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The Truncated Lindley Distribution: Inference and Application

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Abstract: This paper introduced the truncated versions of the Lindley distribution and studied the characteristics of the proposed distributions with showing the monotonicity of the density and hazard functions. The statistical proprieties such as moments, quantile function and order statistics are also discussed. The maximum likelihood estimators are constructed for estimating the unknown parameters of the upper, lower and double truncated Lindley distributions. A set of real data containing the strengths of the glass of aircraft window, is considered to show the applicability of the truncated Lindley distributions.

Keywords: Truncated Lindley distribution, Moments, Quantile function, Order statistics, Maximum likelihood estimator, 62F10

1 Introduction

The truncated distributions are quite effectively used where a random variable is restricted to be observed on some range and these situations are common in various fields. For instance, in survival analysis, failures during the warranty period may not be counted. Items may also be replaced after certain time following the replacement policy, so that failures of the item are ignored.

Many researchers, therefore being attracted to the problem of analysing such truncated data encountered in various disciplines, proposed the truncated versions of the usual statistical distributions. [2] discussed the application of the truncated version of the Birnbaum-Saunders (BS) distribution to improve a forecasting actuarial model and particularly for modelling data from insurance payments that establish a deductible. [1,17] discussed the application of the truncated Pareto distribution to the statistical analysis of masses of stars and of diameters of asteroids. The truncated Weibull distribution has been found being applied in the various fields such as to analyse the diameter data of trees truncate data specific threshold level, to predict the height distribution of small trees based on incomplete laser scanning data, to modelling the diameter distribution of forest, to characterize the observed Portuguese fire size distribution, to seismological data, on the development of the pit depths on a water pipe etc. For more detail on the truncated Weibull distribution and related references readers may refer to book [13] covered the subject of Weibull distribution and recently published article [18] based on the truncated Weibull distribution.

From the above commentary and monitoring the wide applicability of the truncated distributions, we proposed the truncation in the Lindley distribution. The Lindley distribution is mixture of $exponential(\theta)$ and $gamma(2,\theta)$ distributions with their mixing proportions are $(1/(1+\theta))$ and $(\theta/(1+\theta))$, respectively and was first proposed by [11] as counter example of the fiducial statistics. [8] have given the extensive mathematical treatments to study the various properties of the Lindley distribution and advocated the use of Lindley distribution over the exponential distribution considering the waiting times before service of the banks customers. One of the main reasons to prefer the Lindley distribution over the exponential distribution is its time dependent increasing failure rate which is common practice in the survival analysis. Since last decade, Lindley distribution has been attracting the attention of the researchers, scientists and the reliability probationers, and many author extended it to the various parsimonious distributions. To name a few extensions, three parameter generalized Lindley [16], the generalized Lindley [14,15], extended Lindley [3], weighted Lindley [7], power Lindley [6], exponential Poisson-Lindley [4] and the transmuted Lindley [12] distributions.

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Some extensions of the Lindley distributions e.g. power Lindley and generalized Lindley distributions etc. are the good competitors of the Weibull distribution and can be quite effectively used to model the real phenomenon where the Weibull distribution seems to be incompatible to the real data. In this directions, one can also study the properties of truncated versions of these Lindley's generalizations as the alternative models to the truncated Weibull distribution in the literature. Therefore, this article aims to start the discussions with introducing the concept of the truncation in one parameter Lindley distribution.

The rest of the paper is arranged in the following sections. In section 2, the truncated versions of the Lindley distribution, named as the upper truncated Lindley (UTL), lower truncated Lindley (LTL), double truncated Lindley (DTL) distributions are introduced. Particularly, the flexibility of the UTL distribution has been shown demonstrating the characteristics of the probability density (pdf) and hazard functions with different combination of the values of its parameters. The moments, quantile function and order statistics of the UTL distribution are derived in section 3. In section 4, the method of the maximum likelihood is applied to obtain the estimates of the parameters of the UTL, LTL and DTL distributions. In section 5, a set of real data is modelled through the different distributions and their applicability are compared. Finally, the paper is concluded in section 6.

2 The truncated Lindley distributions

A distribution $G(x;\Theta)$ is said to be a double truncated distribution over the interval $[\nu,\zeta]$ if it has the cumulative distribution function (cdf) defined as

$$G(x;\Theta)=\frac{F(x;\Theta)-F(\nu;\Theta)}{F(\zeta;\Theta)-F(\nu;\Theta)},\ \nu\le x\le\zeta,\ -\infty<\nu<\zeta<\infty \tag{1}$$

and probability density function (pdf) is

$$g(x;\Theta)=\frac{f(x;\Theta)}{F(\zeta;\Theta)-F(\nu;\Theta)},\ \nu\le x\le\zeta,\ -\infty<\nu<\zeta<\infty \tag{2}$$

where, $f(x;\Theta)$ and $F(x;\Theta)$ are the pdf and cdf of the baseline model and $\Theta\in\mathscr{R}^n$ denotes the vector parameters of base line model. Here, three cases can be recognized as

(i)When $\nu=0$ and $\zeta\to\infty$, it reduces to baseline model.
(ii)When $\nu=0$, it is called the upper truncated distribution of the baseline model.
(iii)When $\zeta\to\infty$, it is called the lower truncated distribution of the baseline model.

In this article, we consider the Lindley distribution as a baseline model with the following distribution function

$$F(x;\theta)=1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x},x>0,\theta>0 \tag{3}$$

Using (1) and (3), the double truncated Lindley distribution is defined as

$$g_D(x;\theta)=\frac{\theta^2}{(1+\theta)}\frac{(1+x)\exp(-\theta x)}{F(\zeta;\theta)-F(\nu;\theta)};\ 0\le\nu\le x\le\zeta<\infty \tag{4}$$

In the following sections, we will only discuss the properties of the upper truncated Lindley distribution and the same procedure can be applied to study the properties of the lower truncated Lindley distribution as well as double truncated Lindley distribution. The upper truncated Lindley distribution has the following pdf is given by

$$g_U(x;\theta)=\frac{\theta^2(1+x)\exp(-\theta(x-\zeta))}{(1+\theta)(\exp(\theta\zeta)-1)-\theta\zeta};\ 0\le x\le\zeta \tag{5}$$

It is denoted by UTL(θ,ζ). Note that the above pdf will behave like as

$(i)\frac{d}{dx}g(x;\theta)=g'(x)=\frac{\theta^2(1-\theta-\theta x)\exp(-\theta(x-\zeta))}{(1+\theta)(\exp(\theta\zeta)-1)-\theta\zeta}$
(ii)When $\theta\ge1$, $g'(x)<0$, it indicates that $g(x)$ is decreasing in x.
(iii)When $\theta<1$, $g(x)$ is uni-modal and mode values is $x_{Mo}=(1-\theta)/\theta$, see Figure (1).

The corresponding hazard function at epoch t is given by

$$H(t;\theta) = \frac{\theta^2}{(1+\theta)} \frac{(1+t)\exp(-\theta t)}{F(\zeta;\theta) - F(t;\theta)};\ 0 \le t \le \zeta \tag{6}$$

[9] used the term $\eta(x) = -\frac{f'(x)}{f(x)}$ to determine the monotonicity of the hazard function. For UTL distribution, we get

$$\eta_{UTL}(x) = -\frac{g'(x)}{g(x)} = -\frac{f'(x)/F(\zeta)}{f(x)/F(\zeta)} = -\frac{f'(x)}{f(x)} = \eta_L(x)$$

It followed that

$(i) H(0) = \theta^2 / [(1+\theta)(1-\exp(-\theta\zeta)) - \theta\zeta\exp(-\theta\zeta)]$
$(ii) H(\zeta) = \infty$, i.e. as $t \to \zeta$, $H(t) \to \infty$
$(iii) \eta_L'(x) = \frac{1}{1+x^2} > 0\ \forall\ x$, it implies that the hazard rate function of UTL distribution is increasing in x and θ, see Figure (2).

3 Statistical properties

3.1 Moments and related measures

The rth moment under the upper truncated Lindley distribution is defined as

$$E[X^r] = \frac{\theta^2}{(1+\theta)} \frac{1}{F(\zeta;\theta)} \int_0^{zeta} x^r (1+x) e^{-\theta x} dx \tag{7}$$

The rth moment can also be written as

$$\mu_r' = \frac{\theta^2}{1+\theta} \frac{\varphi_r(\theta,\zeta) + \varphi_{r+1}(\theta,\zeta)}{F(\zeta,\theta)}, r = 1,2,\ldots \tag{8}$$

Particularly, if $r = r+1$, we have

$$\mu_{r+1}' = \frac{\theta^2}{1+\theta} \frac{\varphi_{r+1}(\theta,\zeta) + \varphi_{r+2}(\theta,\zeta)}{F(\zeta,\theta)} \tag{9}$$

From (8) and (9), we get

$$\mu_{r+1}' = \mu_r' \frac{1 + k_{r+1}(\theta,\zeta)}{1 + k_r^{-1}(\theta,\zeta)}, r = 1,2,... \tag{10}$$

where,

$$k_i(\theta,\zeta) = \frac{\varphi_{i+1}(\theta,\zeta)}{\varphi_i(\theta,\zeta)}, i = 1,2,...$$
$$\text{and } \varphi_1(\theta,\zeta) = \left(1 - e^{-\theta\zeta}(1+\theta\zeta)\right)/\theta^2,$$
$$\varphi_j(\theta,\zeta) = \left(j\varphi_{j-1}(\theta,\zeta) - \zeta^j e^{-\theta\zeta}\right)/\theta,\ j = 2,3,...$$

The mean and variance of UTL distribution can be easily calculated by

$$\mu = \frac{\theta\left[(\theta+2)\varphi_1(\theta,\zeta) - \zeta^2 e^{-\theta\zeta}\right]}{\left[(\theta+1)\left(1-e^{-\theta\zeta}\right) - \theta\zeta e^{-\theta\zeta}\right]}$$

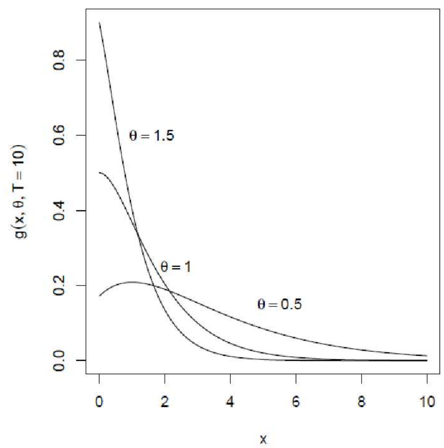

Fig. 1: The density function of UTL distribution for given $\theta = 0.5,\ 1$ & 1.5 and $\zeta = 10$

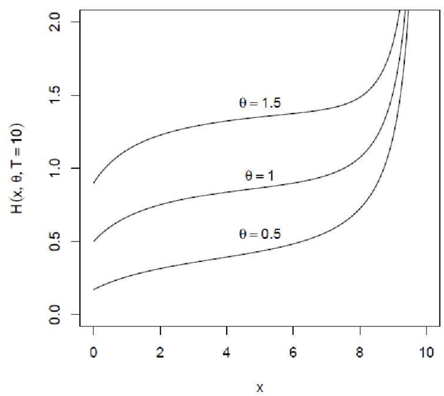

Fig. 2: The hazard function of UTL distribution for given $\theta = 0.5,\ 1$ & 1.5 and $\zeta = 10$

and

$$\sigma^2 = V(X) = E\left[X - E(X)\right]^2 = \mu_2^{'} - \left[\mu_1^{'}\right]^2$$

respectively. We calculated the mean and variance of ULT distribution for given values of θ and ζ and presented in Table 1. It is observed from Table 1 that mean and variance decrease as θ increases while ζ is kept fix. For fixed values of θ, mean and variance increase initially as ζ increases and stabilise at a point. It is due fact that there is no mass to be truncated from the data after a certain point for a given value of θ.

The skewness and kurtosis of the distribution can be simply verified by using the following relationship

$$S_k = \frac{\left(\mu_3^{'} - 3\mu\mu_2^{'} + 2\mu^3\right)^2}{\left(\mu_2^{'} - \mu^2\right)^3}$$

$$K_t = \frac{\left(\mu_4^{'} - 4\mu\mu_3^{'} + 6\mu^2\mu_2^{'} - 3\mu^4\right)}{\left(\mu_2^{'} - \mu^2\right)^2}$$

The skewness and kurtosis of the UTL distribution are sketched in Figure 3 with respect to its parameters θ and ζ.

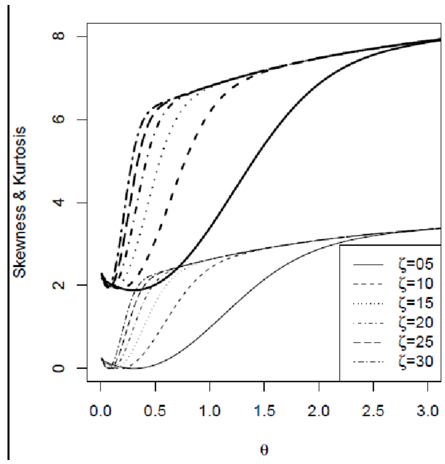

Fig. 3: The skewness (lower) and kurtosis (upper bold) function of UTL distribution with varying θ and T

Table 1: Mean and variance for various choices of the values of θ and ζ.

θ	ζ=5		ζ=10		ζ=15		ζ=20		ζ=25	
	μ	σ^2	μ	σ^2	μ	σ^2	μ	σ^2	μ	σ^2
0.10	1.8221	2.9174	7.0751	5.7116	15.801	8.2046	27.824	10.376	42.647	12.227
0.25	1.8994	2.6373	7.1384	4.6299	14.159	5.8916	20.757	6.5920	25.579	6.9383
0.50	1.8330	2.1657	5.1332	3.0634	6.9202	3.2871	7.4329	3.3268	7.5359	3.3325
1.00	1.5678	1.7376	2.9734	2.0677	3.2093	2.0939	3.2279	2.0952	3.2290	2.0952
1.25	1.2285	1.3879	1.7251	1.4974	1.7495	1.5000	1.7500	1.5000	1.7500	1.5000
1.50	0.9185	1.1207	1.0802	1.1553	1.0825	1.1556	1.0825	1.1556	1.0825	1.1556
1.75	0.6781	0.9227	0.7287	0.9333	0.7289	0.9333	0.7289	0.9333	0.7289	0.9333

3.2 Quantile function

The quantile function is used to describe the percentiles of the distribution and obtained as the solution of the following equation

$$G(\xi_\tau, \Theta) = \tau, \tau \in (0,\ 1) \tag{11}$$

From (3) and (11), we have

$$-(1+\theta+\theta\xi_\tau)\, e^{-(1+\theta+\theta\xi_\tau)} = \frac{(1+\theta)(\tau F(\zeta;\theta)-1)}{\exp(1+\theta)} \tag{12}$$

To solve the above equation for ξ_τ, [10] introduced the use of Lambert W function for the generation of random variables with Lindley or Poisson-Lindley distribution. The Lambert W function is a multivalued complex function defined as the solution of the equation:

$$W(z) exp(W(z)) = z, \tag{13}$$

where, z is a complex number. Now, form (12) and (13), we obtained

$$-(1+\theta+\theta\xi_\tau) = W_{-1}\left(\frac{(1+\theta)(\tau F(\zeta;\theta)-1)}{\exp(1+\theta)}\right) \tag{14}$$

where, W_{-1} is negative branch of the Lambert W function. Thus,

$$\xi_\tau = -1 - \frac{1}{\theta} - \frac{1}{\theta} W_{-1}\left(\frac{(1+\theta)(\tau F(\zeta;\theta)-1)}{\exp(1+\theta)}\right) \tag{15}$$

As $\zeta \to \infty$, from the above equation (15), we get the quantile function of Lindley distribution derived by [10] as

$$\xi_\tau = -1 - \frac{1}{\theta} - \frac{1}{\theta} W_{-1}\left(\frac{(1+\theta)(\tau-1)}{\exp(1+\theta)}\right) \tag{16}$$

The median of the UTL distribution can obtained as

$$Md_x = -1 - \frac{1}{\theta} - \frac{1}{\theta} W_{-1}\left(\frac{(1+\theta)\,(F\,(\zeta;\theta)-2)}{2\exp(1+\theta)} \right) \tag{17}$$

3.3 Order statistics

In this subsection, we derive the pdf of the sth $(1 \le s \le n)$ order statistics $X_{s:n}$, $g_{s:n}$ say, is defined as

$$g_{s:n}(t) = \frac{1}{B\,(s, n-s+1)} g\,(t)\, G^{s-1}\,(t)\, \{1 - G\,(t)\}^{n-s}, \tag{18}$$

where, $B\,(s, n-s+1)$ is the beta function. Expanding the binomial expansion, we get

$$g_{s:n}(t) = \frac{1}{B\,(s, n-s+1)} \sum_{i=1}^{n-s} (-1)^i \binom{n-s}{i} \left(\frac{F\,(t,\theta)}{F\,(\zeta,\theta)} \right)^{s+i} \frac{f\,(t,\theta)}{F\,(t,\theta)} \tag{19}$$

where, $f\,(t,\theta)$ and $F\,(t,\theta)$ are the pdf and cdfs of the Lindley distribution. For $s = 1$, particularly the pdf of the first order statistics is given by

$$g_{1:n}(t) = n \sum_{i=1}^{n-1} (-1)^i \binom{n-1}{i} \frac{f\,(t,\theta)\, F^i\,(t,\theta)}{F^{i+1}\,(\zeta,\theta)} \tag{20}$$

Substituting, the pdf and cdfs of the Lindley distribution, we obtained

$$g_{1:n}(t) = \frac{\theta^2}{1+\theta} \sum_{i=1}^{n-1} \frac{(-1)^i\,(1+t)\, e^{-\theta t} \left(1 - \left(1 + \frac{\theta t}{1+\theta}\right) e^{-\theta t}\right)^i}{B\,(n-i, i+1) \left(1 - \left(1 + \frac{\theta\zeta}{1+\theta}\right) e^{-\theta\zeta}\right)^{i+1}} \tag{21}$$

Similarly, the pdf of $X_{n:n}$ is given by

$$g_{n:n}(t) = \frac{n\theta^2\,(1+t)\, e^{-\theta t} \left(1 - \left(1 + \frac{\theta t}{1+\theta}\right) e^{-\theta t}\right)^{n-1}}{(1+\theta) \left(1 - \left(1 + \frac{\theta\zeta}{1+\theta}\right) e^{-\theta\zeta}\right)^{n}} \tag{22}$$

The mean and variance of the sth order statistics can be obtained by using the formulae used in section 3.1 for UTL distribution.

4 Maximum likelihood estimation

In this section, we describe the procedure to obtain the maximum likelihood estimates (MLE) of the parameters of UTL as well as lower truncated Lindley (LTLD) and double truncated Lindley (DTLD) distributions based on the random sample $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ of size n, so that these distributions can be effectively used to model the real problems depending upon the nature of the data. We fitted these distributions to a set of real data in next section.

4.1 MLEs for UTLD

Let $\mathbf{x}$ be an iid (independent and identically distributed) sample of size n from UTL distribution. The likelihood function based on the observed sample $\mathbf{x}$ is given by

$$\mathrm{L}\,(\theta, \zeta|\mathbf{x}) = \left[\frac{\theta^2}{(1+\theta)\,(\exp(\theta\zeta) - 1) - \theta\zeta} \right]^n \prod_{i=1}^{n} (1 + x_i)\, e^{-\theta \sum\limits_{i=1}^{n} (x_i - \zeta)} \tag{23}$$

It is to be noted here that $S = \sum\limits_{i=1}^{n} x_i$ is the joint sufficient statistics for θ and ζ. The corresponding log-likelihood equation is given by

$$\ln \mathrm{L} = 2n \ln(\theta) - n \ln\left[(1+\theta)\,(\exp(\theta\zeta) - 1) - \theta\zeta\right] + \sum_{i=1}^{n} \ln(1 + x_i) - \theta \sum_{i=1}^{n} x_i + n\theta\zeta \tag{24}$$

Note that in the above log-likelihood equation (27), it is not possible to get an estimate of ζ in terms of observed sample since ζ is free from x. Now, from the order statistics, let $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ be the order sample corresponding to $x_1, x_2, \cdots, x_n$. Then, the MLE $\hat{\zeta}$ of ζ can be taken as $\hat{\zeta} = \max(x_1, x_2, \cdots, x_n)$ i.e. $\hat{\zeta} = x_{(n)}$ largest observation. Once, we get the MLE of ζ, the MLE $\hat{\theta}$ of θ can be obtained as the solution of the following non-linear equation:

$$\frac{2}{\theta} - \frac{(1+\zeta)\left(e^{\theta\zeta}-1\right)+\theta\zeta e^{\theta\zeta}}{(1+\theta)\left(e^{\theta\zeta}-1\right)-\theta\zeta} - \frac{1}{n}\sum_{i=1}^{n} x_i + \zeta = 0 \tag{25}$$

In order to solve the above equation, we need to use the iterative procedure like Newton's method.

4.2 MLEs for LTLD

The likelihood function based on $\mathbf{x}$ from LTL distribution is given by

$$\mathrm{L}(\theta, \nu|\mathbf{x}) = \frac{\theta^{2n}}{(1+\theta+\theta\nu)^n} \prod_{i=1}^{n} (1+x_i)\, e^{-\theta\sum_{i=1}^{n}(x_i-\nu)} \tag{26}$$

The log-likelihood equation is given by

$$\ln \mathrm{L} = 2n\ln(\theta) - n\ln(1+\theta+\theta\nu) + \sum_{i=1}^{n}\ln(1+x_i) - \theta\sum_{i=1}^{n} x_i + n\theta\nu \tag{27}$$

Similarly from the above subsection, the maximum likelihood estimate of ν will be $\hat{\nu} = \min(x_i); i = 1, 2, \ldots, n$ smallest observation. The maximum likelihood estimate $\hat{\theta}$ of θ can be uniquely determined by solving the following log-likelihood equation

$$\frac{2}{\theta} - \frac{(1+\nu)}{(1+\theta+\theta\nu)} - \frac{1}{n}\sum_{i=1}^{n} x_i + \nu = 0 \tag{28}$$

Applying some mathematical treatments on equation (28) yields,

$$\hat{\theta} = \frac{-(\bar{x}-2\nu-1)+\sqrt{\bar{x}^2+2\bar{x}(2\nu+3)-4\nu(\nu+1)+1}}{2(1+\nu)(\bar{x}-\nu)} \tag{29}$$

where, $\bar{x}$ is the mean of the observed sample.

4.3 MLEs for DTLD

The likelihood function under the assumption of the double truncated Lindley distribution for the random variable X, is given by

$$\mathrm{L}(\theta, \nu, \zeta|\mathbf{x}) = \frac{\theta^{2n}}{\phi(\theta)^n} \prod_{i=1}^{n} (1+x_i)\, e^{-\theta\sum_{i=1}^{n} x_i} \tag{30}$$

where, $\phi(\theta) = (1+\theta)\left(e^{-\theta\nu} - e^{-\theta\zeta}\right) + \theta\left(\nu e^{-\theta\nu} - \zeta e^{-\theta\zeta}\right)$. The corresponding log-likelihood function is given by

$$\ln \mathrm{L} = 2n\ln(\theta) - n\ln(\phi(\theta)) + \sum_{i=1}^{n}\ln(1+x_i) - \theta\sum_{i=1}^{n} x_i \tag{31}$$

For given MLEs of ν and ζ as $\hat{\nu} = x_{(1)}$ and $\hat{\zeta} = x_{(n)}$, respectively, the MLE of θ can be obtained by solving the following log-likelihood equation

$$\frac{2}{\theta} - \frac{\phi'(\theta)}{\phi(\theta)} - \bar{x} = 0 \tag{32}$$

where, $\phi'(\theta) = \frac{d\phi(\theta)}{d\theta}$.

5 Real data modelling

In this section, we verified that the truncation of the Lindley distribution improves its applicability taking the strength data of glass of the aircraft window which is reported by [5]. The data are given as:

18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.5, 25.52,
25.80, 26.69, 26.770, 26.78, 27.05, 27.67, 29.90, 31.11, 33.20,
33.73, 33.76, 33.890, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09,
39.58, 44.045, 45.29, 45.381.

The summary of the above data is given by

Units	Minimum	1st Qu.	Median	Mean	3rd Qu.	Maximum
31	18.83	25.51	29.90	30.81	35.83	45.381

We fitted the data by exponential, Weibull, Lindley, and lower, upper and double truncated Lindley distributions. The distribution function of the Weibull model is defined as

$$F(x) = 1 - \exp\left(-\theta x^{p}\right); \theta, p > 0$$

To compare the goodness-of-fit of above models, we used the Akaikes information criterion (AIC), Corrected Akaikes information criterion (AICC), Bayesian information criterion (BIC) and Kolmogorov-Smirnov (K-S) statistic, which are calculated form the following formulae

$$\text{AIC} = -2\log(\text{L}) + 2k,\ \text{AICC=AIC} + \frac{2k(k+1)}{(n-k-1)},$$

$$\text{BIC} = -2\log(\text{L}) + k\log(n) \text{ and } \text{D} = \sup_{x}|F_n(x) - F_0(x)|.$$

where, k is the number of parameters, n is the sample size and $F_n(x)$ is the empirical distribution function.

Based on the data, the fitting summary including the estimates of the parameters, log-likelihood, AIC, AICC, BIC and KS statistics values have been summarised in Tables 2. The probability-probability (P-P) plots for various distributions based on real data are plotted in Figure 4. Figure 5 shows the log-log plot of the survival function of the considered models based on the real data. The above study clearly indicate that the double truncated Lindley distribution gives reasonable fit to the data. From Figure 5, we observed that the usual distributions such as Exp, Lindley and Weibull are trying to capture the data from 0 (zero) as they support the whole positive real line. Whereas, the upper truncated and the lower truncated Lindley distributions capture only right and left tails of the data respectively. The performances based on used criterion

Table 2: Maximum likelihood estimates, AIC, AICC, BIC and KS statistics values under considered models based on real data

Distribution	Estimates	LogL	AIC	AICC	BIC	KS
Exp(α)	$(\hat{\alpha})$=0.03246	137.264	276.529	276.667	277.512	0.426
W(α,p)	$(\hat{\alpha},\hat{p})$=(0.00017, 2.5046)	113.067	230.135	230.563	229.117	0.225
LD(θ)	$(\hat{\theta})$=0.06299	126.994	255.988	256.126	256.971	0.333
UTLD(θ,ζ)	$(\hat{\theta},\hat{\zeta})$=(0.00754, 45.381)	110.216	224.431	224.860	223.414	0.184
LTLD(θ,v)	$(\hat{\theta},\hat{v})$=(0.10974,18.83)	107.103	218.205	218.634	217.188	0.150
DTLD(θ,v,ζ)	$(\hat{\theta},\hat{v},\hat{\zeta})$ =(0.0539,18.83,45.381)	101.090	208.180	209.069	205.163	0.118

(AIC & BIC etc.) of the different truncated forms of the Lindley distribution can be diagrammatically shown as

$$\boxed{\textbf{Worst}\ \text{Lindley} \rightarrow \text{UT Lindley} \rightarrow \text{LT Lindley} \rightarrow \text{DT Lindley}\ \textbf{Best}}$$

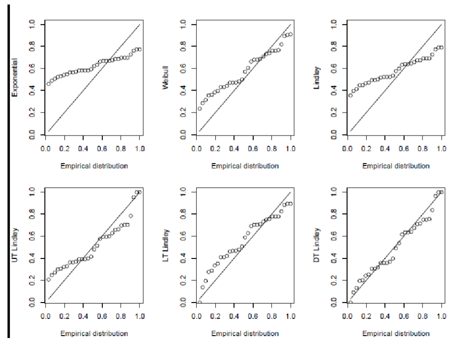

Fig. 4: The probability-probability (P-P) plots of various distributions based on real data

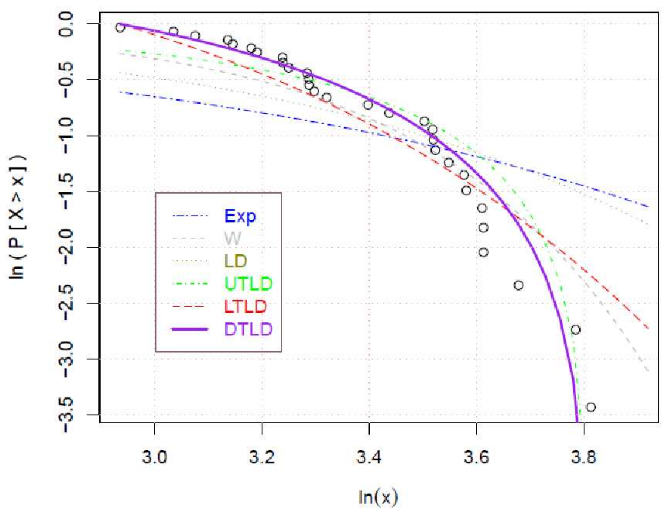

Fig. 5: Log-log plot of the survival function of various models based on real data

6 Conclusions

In this article, we introduced the truncated Lindley distributions called upper truncated, lower truncated and double truncated Lindley distribution. Particularly, the properties of the upper truncated Lindley distribution such as moments, quantile function and order statistics are discussed. The maximum likelihood estimators are constructed for estimating the unknown parameters of the upper truncated Lindley as well as lower truncated and double truncated Lindley distributions. The goodness-of-fits of the exponential, Weibull, Lindley and truncated (lower, upper, double) Lindley distributions have been compared through the AIC, AICC, BIC and KS statistics and found that the double truncated Lindley distribution fits well the data of the window strengths. Finally, it is concluded that the truncated distributions can be quit effectively used to model the real problems and so we can recommend the use of the truncated Lindley distributions in various fields including engineering, medical, finance and demography where such type of truncated data are commonly encountered.

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