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Coupled fixed point theorems for F-invariant set

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Abstract: In this paper, we extend and complement some recent results of coupled fixed point theorems of Luong and Thuan in [N. V. Luong, N. X. Thuan, Coupled fixed point theorems for mixed monotone mappings and an application to integral equations, Computers and Mathematics with Applications 62 (2011) 4238-4248.] by weaken the concept of the mixed monotone property. The example of a nonlinear contraction mapping which is not applied by the results of Luong and Thuan, but can be applied to our results is given. The presented results extend and complement results of Luong and Thuan and some known existence results from the literature.

Keywords: Coupled fixed points; F-invariant sets; partially order metric spaces.

1. Introduction

One of the simplest and the most useful result in the fixed point theory is the Banach's contraction principle [8]. This principle has been generalized in different ways in several spaces by mathematicians over the years (see [2,7,10,11, 19] and references mentioned therein).

In 2004, fixed point theory was extend to metric spaces endowed with a partial ordering by Ran and Reurings [16] and they presented applications of their results to matrix equations. Subsequently, Nieto and Rodríguez-López [15] extended the results in [16] for nondecreasing mappings and obtained a unique solution for a first order ordinary differential equation with periodic boundary conditions (see also, [4,6,12]).

On the other hand, Bhaskar and Lakshmikantham [9] introduced the concept of mixed monotone property. Furthermore, they established the classical coupled fixed point theorems for mappings which satisfy the mixed monotone property and showed some applications in the existence and uniqueness of a solution for a periodic boundary value problem. Because their important role in the study of nonlinear differential equations, nonlinear integral equations and differential inclusions, so a wide discussion on coupled fixed point theorems aimed the interest of many scientists. A number of articles on this topic have been dedicated to the improvement and generalization see in [1,3, 13,18,20–23] and reference therein. Recently, Luong and Thuan [14] extend and generalized the classical coupled fixed point of Bhaskar and Lakshmikantham [9] and some coupled fixed point theorems.

The aim of this paper is to extend and unify the coupled fixed point results in [14], using the concept of F-invariant set due to Samet and Vetro [17] and to study condition to guarantee the uniqueness of coupled fixed points. We also give the example of a nonlinear contraction mapping which is not applied by the results of Luong and Thuan [14], but can be applied to our results. The presented results extend and complement some recent results of Luong and Thuan [14] and some known existence results from the literature.

2. Preliminaries

Throughout this paper $(X, \preceq)$ denotes a partially ordered set. By $x \prec y$, we mean $x \preceq y$ but $x \neq y$. A mapping $f : X \to X$ is said to be non-decreasing (non-increasing) if for all $x, y \in X$, $x \preceq y$ implies $f(x) \preceq f(y)$ ($f(y) \preceq f(x)$ respectively).

Definition 21 ([9]) *Let $(X, \preceq)$ be a partially ordered set. A mapping $F : X \times X \to X$ is said to has the a* mixed monotone property *if F is monotone non-decreasing in its first argument and is monotone non-increasing in its sec-*

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ond argument, that is, for any $x, y \in X$

$$x_1, x_2 \in X, x_1 \preceq x_2 \implies F(x_1, y) \preceq F(x_2, y) \tag{2.1}$$

and

$$y_1, y_2 \in X, y_1 \preceq y_2 \implies F(x, y_1) \succeq F(x, y_2). \tag{2.2}$$

Definition 22 *[[9]] Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a* coupled fixed point *of mapping $F : X \times X \to X$ if*

$$x = F(x, y) \text{ and } y = F(y, x).$$

Definition 23 *[[17]] Let (X, d) be a metric space and $F : X \times X \to X$ be a given mapping. Let M be a nonempty subset of X^4. We say that M is F-invariant subset of X^4 if and only if for all $x, y, z, w \in X$, we have*

(a) $(x, y, z, w) \in M \iff (w, z, y, x) \in M$;
(b) $(x, y, z, w) \in M$
$\implies (F(x, y), F(y, x), F(z, w), F(w, z)) \in M$.

Remark 24 *We can easily to check that the set $M = X^4$ is trivially F-invariant.*

Example 25 *Let $X = \{1, 2, 3, 4\}$ endowed with the usual metric and $F : X \times X \to X$ be defined by*

$$F(x, y) = \begin{cases} 2, & x, y \in \{2, 4\} \\ 3, & \text{otherwise.} \end{cases}$$

It easy to see that $M = \{2, 4\}^4 \subseteq X^4$ is F-invariant.

Next example plays a key role in the proof of our main results in partially ordered set.

Example 26 *Let (X, d) be a metric space endowed with a partial order $\preceq$. Let $F : X \times X \to X$ be a mapping satisfying the mixed monotone property. Define set M by*

$$M = \{(a, b, c, d) \in M^4 : c \preceq a, b \preceq d\}.$$

Then, M is F-invariant subset of X^4.

3. Coupled fixed point for F-invariant set

Let Θ denote the class of all functions $\theta : [0, \infty) \times [0, \infty) \to [0, 1)$ which satisfies following condition:

For any two sequences $\{t_n\}$ and $\{s_n\}$ of nonnegative real numbers,

$$\theta(t_n, s_n) \to 1 \implies t_n, s_n \to 0.$$

Following are examples of some function in Θ.

$\theta_1(s, t) = k$ for $s, t \in [0, \infty)$, where $k \in [0, 1)$.

$$\theta_2(s, t) = \begin{cases} \dfrac{\ln(1 + ks + lt)}{ks + lt} & ; s > 0 \text{ or } t > 0, \\ r \in [0, 1) & ; s = 0, t = 0, \end{cases}$$

where $k, l \in (0, 1)$

$$\theta_3(s, t) = \begin{cases} \dfrac{\ln(1 + \max\{s, t\})}{\max\{s, t\}} & ; s > 0 \text{ or } t > 0, \\ r \in [0, 1) & ; s = 0, t = 0. \end{cases}$$

Now, we prove our main result under the concept of F-invariant.

Theorem 31 *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X, $F : X \times X \to X$ be a continuous mapping and M be a nonempty subset of X^4. We assume that*

(a) M is F-invariant;
(b) there exists $(x_0, y_0) \in X \times X$ such that

$$(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M;$$

(c) there exists $\theta \in \Theta$ such that for all $(x, y, u, v) \in M$, we have

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ \leq \theta(d(x, u), d(y, v))(d(x, u) + d(y, v)).$$

Then there exists $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

Proof As $F(X \times X) \subseteq X$, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_n = F(x_{n-1}, y_{n-1}) \text{ and } y_n = F(y_{n-1}, x_{n-1}) \tag{3.1}$$

for all $n \in \mathbb{N}$. Since

$$(F(x_0, y_0), F(y_0, x_0), x_0, y_0) = (x_1, y_1, x_0, y_0) \in M,$$

we using contractive condition (c), we get

$$\begin{aligned} & d(x_2, x_1) + d(y_2, y_1) \\ &= d(F(x_1, y_1), F(x_0, y_0)) + d(F(y_1, x_1), F(y_0, x_0)) \\ &\leq \theta(d(x_1, x_0), d(y_1, y_0))(d(x_1, x_0) + d(y_1, y_0)) \\ &< d(x_1, x_0) + d(y_1, y_0). \end{aligned}$$

Since $(x_1, y_1, x_0, y_0) \in M$, we have

$$(F(x_1, y_1), F(y_1, x_1), F(x_0, y_0), F(y_0, x_0)) \in M$$

that is

$$(x_2, y_2, x_1, y_1) \in M.$$

Again, using the contractive condition, we get

$$\begin{aligned} & d(x_3, x_2) + d(y_3, y_2) \\ &= d(F(x_2, y_2), F(x_1, y_1)) + d(F(y_2, x_2), F(y_1, x_1)) \\ &\leq \theta(d(x_2, x_1), d(y_2, y_1))(d(x_2, x_1) + d(y_2, y_1)) \\ &< d(x_2, x_1) + d(y_2, y_1). \end{aligned}$$

Using a similar argument to the above, we get

$$\begin{aligned} & d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \\ &\leq \theta(d(x_n, x_{n-1}), d(y_n, y_{n-1}))(d(x_n, x_{n-1}) \\ &\quad + d(y_n, y_{n-1})) \\ &< d(x_n, x_{n-1}) + d(y_n, y_{n-1}) \end{aligned} \tag{3.2}$$

for all $n \in \mathbb{N}$. This implies that the sequence

$$\{d(x_{n+1}, x_n) + d(y_{n+1}, y_n)\}$$

is monotone decreasing and bounded below. Therefore, there is some $d \geq 0$ such that

$$d_n := d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \to d \text{ as } n \to \infty$$

Next, we show that $d = 0$. Assume, to the contrary, that $d > 0$, then from (3.2), we have

$$\theta(d(x_{n+1}, x_n), d(y_{n+1}, y_n)) \to 1 \text{ as } n \to \infty$$

Since $\theta \in \Theta$, we get

$$d(x_{n+1}, x_n) \to 0 \text{ and } d(y_{n+1}, y_n) \to 0$$

as $n \to \infty$. So

$$d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \to 0 \text{ as } n \to \infty,$$

which is a contradiction. Therefore, $d = 0$, that is,

$$d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \to 0 \text{ as } n \to \infty$$

Next, we show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. On contrary, assume that at least one of $\{x_n\}$ or $\{y_n\}$ is not a Cauchy sequence. Therefore, there exists $\epsilon > 0$ and two subsequences of integers $n(k)$ and $m(k)$ with $n(k) > m(k) \geq k$ such that

$$d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \geq \epsilon. \tag{3.3}$$

for all $k \in \mathbb{N}$. Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ satisfying (3.3). Then we have

$$d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \geq \epsilon. \tag{3.4}$$

and

$$d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)-1}, y_{m(k)}) < \epsilon. \tag{3.5}$$

Using (3.4), (3.5) and the triangle inequality, we have

$$\begin{aligned}
\epsilon \leq r_k &:= d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \\
&\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) \\
&\quad + d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)}) \\
&< d(x_{n(k)}, x_{n(k)-1}) + d(y_{n(k)}, y_{n(k)-1}) + \epsilon.
\end{aligned}$$

On taking limit as $k \to \infty$, we have

$$r_k = d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \to \epsilon. \tag{3.6}$$

By the triangle inequality, we get

$$\begin{aligned}
r_k &= d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \\
&\leq d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) \\
&\quad + d(x_{m(k)+1}, x_{m(k)}) + d(y_{n(k)}, y_{n(k)+1}) \\
&\quad + d(y_{n(k)+1}, y_{m(k)+1}) + d(y_{m(k)+1}, y_{m(k)}) \\
&= [d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1})] \\
&\quad + [d(x_{n(k)}, x_{n(k)+1}) + d(y_{n(k)}, y_{n(k)+1})] \\
&\quad + [d(x_{m(k)+1}, x_{m(k)}) + d(y_{m(k)+1}, y_{m(k)})] \\
&= [d(x_{m(k)+1}, x_{n(k)+1}) + d(y_{m(k)+1}, y_{n(k)+1})] \\
&\quad + d_{n(k)} + d_{m(k)} \\
&= \Big[d(F(x_{m(k)}, y_{m(k)}), F(x_{n(k)}, y_{n(k)})) \\
&\quad + d(F(y_{m(k)}, x_{m(k)}), F(y_{n(k)}, x_{n(k)})) \Big] \\
&\quad + d_{n(k)} + d_{m(k)} \\
&\leq \theta(d(x_{m(k)}, x_{n(k)}), d(y_{m(k)}, y_{n(k)})) \\
&\quad (d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})) \\
&\quad + d_{n(k)} + d_{m(k)} \\
&= \theta(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) r_k \\
&\quad + d_{n(k)} + d_{m(k)}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
r_k &\leq \theta(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) r_k \\
&\quad + d_{n(k)} + d_{m(k)}.
\end{aligned}$$

This further implies that

$$\begin{aligned}
&\frac{r_k - d_{n(k)} - d_{m(k)}}{r_k} \\
&\leq \theta(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) \\
&< 1.
\end{aligned}$$

On taking limit as $k \to \infty$, we obtain

$$\theta(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) \to 1.$$

Since $\theta \in \Theta$, we have

$$d(x_{n(k)}, x_{m(k)}) \to 0 \text{ and } d(y_{n(k)}, y_{m(k)}) \to 0$$

as $k \to \infty$, that is

$$d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \to 0$$

as $k \to \infty$, which is a contradiction. Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence. By completeness of X, there exists $x, y \in X$ such that $\{x_n\}$ and $\{y_n\}$ converges to x and y respectively.

Now, we show that F has a coupled fixed point. Since F is a continuous, taking $n \to \infty$ in (3.1), we get

$$\begin{aligned}
x &= \lim_{n\to\infty} x_{n+1} \\
&= \lim_{n\to\infty} F(x_n, y_n) \\
&= F(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) \\
&= F(x, y)
\end{aligned}$$

and

$$\begin{aligned} y &= \lim_{n\to\infty} y_{n+1} \\ &= \lim_{n\to\infty} F(y_n, x_n) \\ &= F(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) \\ &= F(y, x). \end{aligned}$$

Therefore, $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

Theorem 32 *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X, $F : X \times X \to X$ be a mapping and M be a nonempty subset of X^4. We assume that*

(a) M is F-invariant;
(b) there exists $(x_0, y_0) \in X \times X$ such that

$$(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M;$$

(c) there exists $\theta \in \Theta$ such that for all $(x, y, u, v) \in M$, we have

$$\begin{aligned} &d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ &\leq \theta(d(x, u), d(y, v))(d(x, u) + d(y, v)). \end{aligned}$$

If two sequences $\{x_n\}, \{y_n\}$ with

$$(x_{n+1}, y_{n+1}, x_n, y_n) \in M$$

for all $n \in \mathbb{N}$ and $\{x_n\} \to x$, $\{y_n\} \to y$, then

$$(x, y, x_n, y_n) \in M$$

for all $n \in \mathbb{N}$. Then there exists $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

Proof Following arguments similar to those given in Theorem 31, we obtain a sequence $\{x_n\}$ converges to x and a sequence $\{y_n\}$ converges to y for some $x, y \in X$. By assumption, we have $(x, y, x_n, y_n) \in M$ for all $n \in \mathbb{N}$. By the contractive condition, we obtain

$$\begin{aligned} &d(F(x, y), x) + d(F(y, x), y) \\ &\leq d(F(x, y), F(x_n, y_n)) + d(F(x_n, y_n), x) \\ &\quad + d(F(y, x), F(y_n, x_n)) + d(F(y_n, x_n), y) \\ &\leq \theta(d(x, x_n), d(y, y_n))(d(x, x_n) + d(y, y_n)) \\ &\quad + d(x_{n+1}, x) + d(y_{n+1}, y) \\ &< d(x, x_n) + d(y, y_n) + d(x_{n+1}, x) + d(y_{n+1}, y). \end{aligned}$$

On taking limit as $n \to \infty$, we have

$$d(F(x, y), x) + d(F(y, x), y) = 0.$$

Thus $x = F(x, y)$ and $y = F(x, y)$ that is (x, y) is a coupled fixed point of F.

From Example 26, we can apply Theorem 31 and 32 with $M = \{(a, b, c, d) \in X^4 : c \preceq a, b \preceq d\}$ to the main result of Luong and Thuan [14].

Corollary 33 *[14, Theorem 2.1] Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X and $F : X \times X \to X$ be a mapping having the mixed monotone property on X such that there exist two element $x_0, y_0 \in X$ such that*

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0).$$

Suppose that there exists $\theta \in \Theta$ such that

$$\begin{aligned} &d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ &\leq \theta(d(x, u), d(y, v))(d(x, u) + d(y, v)) \end{aligned}$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either

(a) F is continuous or
(b) X has the following property:
  1.if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$,
  2.if a non-increasing sequence $\{y_n\} \to y$, then $y \succeq y_n$ for all $n \in \mathbb{N}$,

then there exists $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

Theorem 34 *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X and $F : X \times X \to X$ be a continuous mapping and M be a nonempty subset of X^4. We assume that*

(a) M is F-invariant;
(b) there exists $(x_0, y_0) \in X \times X$ such that

$$(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M;$$

(c) there exists $\eta \in \Theta$ such that for all $(x, y, u, v) \in M$, we have

$$\begin{aligned} &d(F(x, y), F(u, v)) \\ &\leq \frac{1}{2}\eta(d(x, u), d(y, v))(d(x, u) + d(y, v)). \end{aligned}$$

Suppose either

(d) F is continuous or
(e) If two sequences $\{x_n\}, \{y_n\}$ with

$$(x_{n+1}, y_{n+1}, x_n, y_n) \in M$$

for all $n \in \mathbb{N}$ and $\{x_n\} \to x$, $\{y_n\} \to y$, then

$$(x, y, x_n, y_n) \in M$$

for all $n \in \mathbb{N}$,

then there exists $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

Proof For $(x, y, u, v) \in M$, from (c), we have

$$\begin{aligned} &d(F(x, y), F(u, v)) \\ &\leq \frac{1}{2}\eta(d(x, u), d(y, v))(d(x, u) + d(y, v)). \end{aligned} \tag{3.7}$$

Since $(x, y, u, v) \in M$, we get $(v, u, y, x) \in M$ and then

$$\begin{aligned} & d(F(y,x), F(v,u)) \\ &= d(F(v,u), F(y,x)) \\ &\leq \frac{1}{2}\eta(d(v,y), d(u,x))(d(v,y) + d(u,x)). \end{aligned} \tag{3.8}$$

Now, combining (3.7) and (3.8), we get

$$\begin{aligned} & d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \\ &\leq \frac{1}{2}[\eta(d(x,u), d(y,v)) + \eta(d(v,y), d(u,x))] \\ &\quad (d(x,u) + d(y,v)) \\ &= \theta(d(x,u), d(y,v))(d(x,u) + d(y,v)) \end{aligned} \tag{3.9}$$

for $(x, y, u, v) \in M$, where

$$\theta(t_1, t_2) = \frac{1}{2}[\eta(t_1, t_2) + \eta(t_2, t_1)]$$

for all $t_1, t_2 \in [0, \infty)$. It is easy to verify that $\theta \in \Theta$ and we can apply Theorems 31 and 32. Hence F has a coupled fixed point.

Corollary 35 *[14, Corollary 2.2] Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X and $F : X \times X \to X$ be a mapping having the mixed monotone property. Suppose that there exists a $\eta \in \Theta$ such that*

$$\begin{aligned} & d(F(x,y), F(u,v)) \\ &\leq \frac{1}{2}\eta(d(x,u), d(y,v))(d(x,u) + d(y,v)). \end{aligned}$$

for all $x, y, u, v \in X$ for which $x \succeq u$ and $y \preceq v$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0)$$

and either

(a) F is continuous or
(b) X has the following property:

1. if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$,
2. if a non-increasing sequence $\{y_n\} \to y$, then $y \succeq y_n$ for all $n \in \mathbb{N}$,

then there exists $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

Corollary 36 *[9, Theorem 1.3 and 1.4] Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X and $F : X \times X \to X$ be a mapping having the mixed monotone property. Suppose that there exists a $k \in [0, 1)$ such that*

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}(d(x,u) + d(y,v))$$

for all $x, y, u, v \in X$ for which $x \succeq u$ and $y \preceq v$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0)$$

and either

(a) F is continuous or
(b) X has the following property:

1. if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$,
2. if a non-increasing sequence $\{y_n\} \to y$, then $y \succeq y_n$ for all $n \in \mathbb{N}$,

then there exists $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

Proof Taking $\eta(t_1, t_2) = k$ with $k \in [0, 1)$ for all $t_1, t_2 \in [0, \infty)$ in Corollary 35 , result follows immediately.

Let Ω denote the class of those functions $\omega : [0, \infty) \to [0, 1)$ which satisfies the condition:

For any sequences $\{t_n\}$ of nonnegative real numbers,

$$\omega(t_n) \to 1 \Longrightarrow t_n \to 0.$$

Theorem 37 *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X and $F : X \times X \to X$ be a mapping and M be a nonempty subset of X^4. We assume that*

(a) M is F-invariant;
(b) there exists $(x_0, y_0) \in X \times X$ such that

$$(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M;$$

(c) there exists $\omega \in \Omega$ such that for all $(x, y, u, v) \in M$, we have

$$\begin{aligned} & d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \\ &\leq \omega(d(x,u) + d(y,v))(d(x,u) + d(y,v)). \end{aligned}$$

Suppose either

(d) F is continuous or
(e) If two sequences $\{x_n\}, \{y_n\}$ with

$$(x_{n+1}, y_{n+1}, x_n, y_n) \in M$$

for all $n \in \mathbb{N}$ and $\{x_n\} \to x$, $\{y_n\} \to y$, then

$$(x_n, y_n, x, y) \in M$$

for all $n \in \mathbb{N}$,

then there exists $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

Proof Taking $\theta(t_1, t_2) = \omega(t_1 + t_2)$ for all $t_1, t_2 \in [0, \infty)$ in Theorem 31 and Theorem 32, result follows.

Corollary 38 *[14, Corollary 2.3] Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X and $F : X \times X \to X$ be a mapping with mixed monotone property. Suppose that there exists $\omega \in \Omega$ such that*

$$\begin{aligned} & d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \\ &\leq \omega(d(x,u) + d(y,v))(d(x,u) + d(y,v)) \end{aligned}$$

for all $x, y, u, v \in X$ for which $x \succeq u$ and $y \preceq v$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0)$$

and either

(a) F is continuous or
(b) X has the following property:

1. if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$,
2. if a non-increasing sequence $\{y_n\} \to y$, then $y \succeq y_n$ for all $n \in \mathbb{N}$,

then there exists $x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$, that is, F has a coupled fixed point.

Taking $\omega(t) = k$ with $k \in [0,1)$ for all $t \in [0,\infty)$ in Corollary 38, we obtain the following corollary.

Corollary 39 *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X and $F : X \times X \to X$ be a mapping with mixed monotone property. Suppose that there exists $k \in [0,1)$ such that*

$$d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))$$
$$\leq k(d(x,u) + d(y,v))$$

for all $x, y, u, v \in X$ for which $x \succeq u$ and $y \preceq v$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0,y_0) \text{ and } y_0 \succeq F(y_0,x_0)$$

and either
(a) F is continuous or
(b) X has the following property:

1. if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$,
2. if a non-increasing sequence $\{y_n\} \to y$, then $y \succeq y_n$ for all $n \in \mathbb{N}$,

then there exists $x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$, that is, F has a coupled fixed point.

Theorem 310 *In addition to the hypotheses in Theorem 31, suppose that*

$$(y_0,x_0,x_0,y_0) \in M \text{ or } (x_0,y_0,y_0,x_0) \in M,$$

then $x = y$ that is $x = F(x,x)$.

Proof We construct the sequences $\{x_n\}$ and $\{y_n\}$ as Theorem 31. So we get the sequences $\{x_n\}$ converges to x for some $x \in X$ and the sequences $\{y_n\}$ converges to y for some $y \in X$. Assume that $(y_0,x_0,x_0,y_0) \in M$. Since M is F-invariant, we have $(y_n,x_n,x_n,y_n) \in M$ for all $n \in \mathbb{N}$. From (c), we have

$$\begin{aligned} &d(y_{n+1},x_{n+1}) + d(x_{n+1},y_{n+1}) \\ &= d(F(y_n,x_n),F(x_n,y_n)) + d(F(x_n,y_n),F(y_n,x_n)) \\ &\leq \theta(d(y_n,x_n),d(x_n,y_n))[d(y_n,x_n) + d(x_n,y_n)] \end{aligned} \tag{3.10}$$

for all $n \in \mathbb{N}$. From (3.10), we get

$$\begin{aligned} d(y_{n+1},x_{n+1}) &\leq \theta(d(y_n,x_n),d(x_n,y_n))d(y_n,x_n) \\ &< d(y_n,x_n). \end{aligned} \tag{3.11}$$

This implies that the sequence $\{d(y_n,x_n)\}$ is monotone decreasing and bounded below. Therefore, there is some $d \geq 0$ such that

$$d(y_n,x_n) \to d \text{ as } n \to \infty$$

Next, we show that $d = 0$. Assume, to the contrary, that $d > 0$, then from (3.11), we have

$$\theta(d(y_n,x_n),d(x_n,y_n)) \to 1 \text{ as } n \to \infty$$

Since $\theta \in \Theta$, we get $d(y_n,x_n) \to 0$ as $n \to \infty$, which is a contradiction. Therefore, $d = 0$, that is,

$$d(y_n,x_n) \to 0 \text{ as } n \to \infty .$$

Now, we have

$$d(y,x) \leq d(y,y_n) + d(y_n,x_n) + d(x_n,x). \tag{3.12}$$

From (3.12), taking limit as $n \to \infty$ we get $d(y,x) = 0$ and hence $x = y$. For case of $(x_0,y_0,y_0,x_0) \in M$, we can similar prove in first case.

Theorem 311 *In addition to the hypotheses of Theorem 32, suppose that*

$$(y_0,x_0,x_0,y_0) \in M \text{ or } (x_0,y_0,y_0,x_0) \in M$$

then $x = y$ that is $x = F(x,x)$.

Proof Following arguments similar to those given in Theorem 310 and then by applying Theorem 32, result follows.

Now, reasoning on Theorem 31 and 32, some questions arise naturally. To be precise, one can ask himself

Is it possible to guarantee the uniqueness of the coupled fixed point of F ?

Motivated by the interest in this research, we give positive answers to these questions adding to Theorem 31 and 32 some hypotheses. We proceed with order. Then, to have the uniqueness, we state and prove the following theorem.

Theorem 312 *In addition to the hypotheses in Theorem 31, suppose that for every $(x,y),(z,t) \in X \times X$, there exists a point $(u,v) \in X \times X$ such that $(x,y,u,v) \in M$ and $(z,t,u,v) \in M$. Then F has a unique coupled fixed point.*

Proof From Theorem 31, F has a coupled fixed point. Suppose (x,y) and (z,t) are coupled fixed points of F, that is,

$$x = F(x,y), y = F(y,x), z = F(z,t) \text{ and } t = F(t,z).$$

Next, we claim that $x = z$ and $y = t$. By given hypothesis, there exists $(u,v) \in X \times X$ such that $(x,y,u,v) \in M$ and $(z,t,u,v) \in M$. We put $u_0 = u$ and $v_0 = v$ and construct sequences $\{u_n\}$ and $\{v_n\}$ by

$$u_n = F(u_{n-1},v_{n-1}) \text{ and } v_n = F(v_{n-1},u_{n-1})$$

for all $n \in \mathbb{N}$.

Since $(x,y,u,v) \in M$, we get $(x,y,u_{n-1},v_{n-1}) \in M$ for all $n \in \mathbb{N}$ and then

$$\begin{aligned} &d(x,u_n) + d(y,v_n) \\ &= d(F(x,y),F(u_{n-1},v_{n-1})) + d(F(y,x),F(v_{n-1},u_{n-1})) \\ &\leq \theta(d(x,u_{n-1}),d(y,v_{n-1}))[d(x,u_{n-1}) + d(y,v_{n-1})] \\ &< d(x,u_{n-1}) + d(y,v_{n-1}). \end{aligned} \tag{3.13}$$

Consequently, sequence $\{d(x,u_n)+d(y,v_n)\}$ is non-negative decreasing and bounded below, so

$$d(x,u_n)+d(y,v_n)\to d$$

as $n\to\infty$, for some $d\geq 0$. We claim that $d=0$. Indeed, if $d>0$ then following similar arguments to those given in the proof of Theorem 31, we conclude that

$$\theta(d(x,u_n),d(y,v_n))\to 1 \text{ as } n\to\infty.$$

Since $\theta\in\Theta$, we obtain $d(x,u_n)\to 0$ and $d(y,v_n)\to 0$ as $n\to\infty$. Therefore,

$$d(x,u_n)+d(y,v_n)\to 0 \text{ as } n\to\infty$$

which is a contradiction. Hence

$$d(x,u_n)+d(y,v_n)\to 0 \text{ as } n\to\infty.$$

Similarly, one can prove that

$$d(z,u_n)+d(t,v_n)\to 0 \text{ as } n\to\infty.$$

Finally, we have

$$\begin{aligned} d(z,x)+d(y,t) &\leq [d(z,u_n)+d(u_n,x)] \\ &\quad +[d(y,v_n)+d(v_n,t)] \\ &= [d(x,u_n)+d(y,v_n)] \\ &\quad +[d(z,u_n)+d(t,v_n)]. \end{aligned}$$

Taking $n\to\infty$, we have $d(z,x)=0$ and $d(y,t)=0$,that is, $z=x$ and $y=t$. This completes the proof.

Theorem 313 *In addition to the hypotheses in Theorem 32, suppose that for every $(x,y),(z,t)\in X\times X$, there exists a point $(u,v)\in X\times X$ such that $(x,y,u,v)\in M$ and $(z,t,u,v)\in M$. Then F has a unique coupled fixed point.*

Proof The proof is straightforward, following the same lines of the proof of Theorem 312. Then, in order to avoid repetition, the details are omitted.

Remark 314*In Theorem 312 and 313, taking M similar to Example 26, we obtain the results of Luong and Thuan [14, Theorem 2.7].*

Next, we give example to validate Theorem 312.

Example 315*Let $X=\mathbb{R}$ endowed with the usual metric $d(x,y)=|x-y|$ for all $x,y\in X$ and endowed with the usual partial order as $x\preceq y \iff x\leq y$. Define the mapping $F:X\times X\to X$ by*

$$F(x,y)=\frac{x+y+2}{3},\quad \forall (x,y)\in X\times X.$$

Consider $y_1=2$ and $y_2=3$, we have $y_1\preceq y_2$ but $F(x,y_1)\preceq F(x,y_2)$ for all $x\in X$. So the mapping F does not satisfy the mixed monotone property. Hence main results of Luong and Thuan [14] can not be applied to this example. But, by simple calculation, we see that for all $x,y,u,v\in X^4$, we have

$$\begin{aligned} &d(F(x,y),F(u,v))+d(F(y,x),F(v,u)) \\ &=\left|\frac{x+y+2}{3}-\frac{u+v+2}{3}\right|+\left|\frac{y+x+2}{3}-\frac{v+u+2}{3}\right| \\ &\leq \frac{1}{3}[d(x,u)+d(y,v)]+\frac{1}{3}[d(y,v)+d(x,u)] \\ &=\theta(d(x,u),d(y,v))(d(x,u)+d(y,v)), \end{aligned}$$

where $\theta(s,t)=\frac{2}{3}$. Now, we can applying Theorem 312 with $M=X^4$. Therefore, F has a unique coupled fixed point that is a point $(2,2)$.

Remark 316*Although main results of Luong and Thuan [14] is essential tool in the partially ordered metric spaces to claim the existence of coupled fixed point. However, the mapping do not have the mixed monotone property in general case such as the mapping in the above example. Therefore, it is very interesting to our Theorem as another auxiliary tool to claim the existence of a coupled fixed point.*

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