Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/080603

Shock-Waves and other Solutions to the Sharma-Tasso-Olver Equation with Lie Point Symmetry and Travelling-Waves Approach

B. S. Ahmed [1], *R. M. Morris* [2], *E. V. Krishnan* [3], *P. G. L. Leach* [4] *and Anjan Biswas* [2,5,*]

[1] Department of Mathematics, College of Girls, Ain Shams University, Cairo, Egypt
[2] Department of Mathematical Sciences, Delaware State University, Dover, DE 19901-2277, USA
[3] Department of Mathematics and Statistics, Sultan Qaboos University, P. O. Box 36, Al Khod 123, Muscat, Oman
[4] School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, PO Box X54001, Durban 4000, Republic of South Africa
[5] Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah-21589, Saudi Arabia

Received: 5 Oct. 2013, Revised: 3 Jan. 2014, Accepted: 4 Jan. 2014
Published online: 1 Nov. 2014

Abstract: This paper addresses the Sharma-Tasso-Olver equation from an integrability perspective. There are three integration tools that are applied to extract the solutions to this nonlinear evolution equation. The ansatz method is applied to the generalised equation with power-law nonlinearity to obtain shock-wave solutions. Subsequently, the traveling-wave hypothesis leads to another set of solutions in the complex domain. Finally, Lie symmetry analysis leads to a third set of solutions. Several constraint conditions emerge from the various analyses.

Keywords: Sharmo-Tasso-Olver equation, travelling-wave, shock-wave, Lie point symmetry

1 Introduction

The theory of solitons is a very important area of research in applied mathematics, information sciences and theoretical physics [1,2,3,4,5,6,7,9,10,11,13,14,16,19,20,21,22,23,24,25]. In the context of information sciences, solitons act as information carrier bits through optical fibers across transcontinental and transoceanic distances. The theory of solitons is also studied in several other forms such as analysing topological solitons, which are also known as shock-waves, singular solitons that are also known as rogue waves in oceanography and optical rogons in nonlinear optics. This paper considers one such nonlinear evolution equation (NLEE) that leads to shock-wave solutions or topological soliton solutions. This is the Sharma-Tasso-Olver (STO) equation. This NLEE is an odd-ordered hierarchy of the well-known Burgers equation, that also produces shock-wave solutions [7]. The STO equation under study appears with dual nonlinear terms and a single dispersion term.

Several methods of integration will be applied to extract soliton and other solutions to this equation. The ansatz approach is firstly applied with power-law nonlinearity. In this context, it will be proved that the power-law nonlinearity parameter collapses to unity for this equation to support shock-waves. The constraint conditions will be identified for the waves to exist. Traveling-wave will subsequently be employed with a specialised form of a solution structure. This will lead to solutions that are valid in the complex domain. Finally, Lie symmetry analysis will lead to an additional set of solutions that are exhibited. Some numerical simulations will support the analysis developed in this paper.

2 Ansatz approach

The version of the STO equation we consider is given by

$$u_t + a\left(u^{2n+1}\right)_x + b\left(u^{2n}\right)_{xx} + cu_{xxx} = 0, \qquad (1)$$

* Corresponding author e-mail: biswas.anjan@gmail.com

where $u(x,t)$ is the dependent variable and is the profile of the wave. The independent variables are x and t, while a, b and c are all real-valued constants. The parameter n dictates power-law nonlinearity.

It must be noted that this approach of integrability was already studied earlier for the case $n=1$ in [10]. In this paper, we adopt this approach on eq. (1) for a general value of n thus keeping it on a more generalised setting from any previous work.

In order to seek shock-wave solutions to eq. (1), the starting hypothesis is given by

$$u(x,t) = A\tanh^p[B(x-vt)], \tag{2}$$

where A and B are free parameters and v represents speed of the shock-wave. The value of the unknown exponent p will become evident during the course of derivation of the shock-wave solution.

The substitution of eq. (2) into eq. (1) leads to

$$\begin{aligned}
&pvA\left(\tanh^{p+1}\tau - \tanh^{p-1}\tau\right)\\
+\;&(2n+1)paA^{2n+1}\left(\tanh^{2np+p+1}\tau - \tanh^{2np+p-1}\tau\right)\\
+\;&2npbA^{2n}B\left\{(2np-1)\tanh^{2np-2}\tau - 2np\tanh^{2np}\tau\right.\\
+\;&\left.(2np+1)\tanh^{2np+2}\tau\right\}\\
+\;&cAB^2\left\{p(p-1)(p-2)\tanh^{p-3}\tau\right.\\
-\;&\left[p(p-1)(p-2)+2p^3\right]\tanh^{p-1}\tau\\
+\;&\left[p(p+1)(p+2)+2p^3\right]\tanh^{p+1}\tau\\
-\;&\left.p(p+1)(p+2)\tanh^{p+3}\tau\right\} = 0,
\end{aligned} \tag{3}$$

where

$$\tau = B(x-vt). \tag{4}$$

From eq. (3), by the balancing principle, equating the exponents $2np+p+1$ and $p+3$ leads to

$$p = \frac{1}{n}. \tag{5}$$

Again equating the exponents $2np+2$ and $p+3$ gives

$$p = \frac{1}{2n-1}. \tag{6}$$

Equating the two values of p from eqs. (5) and (6) imply

$$n = 1, \tag{7}$$

and hence from eq. (5)

$$p = 1. \tag{8}$$

The same value of p is obtained upon setting the coefficient of the stand-alone linearly independent function $\tanh^{p-3}\tau$ to zero. Therefore eq. (1) reduces to

$$u_t + a\left(u^3\right)_x + b\left(u^2\right)_{xx} + cu_{xxx} = 0. \tag{9}$$

Now setting the coefficients of the linearly independent functions $\tanh^j\tau$ for $j = 0,2,4$ in eq. (3) leads to

$$v = 2bAB, \tag{10}$$

$$v = 3aA^2 + 4bAB - 8cB^2, \tag{11}$$

$$aA^2 + 2bAB - 2cB^2 = 0. \tag{12}$$

Equating the two values of shock-wave velocity (v) from eqs. (10) and (11) leads to

$$3aA^2 + 2bAB - 8cB^2 = 0. \tag{13}$$

From eqs. (12) and (13) one easily obtains

$$A = \sqrt{\frac{3c}{a}}B, \tag{14}$$

which prompts the constraint condition

$$ac > 0, \tag{15}$$

which must hold for shock waves to exist. Finally, the shock-wave solution to eq. (9) is given by

$$u(x,t) = A\tanh[B(x-vt)], \tag{16}$$

where the free parameters A and B are connected as given by eq. (14).

Figure 1 displays a shock-wave solution to eq. (9) with parameter values $a = b = c = 1$, $B = 0.5$, with ranges being $-50 \le x \le 50$ and $0 \le t \le 40$.

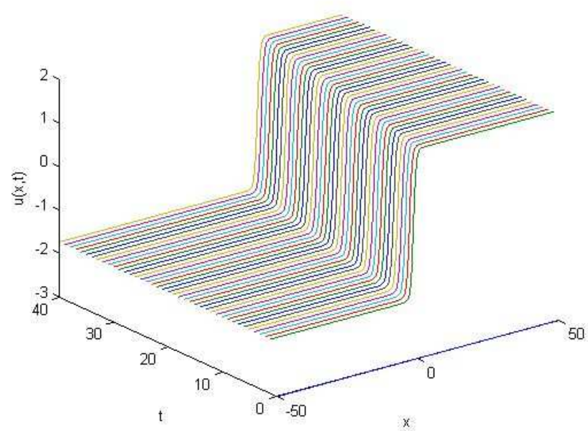


Fig. 1: Topological solution for $u(x,t)$

3 Travelling-waves

Assuming a travelling-wave solution in the form given by (4), eq. (9) reduces to the ordinary differential equation (ODE)

$$-\nu u_\tau + a(u^3)_\tau + bB(u^2)_{\tau\tau} + cB^2 u_{\tau\tau\tau} = 0. \quad (17)$$

Integrating eq. (17) with respect to τ and keeping the integration constant zero, we arrive at the ODE

$$-\nu u + au^3 + 2bBuu' + cB^2 u'' = 0, \quad (18)$$

where $(')$ denotes differentiation with respect to τ.

We assume a solution of eq. (18) in the form

$$u(\tau) = \frac{l\sinh\tau}{m+n\cosh\tau}. \quad (19)$$

Substituting for u, u' and u'' in eq. (18), we obtain an algebraic equation in powers of $\cosh\tau$ given by

$$\begin{aligned}&-\nu(m+n\cosh\tau)^2 + al^2(\cosh^2\tau - 1)\\ &+2bB(lm\cosh\tau + ln)\\ &-cB^2(2n^2 - m^2 + mn\cosh\tau) = 0. \quad (20)\end{aligned}$$

Equating the coefficients of different powers of $\cosh\tau$ to zero, we get

$$-\nu m^2 - al^2 + 2bBln - 2cB^2n^2 + cB^2m^2 = 0, \quad (21)$$

$$-2\nu mn + 2bBlm - cB^2mn = 0, \quad (22)$$

$$-\nu n^2 + al^2 = 0. \quad (23)$$

From eqs. (22) and (23) we get a constraint relation

$$4b^2B^2\nu = a(cB^2 + 2\nu)^2. \quad (24)$$

Using eq. (23) and the constraint relation (24), eq. (21) reduces to

$$al^2 = \nu m^2. \quad (25)$$

From eqs. (23) and (25) one can conclude that $m = n$.

The solution (19) can be written as

$$u(\tau) = \pm\sqrt{\frac{\nu}{a}}\,\frac{\sinh\tau}{1+\cosh\tau}. \quad (26)$$

Therefore the solution of eq. (9) can be written as

$$u(x,t) = \pm\sqrt{\frac{\nu}{a}}\,\frac{\sinh[B(x-\nu t)]}{1+\cosh[B(x-\nu t)]}, \quad (27)$$

which is a kink-wave. Now we assume a solution of eq. (18) in the form

$$u(\tau) = \frac{l\cosh\tau}{m+n\sinh\tau}. \quad (28)$$

Substituting for u, u' and u'' in eq. (18), we obtain an algebraic equation in powers of $\sinh\tau$ given by

$$\begin{aligned}&-\nu(m+n\sinh\tau)^2 + al^2(1+\sinh^2\tau)\\ &+2bB(lm\sinh\tau - ln)\\ &+cB^2(2n^2 + m^2 - mn\sinh\tau) = 0. \quad (29)\end{aligned}$$

Equating the coefficients of different powers of $\sinh\tau$ to zero, we get

$$-\nu m^2 + al^2 - 2bBln + 2cB^2n^2 + cB^2m^2 = 0, \quad (30)$$

$$-2\nu mn + 2bBlm - cB^2mn = 0, \quad (31)$$

$$-\nu n^2 + al^2 = 0. \quad (32)$$

From eqs. (31) and (32) we get the constraint relation

$$4b^2B^2\nu = a(cB^2 + 2\nu)^2, \quad (33)$$

which is the same as eq. (24). Using eq. (32) and the constraint relation (33), eq. (30) reduces to

$$al^2 = -\nu m^2. \quad (34)$$

From eqs. (32) and (34) one can conclude that $n = \pm im$. The solution (28) can be written as

$$u(\tau) = \pm\sqrt{\frac{\nu}{a}}\,\frac{i\cosh\tau}{1\pm i\sinh\tau}. \quad (35)$$

Therefore the solution of eq. (9) can be written as

$$u(x,t) = \pm\sqrt{\frac{\nu}{a}}\,\frac{i\cosh[B(x-\nu t)]}{1\pm i\sinh[B(x-\nu t)]}. \quad (36)$$

Eqs. (26), (27), (35) and (36) imply

$$a\nu > 0. \quad (37)$$

This shows that the coefficient of the first nonlinear term and the speed of the wave must carry the same sign.

4 Lie symmetry analysis

The study and analysis of differential equations through the realm of group theory is associated with the great mathematician Sophus Lie [12]. The method of determining the Lie point symmetry generators via the classical approach for partial differential equations (PDEs) is well-known [6,9,11,13,15,17]. In this section, we calculate the Lie point symmetries admitted by eq. (9).

We define

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}, \quad (38)$$

to be the vector field that leaves eq. (9) invariant, i.e.,

$$X^{[3]}\left[u_t+a\left(u^3\right)_x+b\left(u^2\right)_{xx}+cu_{xxx}\right]=0, \tag{39}$$

where $\xi=\xi(x,t,u),\ \tau=\tau(x,t,u),\ \phi=\phi(x,t,u)$ and $X^{[3]}$ is the third-order prolonged operator of X defined by

$$X^{[3]}=X+\phi^t\frac{\partial}{\partial u_t}+\phi^x\frac{\partial}{\partial u_x}+\phi^{xx}\frac{\partial}{\partial u_{xx}}+\phi^{xxx}\frac{\partial}{\partial u_{xxx}}.$$

Eq. (39) is the symmetry condition

$$\begin{aligned}&\phi^t+2\phi(3auu_x+bu_{xx})+\phi^x(3au^2+4bu_x)+2b\phi^{xx}u\\&\quad+c\phi^{xxx}=0,\end{aligned} \tag{40}$$

where ϕ^t, ϕ^x, ϕ^{xx} and ϕ^{xxx} are given by

$$\begin{aligned}\phi^t&=D_t\phi-u_tD_t\tau-u_xD_t\xi,\\ \phi^x&=D_x\phi-u_tD_x\tau-u_xD_x\xi,\\ \phi^{xx}&=D_x\phi^x-u_{xt}D_x\tau-u_{xx}D_x\xi,\\ \phi^{xxx}&=D_x\phi^{xx}-u_{xxt}D_x\tau-u_{xxx}D_x\xi,\end{aligned}$$

and D_i denotes the differentiation operator with respect to x^i given by

$$D_i=\frac{\partial}{\partial x^i}+u_i^\alpha\frac{\partial}{\partial u^\alpha}+u_{ij}^\alpha\frac{\partial}{\partial u_j^\alpha}+\cdots,\qquad i=1,2,$$

with $(x^1,x^2)=(x,t)$.

To determine the governing equations from eq. (40), we use the package SYM [8] in Mathematica to separate the monomials in the derivatives of u, since the coefficient functions ξ, τ and ϕ are independent of the derivatives of u, and replace u_t by

$$u_t=-\left[a\left(u^3\right)_x+b\left(u^2\right)_{xx}+cu_{xxx}\right].$$

This process leads to solving the system of PDEs

$$\begin{aligned}&\tau_x=0,\quad \tau_u=0,\quad \tau=\tau(t),\quad \xi_u=0,\quad \phi_{uu}=0,\\&-3\xi_x+\tau_t=0,\quad \phi_u-2\xi_x+\tau_t=0,\\&3au^2\phi_x+2bu\phi_{xx}+c\phi_{xxx}+\phi_t=0,\\&2b\left(\phi+u\tau_t-2u\xi_x\right)+3c\left(\phi_{xu}-\xi_{xx}\right)=0,\\&6au\phi+4b\phi_x-3au^2\xi_x+4bu\phi_{xu}-2bu\xi_{xx}-c\xi_{xxx}\\&\quad+3au^2\tau_t-\xi_t=0.\end{aligned} \tag{41}$$

The computation of eq. (41) reveals that

$$\tau=3m_1t+m_2,\quad \xi=m_1x+m_3,\quad \phi=-m_1u. \tag{42}$$

Thus the three-dimensional Lie point symmetry algebra is spanned by the vector fields

$$X_1=\frac{\partial}{\partial t},\quad X_2=\frac{\partial}{\partial x},\quad X_3=3t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}-u\frac{\partial}{\partial u}. \tag{43}$$

We now perform symmetry reductions to determine solutions for eq. (9) using the Lie point symmetries from eq. (43). This entails applying the well-known method of invariance also known as the equation for characteristics.

(I) X_2

We solve the characteristic equation

$$\frac{dt}{0}=\frac{dx}{1}=\frac{du}{0}. \tag{44}$$

The invariants from eq. (44) are given by

$$y=t,\quad w=u, \tag{45}$$

where $w=w(y)$ and

$$u_t=w',\quad u_x=0, \tag{46}$$

where $(')$ denotes the derivative of w with respect to y.

The substitution of eq. (46) into eq. (9) and then solving the resulting ODE for w gives

$$w=A, \tag{47}$$

where A is a constant of integration.

Thus the substitution of eq. (47) into eq. (45) gives the trivial solution for eq. (9) as

$$u=A. \tag{48}$$

(II) X_1

The characteristic equation is

$$\frac{dt}{1}=\frac{dx}{0}=\frac{du}{0}. \tag{49}$$

The invariants from eq. (49) are given by

$$y=x,\quad w=u, \tag{50}$$

with

$$u_t=0,\quad u_x=w',\quad u_{xx}=w'',\quad u_{xxx}=w'''. \tag{51}$$

The substitution of eq. (51) into eq. (9) and then integrating once with respect to y (setting the integration constant to be zero) results in the ODE

$$aw^3+2b\left(w^2\right)'+cw''=0. \tag{52}$$

Since

$$Y=\frac{\partial}{\partial y}$$

is a Lie point symmetry of eq. (52), we have the zeroth-, first-order and second-order invariants

$$p=w,\quad q=w',\quad \frac{dq}{dp}=\frac{w''}{w'}, \tag{53}$$

where $q = q(p)$.

The substitution of eq. (53) into eq. (52) results in the first-order ODE

$$\frac{dq}{dp} = -\frac{1}{c}\left(\frac{ap^3}{q} + 4bp\right). \tag{54}$$

We obtain the solution for eq. (54) using Mathematica as

$$2b\alpha \tanh^{-1}\left[\alpha\left(b + \frac{cq}{p^2}\right)\right] + \ln\left[ap^4 + 4bp^2q + 2cq^2\right] = A, \tag{55}$$

where A is a constant of integration and

$$\alpha = \sqrt{\frac{2}{2b^2 - ac}}.$$

To analyse eq. (55) further, we set $A = 0$ and apply the identity

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

This results in

$$\alpha\left(b + \frac{cq}{p^2}\right) = \frac{1 - (ap^4 + 4bp^2q + 2cq^2)^{\frac{1}{b\alpha}}}{1 + (ap^4 + 4bp^2q + 2cq^2)^{\frac{1}{b\alpha}}}, \tag{56}$$

so that the substitution of eq. (53) into eq. (56) gives

$$\alpha\left(b + \frac{cw'}{w^2}\right) = \frac{1 - \left[aw^4 + 4bw^2w' + 2c(w')^2\right]^{\frac{1}{b\alpha}}}{1 + \left[aw^4 + 4bw^2w' + 2c(w')^2\right]^{\frac{1}{b\alpha}}}. \tag{57}$$

Remarks: (1) The solution (55) is compatible with the solution of eq. (9).
(2) Eq. (52) has an additional symmetry

$$Z = -y\frac{\partial}{\partial y} + w\frac{\partial}{\partial w}$$

for general values of a, b and c, and altogether has eight Lie point symmetries when the parameters are related by a single constraint.

(III) $X = X_1 + kX_2$

In this case, a linear combination of X_1 and X_2, where k is an arbitrary constant leads to travelling-wave solutions of eq. (9).

The characteristic equation is

$$\frac{dt}{1} = \frac{dx}{k} = \frac{du}{0}. \tag{58}$$

The invariants from eq. (58) are given by

$$y = x - kt, \quad w = u, \tag{59}$$

with

$$u_t = -kw', \quad u_x = w', \quad u_{xx} = w'', \quad u_{xxx} = w'''. \tag{60}$$

The substitution of eq. (60) into eq. (9) and then integrating once (setting the integration constant to be zero) results in eq. (18) (with redefined constants).

(IV) X_3

The characteristic equation is

$$\frac{dt}{3t} = \frac{dx}{x} = \frac{du}{-u}. \tag{61}$$

The invariants from eq. (61) are given by

$$y = \frac{x^3}{t}, \quad w = xu, \tag{62}$$

with

$$u_t = -\frac{yw'}{xt}, \quad u_x = -\frac{w}{x^2} + \frac{3xw'}{t}, \quad u_{xx} = \frac{2w}{x^3} + \frac{9x^3w''}{t^2},$$
$$u_{xxx} = -\frac{6w}{x^4} + \frac{6w'}{xt} + \frac{27x^2}{t^2}\left(w'' + yw'''\right). \tag{63}$$

The substitution of eq. (63) into eq. (9) and multiplying both sides by x^4 results in the ODE

$$-y^2w' + 3aw^2\left(3yw' - w\right) + 6b\left[-2yww' + 3y^2(ww'' + (w')^2) + w^2\right] + 3c\left[-2(w - yw') + 9y^2(w'' + yw''')\right] = 0. \tag{64}$$

Figure 2 is a typical representative of a numerical simulation for eq. (64) via Mathematica, where the parameter values were chosen as $a = c = \frac{1}{3}$ and $b = \frac{1}{6}$ for $y \in [0, 200]$, and the initial conditions were given as $w(1) = 0$, $w'(1) = w''(1) = -1$.

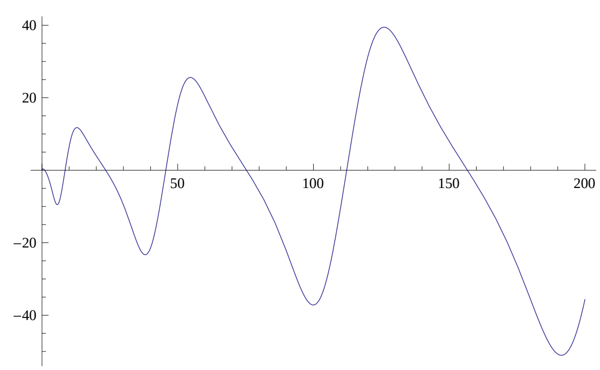


Fig. 2: Profile of solution for $w(y)$

5 Conclusion

This paper addressed the STO equation from an integration standpoint, where several solutions were exhibited. The traveling-wave hypothesis, ansatz method and Lie symmetry analysis lead to exact solutions for this equation. For the ansatz method, it was established that the power-law nonlinearity parameter condenses to unity for shock wave solutions to exist. Some numerical simulations were also included to support the analysis. The results of this paper stands on a strong footing for further research development. In the future, perturbation terms will be added in order to establish the integrability aspects of the perturbed STO equation. Numerical simulations, as well as additional integration tools will be applied to address this version of the equation. A few of them to mention are the G'/G-expansion approach, Kudryashov's method, variational iteration method, simplest equation approach, Lie symmetry approach and several others. These results will ultimately be reported in the future.

References

[1] M. Akbari. Exact solutions of the coupled Higgs equation and the Maccari system using the modified simplest equation method. Information Sciences Letters, **2**, 155-158 (2013).

[2] JI D$\bar{I}$az, Nonlinear partial differential equations and free boundaries, Pitman Advanced Pub. Program (Boston), (1985).

[3] Martin Kruskal, Nonlinear wave equations, Dynamical Systems, Theory and Applications, Springer, **38**, 310-354 (1975).

[4] A. Biswas. Soliton solutions of the pertubed resonant nonlinear Schrödinger's equation with full nonlinearity by semi-inverse variational principle. Quantum Physics Letters, **1**, 79-83 (2013).

[5] Ji-Huan He, Variational principles for some nonlinear partial differential equations with variable coefficients, Chaos, Solitons & Fractals, Elsevier, **19**, 847-851 (2004).

[6] G. W. Bluman and S. Kumei. Symmetries and Differential Equations, in: Applied Mathematical Sciences. Volume **81**, Springer-Verlag, New York, (1989).

[7] X. Bo and W. Chen-Ming. Conservation laws and Darboux transformation for Sharma-Tasso-Olver equation. Communications in Theoretical Physics, **58**, 317-322 (2012).

[8] S. Dimas and D. Tsoubelis, SYM: A new symmetry finding package for Mathematica. In N. H. Ibragimov, C. Sophocleous and P. A. Damianou, editors, The 10th International Conference in Modern Group Analysis, (University of Cyprus, Nicosia) 64-70 (2005), $http://www.math.upatras.gr/spawn$.

[9] N. H. Ibragimov, Transformation Groups Applied to Mathematical Physics, Reidel, Dordrecht, The Netherlands, (1985).

[10] A. J. M. Jawad, M. D. Petković and A. Biswas. Modified simple equation method for nonlinear evolution equations. Applied Mathematics and Computation, **217**, 869-877 (2010).

[11] E. V. Krishnan, S. Kumar and A. Biswas. Solitons and other nonlinear waves of the Boussinesq equation. Nonlinear Dynamics, **70**, 1213-1221 (2012).

[12] S. Lie, Lectures on differential equations with known infinitesimal transformations, (in German, written with the help of G. Scheffers), B. G. Teubner, Leipzig, (1891).

[13] H. Liu, J. Li and Q. Zhang. Lie symmetry analysis and exact explicit solutions for general Burgers' equation. Journal of Computational and Applied Mathematics, **228**, 1-9 (2009).

[14] M. Mirzazadeh. Modified simple equation method and its applications to nonlinear partial differential equations. Information Sciences Letters, **3**, 1-9 (2014).

[15] P. J. Olver. Applications of Lie Groups to Differential Equations. Graduate Texts in Mathematics, Volume **107**, Springer-Verlag, Berlin, (1986).

[16] E. Osman, M. Khalfallah and H. Sapoor. Improved (G'/G) -expansion method for elliptic-like equation and general traveling wave solutions of some class of NLPDEs. Information Sciences Letters, **2**, 123-127 (2013).

[17] L. V. Ovsiannikov. Group Analysis of Differential Equations. Academic Press, New York, (1982).

[18] Z Dutton, M Budde, C Slowe, LV Hau, Observation of quantum shock waves created with ultra-compressed slow light pulses in a Bose-Einstein condensate, Science, (2001).

[19] Y. Shang, Y. Huang and W. Yuan. Bäcklund transformations and abundant exact explicit solutions of the Sharma-Tasso-Olver equation. Applied Mathematics and Computation, **217**, 7172-7183 (2011).

[20] K. Tabatabaei and E. Gunerhan. Numerical solution of Duffing equation by the differential transform method. Applied Mathematics & Information Sciences Letters, **2**, 1-6 (2014).

[21] N. Taghizadeh, M. Mirzazadeh and A. S. Paghaleh. Application of first integral method to some special nonlinear partial differential equations. Applied Mathematics & Information Sciences Letters, **1**, 89-95 (2013).

[22] S. Wang, X. Y. Tang, S. Y. Lou. Soliton fission and fusion: Burgers equation and Sharma-Tasso-Olver equation. Chaos, Solitons and Fractals, **21**, 231-239 (2004).

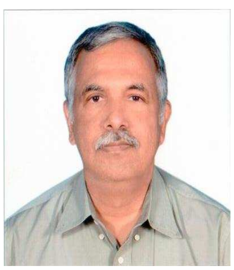
[23] A-M. Wazwaz. New solitons and kinks solutions to the Sharma-Tasso-Olver equation. Applied Mathematics and Computation, **188**, 1205-1213 (2007).

[24] A-M. Wazwaz. New solutions of distinct physical structures to high-dimensional nonlinear evolution equations. Applied Mathematics and Computation, **196**, 363-370 (2008).

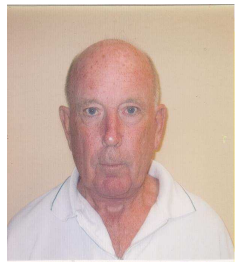
[25] A-M. Wazwaz. Multiple kink solutions for the (2+1)-dimensional Sharma–Tasso–Olver and the Sharma–Tasso–Olver–Burgers equations. Journal of Applied Nonlinear Dynamics, **2**, 95-102 (2013).

Bouthina S. Ahmed Doctor of Numerical Analysis in Mathematics Department - Women Science -Ain Shams University - Cairo - Egypt. She obtained her MS degree in differential equations and her PhD degree in numerical analysis. Her research interests are in Applied Mathematics.

Richard M. Morris obtained his BSc(Honours), MSc and PhD in mathematics at the University of the Witwatersrand in Johannesburg, South Africa. He is currently a Post-doctoral researcher at Delaware State University in Dover, Delaware, USA. His research interests are in the areas of Applied Mathematics relating to symmetries and conservation laws of higher-order differential equations.



Edamana V. Krishnan did his PhD from the Indian Institute of Science, Bangalore, India in 1979. He did his Post-doctoral work in the Dept. of Mathematics, University of Manchester Institute of Science and Technology, UK and in the Dept. of Physics, Eindhoven University of Technology, The Netherlands from 1979 to 1981. He taught in the Dept. of Mathematics, University of Benin, Nigeria from 1981 to 1987 and in the Dept. of Mathematics and Statistics, Sultan Qaboos University, Oman since 1987.



Peter G. L. Leach obtained his PhD from Latrobe University, Melbourne, Australia, in 1979 and his DSc from the University of Natal, Durban, South Africa, in 1996. He was Professor of Applied Mathematics at the University of the Witwatersrand, Johannesburg, from 1983 to 1990 and Professor of Mathematics in Durban until his retirement. His current official affiliation is Professor Emeritus, University of KwaZulu-Natal. He is widely traveled and has had joint papers with collaborators in fourteen countries. He is on a number of editorial boards, including AMIS, has refereed for over fifty journals and is the author of over 300 papers. His essential interest is the resolution of differential equations with a particular emphasis on symmetry and singularity methods. The areas of application of these differential equations have ranged from Classical Mechanics, Quantum Mechanics, Cosmology, Ecology, Epidemiology to Financial Mathematics. He has been the recipient of a number of awards, is a Fellow of the Royal Society of South Africa and a Member of the Academy of Nonlinear Sciences (Russian Federation).

Anjan Biswas earned his BSc (Honours) degree from Saint Xavier's College in Calcutta, India. Subsequently, he received his MSc and M. Phil degrees in Applied Mathematics from University of Calcutta. Then, he earned MA and PhD degrees in Applied Mathematics from the University of New Mexico in Albuquerque, New Mexico, USA. After that he carried out his Post-doctoral studies in Applied Mathematics from the University of Colorado, Boulder, USA. He was then an Assistant Professor of Mathematics at Tennessee State University that is located in Nashville, TN. Currently, he is working as an Associate Professor of Mathematics at Delaware State University in Dover, DE. His research interest is on Theory of Solitons.