardioid cross sectional cylindrical bar with respect to the parameter $s = 0.3$ in Fig.3 and 4. The displacement of energy in the first mode and second mode of vibration of longitudinal and flexural (antisymmetric) increases linearly as the frequencies increases. It is also observed that there is a small dispersion among the modes of vibration in case of the bar immersed in fluid.

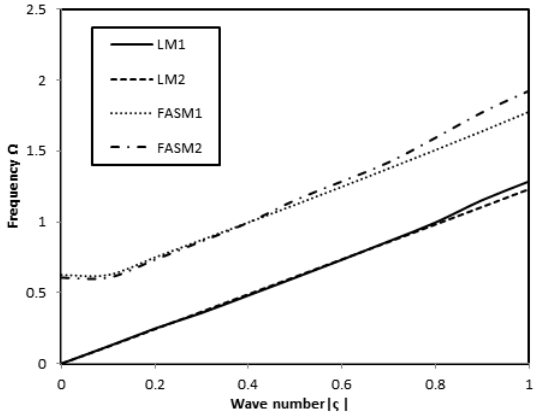

Fig. 3: Non-dimensional wave number $|\varsigma|$ versus dimensionless frequency Ω for free solid bar with $s = 0.3$

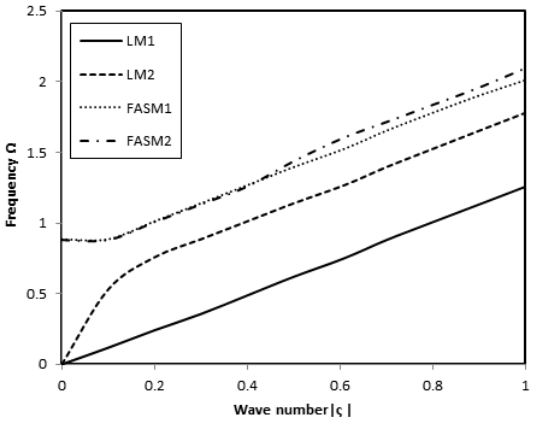

Fig. 4: Non-dimensional wave number $|\varsigma|$ versus dimensionless frequency Ω for immersed solid bar with $s = 0.3$

The dispersion curve is drawn between the dimensionless wave number $|\varsigma|$ versus non-dimensional frequency Ω of transversely isotropic cardioid cross sectional cylindrical bar for $s = 0.3$ in Fig.5 and 6. It is observed that the behavior of the wave propagation is linear in both the cases of vibrational modes except the small deviation in Fig.6. Therefore, the dynamic response of the solid bar with the fluid interaction behave irregular in both the cases of vibrational modes.So, it is clear that the frequency profile in some of the modes exhibits oscillating nature due to the fact that the fluid is acted as extra added mass. The cross over points in the vibrational modes indicates the energy transfer between the solid and fluid medium.

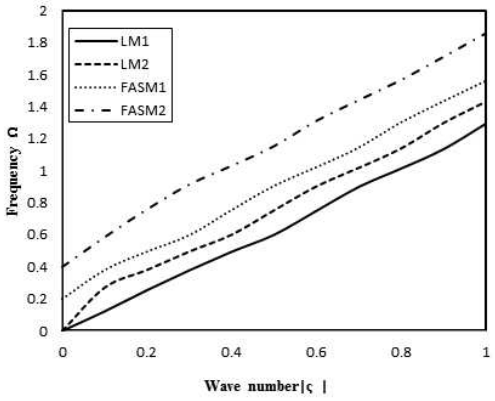

Fig. 5: Non-dimensional wave number $|\varsigma|$ versus dimensionless frequency Ω for free solid bar with $s = 0.5$

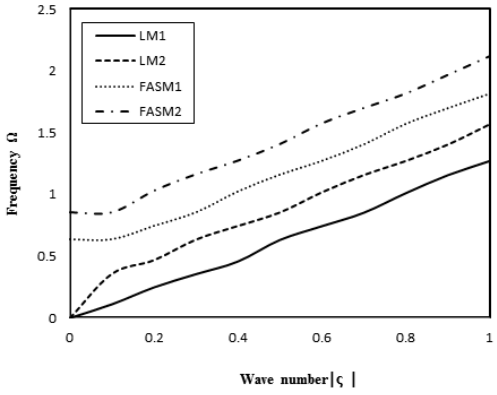

Fig. 6: Non-dimensional wave number $|\varsigma|$ versus dimensionless frequency Ω for immersed solid bar with $s = 0.5$

7 Conclusions

In this paper, the dynamic response of a transversely isotropic solid bar of cardioidal cross sections immersed

in a fluid is analyzed by satisfying the boundary condition on the irregular boundary using the Fourier expansion collocation method and the frequency equation for the longitudinal and flexural anti symmetric modes of vibrations are obtained. The results are presented as dispersion curves. It is clear that the energy radiation is increasing as the waves penetrate deeper in to the medium (higher wave number). The cross over points in the vibrational modes indicates the energy transfer between the solid and fluid medium. The method proposed in this paper can be used to analyze the vibration of a cylindrical bar of any cross section with appropriate geometric relation.

Appendix A

The equations for $e_n^i \sim g_n^i$ referred in the equations (30) and (31) are as follows:

$$\begin{aligned} e_n^i = 2\bar{c}_{66}\Big[&\Big\{n(n-1)J_n(\alpha_i ax) \\ &+(\alpha_i ax)J_{n+1}(\alpha_i ax)\Big\}\cos 2(\theta-\gamma_i) \\ &-\Big\{(\alpha_i a)^2\Big[\bar{c}_{11}\cos^2(\theta-\gamma_i)+\bar{c}_{12}\sin^2(\theta-\gamma_i)\Big] \\ &+\bar{c}_{13}\zeta d_i\Big\}J_n(\alpha_i ax)\Big]\cos n\theta \\ &+2n\bar{c}_{66}\Big\{(n-1)J_n(\alpha_i ax) \\ &-(\alpha_i a)J_{n+1}(\alpha_i ax)\Big\}\sin 2(\theta-\gamma_i)\sin n\theta, \\ &\qquad i=1,2 \end{aligned} \tag{A1}$$

$$\begin{aligned} e_n^3 = 2n\bar{c}_{66}&\Big\{(n-1)J_n(\alpha_3 ax) \\ &-(\alpha_3 a)J_{n+1}(\alpha_3 ax)\Big\}\cos 2(\theta-\gamma_i)\cos n\theta \\ &+\bar{c}_{66}\Big[2(\alpha_3 a)J_{n+1}(\alpha_3 ax)-\Big\{(\alpha_3 a)^2 \\ &-2n(n-1)J_n(\alpha_3 ax)\Big\}\Big]\sin n\theta\sin 2(\theta-\gamma_i) \end{aligned} \tag{A2}$$

$$\begin{aligned} e_n^i = 2\bar{c}_{66}\Big[&\Big\{n(n-1)Y_n(\alpha_i ax)+(\alpha_i ax)\Big\}\cos 2(\theta-\gamma_i) \\ &-\Big\{(\alpha_i a)^2\Big[\bar{c}_{11}\cos^2(\theta-\gamma_i)+\bar{c}_{12}\sin^2(\theta-\gamma_i)\Big] \\ &+\bar{c}_{13}\varsigma d_i\Big\}Y_n(\alpha_i ax)\Big]\cos n\theta \\ &+2n\bar{c}_{66}\Big\{(n-1)Y_n(\alpha_i ax) \\ &-(\alpha_i a)Y_{n+1}(\alpha_i ax)\Big\}\sin 2(\theta-\gamma_i)\sin n\theta, \\ &\qquad i=5,6. \end{aligned} \tag{A3}$$

$$\begin{aligned} e_n^7 = 2n\bar{c}_{66}&\Big\{(n-1)Y_n(\alpha_7 ax) \\ &-(\alpha_7 ax)Y_{n+1}(\alpha_7 ax)\Big\}\cos 2(\theta-\gamma_i)\cos n\theta \\ &+\bar{c}_{66}\Big[2(\alpha_7 a)Y_{n+1}(\alpha_7 ax)-\Big\{(\alpha_7 a)^2 \\ &-2n(n-1)Y_n(\alpha_7 ax)\Big\}\Big]\sin n\theta\sin 2(\theta-\gamma_i) \end{aligned} \tag{A4}$$

$$e_n^4 = \Omega^2\bar{\rho}_1 J_n(\delta_1 ax)\cos n\theta \tag{A5}$$

$$e_n^8 = \Omega^2\bar{\rho}_2 H_n^{(2)}(\delta_2 ax)\cos n\theta \tag{A6}$$

$$\begin{aligned} f_n^i = &\Big[2(\alpha_i a)J_{n+1}(\alpha_i ax)-\Big\{\Big((\alpha_i a)^2 \\ &-2n(n-1)\Big)J_n(\alpha_i ax)\Big\}\Big]\sin 2(\theta-\gamma_i)\cos n\theta \\ &+2n\{(n-1)J_n(\alpha_i ax) \\ &+(\alpha_i a)J_{n+1}(\alpha_i ax)\}\cos(\theta-\gamma_i)\sin n\theta, \\ &\qquad i=1,2 \end{aligned} \tag{A7}$$

$$\begin{aligned} f_n^3 = &2n\{(n-1)J_n(\alpha_3 ax) \\ &-(\alpha_3 a)J_{n+1}(\alpha_3 ax)\}\sin 2(\theta-\gamma_i)\cos n\theta \\ &+\Big[\Big\{(\alpha_3 a)^2-2n(n-1)\Big\}J_n(\alpha_3 ax) \\ &-2(\alpha_3 a)J_{n+1}(\alpha_3 ax)\Big]\cos 2(\theta-\gamma_i)\sin n\theta \end{aligned} \tag{A8}$$

$$\begin{aligned} f_n^i = &\Big[2(\alpha_i a)Y_{n+1}(\alpha_i ax)-\Big\{\Big((\alpha_i ax)^2 \\ &-2n(n-1)\Big)Y_n(\alpha_i ax)\Big\}\Big]\sin 2(\theta-\gamma_i)\cos n\theta \\ &+2n\{(n-1)Y_n(\alpha_i ax) \\ &+(\alpha_i a)Y_{n+1}(\alpha_i ax)\}\cos(\theta-\gamma_i)\sin n\theta, \\ &\qquad i=5,6. \end{aligned} \tag{A9}$$

$$\begin{aligned} f_n^7 = &2n\{(n-1)Y_n(\alpha_7 ax) \\ &-(\alpha_7 a)Y_{n+1}(\alpha_7 ax)\}\sin 2(\theta-\gamma_i)\cos n\theta \\ &+\Big[\Big\{(\alpha_7 a)^2-2n(n-1)\Big\}Y_n(\alpha_7 ax) \\ &-2(\alpha_7 a)Y_{n+1}(\alpha_7 ax)\Big]\cos 2(\theta-\gamma_i)\sin n\theta \end{aligned} \tag{A10}$$

$$f_n^4 = 0 \quad \text{(A11)}$$

$$f_n^8 = 0 \quad \text{(A12)}$$

$$g_n^i = \bar{c}_{44}(\varsigma + d_i)\Big\{n\cos\left(\overline{n-1}\theta + \gamma_i\right)J_n(\alpha_i a x) - (\alpha_i a)J_{n+1}(\alpha_i a x)\cos(\theta - \gamma_i)\cos n\theta\Big\}, \quad i = 1, 2 \quad \text{(A13)}$$

$$g_n^3 = \varsigma\Big\{n\cos\left(\overline{n-1}\theta + \gamma_i\right)J_n(\alpha_3 a x) - (\alpha_3 a)J_{n+1}(\alpha_3 a x)\sin(\theta - \gamma_i)\sin n\theta\Big\} \quad \text{(A14)}$$

$$g_n^i = \bar{c}_{44}(\varsigma + d_i)\Big\{n\cos\left(\overline{n-1}\theta + \gamma_i\right)J_n(\alpha_i a x) - (\alpha_i a)J_{n+1}(\alpha_i a x)\cos(\theta - \gamma_i)\cos n\theta\Big\}, \quad i = 5, 6 \quad \text{(A15)}$$

$$g_n^7 = \varsigma\Big\{n\cos\left(\overline{n-1}\theta + \gamma_i\right)J_n(\alpha_7 a x) - (\alpha_7 a)J_{n+1}(\alpha_7 a x)\sin(\theta - \gamma_i)\sin n\theta\Big\} \quad \text{(A16)}$$

$$g_n^4 = 0 \quad \text{(A17)}$$

$$g_n^8 = 0 \quad \text{(A18)}$$

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