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The Bäcklund Transformation and Exact Solutions for the Equivalent Dodd-Bullough-Mikhailov (E-DBM) Equation

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Abstract: In this paper, the E-DBM equation is studied. The Bäcklund transformation for this equation is obtained by using the homogeneous balance (HB) method. And some new solitary wave solutions, including double peakon solitary wave solutions and peakon singular solitary wave solutions, are given by the transformation.

Keywords: Bäcklund transformation, E-DBM equation, Solitary wave solutions

1 Introduction

A large variety of physical, chemical, and biological phenomena are governed by nonlinear partial differential equations equations (NPDEs). Exact solutions of nonlinear equations play an important role in the study of nonlinear physical phenomena.

The Dodd-Bullough-Mikhailov (DBM) equation [1]-[2], which appears in fluid flow and quantum field theory, is given by

$$v_{xt}+pe^{v}+qe^{-2v}=0, \quad (1)$$

where $v=v(x,t), p$ and q are real number and $p^2+q^2\neq 0$.When $p\neq 0, q=0$,equation (1) become Liouville equation

$$v_{xt}+pe^{v}=0,$$

Using the tanh method, Wazwaz [1] considered some solitary and periodic wave solutions for the following special DBM equation

$$v_{xt}+e^{v}+e^{-2v}=0,$$

Using the extended tanh method, Fan [2] build some exact explicit traveling wave solutions for the DBM equation.

By means of the transformation

$$v=\ln u, \quad (2)$$

the DBM equation (1) has the equivalent form

$$uu_{xt}-u_xu_t+pu^3+q=0, \quad (3)$$

where $u=u(x,t)$.Here, equation (3) is called the E-DBM equation for short (the equivalent form DBM equation). Clearly, if we know the solutions of E-DBM equation (3), then we can obtain the solutions of DBM equation (1) by the transformation (2).Thus the focus of the present work is the E-DBM equation (3).

In this paper, the homogeneous balance (HB) method is used to obtain the Bäcklund transformation for this equation and some new solitary wave solutions, including peakon solitary wave solutions, peakon singular solitary wave solutions, are given by the Bäcklund transformation.

2 Bäcklund transformation

As well known, there exist some techniques to seek Bäcklund transformations of NPDEs . For example, Tian and Gao [3] and Yan [4] presented some Bäcklund transformations of some nonlinear evolution equations by truncating the Painleve expansion. Wang and Zhang [5] obtained Auto-Bäcklund transformation of (2+1)-dimensional Nizhnik-Novikov-Veselov Equation by Laurent series expansion.

The HB method is a powerful tool to find Bäcklund transformations of NPDEs [6]-[11], which idea came

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from Frobenius's idea to transform a nonlinear PDE into solvable ODEs [12]. In this section, we will study the Bäcklund transformation of the E-DBM equation (3) by using the HB method. According to the idea of the HB method [6]-[11], and in order to balance the derivatives term $uu_{xt} - u_x u_t$ and the nonlinear term u^3,we suppose that the solution of the eq.(3) is as follows :

$$u = \partial_x \partial_t f(w) + v = f'' w_x w_t + f' w_{xt} + v, \quad (4)$$

where $'$ denote $\partial/\partial w$, $f = f(w), w = w(x,t)$ which will be determined later, and $v = v(x,t)$ is a given solution to Eq.(3). With the help of Mathematica, substituting (4) into (3), we obtain the following equation:

$$[f'' f^{(4)} - (f^{(3)})^2 + p(f'')^3] w_x^3 w_t^3 + \cdots = 0, \quad (5)$$

Setting the coefficient of $w_x^3 w_t^3$ to zero, we can obtain an ordinary differential equation for f

$$f'' f^{(4)} - (f^{(3)})^2 + p(f'')^3 = 0, \quad (6)$$

Solving (6) , we can obtain a solution

$$f = \alpha \ln w, \quad (7)$$

where $\alpha = \frac{2}{p}$.According to (7) , we can easily obtain

$$\begin{aligned}
&(f')^2 = -\alpha f'', f' f'' = -\frac{\alpha}{2} f^{(3)}, (f')^2 f'' = \frac{\alpha^2}{6} f^{(4)}, \\
&(f'')^2 = -\frac{\alpha}{6} f^{(4)}, f'(f'')^2 = \frac{\alpha^2}{24} f^{(5)}, (f'')^3 = \frac{\alpha^2}{120} f^{(6)}, \\
&f' f^{(3)} = -\frac{\alpha}{3} f^{(4)}, (f^{(3)})^2 = -\frac{\alpha}{30} f^{(6)}, f' f^{(4)} = -\frac{\alpha}{4} f^{(5)}, \\
&f'' f^{(4)} = -\frac{\alpha}{120} f^{(6)}, (f')^{(3)} = \frac{\alpha}{2} f^{(3)}. \quad (8)
\end{aligned}$$

Substituting (8) into (5) and using (7) , then (5) can be simplified to a linear polynomial of $f^{(6)}, f^{(5)}, f^{(4)}, f^{(3)}, f'', f', 1$ as follows :

$$\begin{aligned}
&(-\frac{1}{60}\alpha w_t^3 w_x^3 + \frac{1}{120} p\alpha w_t^3 w_x^3) \cdot f^{(6)} \\
&+(-\frac{1}{4}\alpha w_t^2 w_x^t w_{xt} + \frac{1}{8} p\alpha^2 w_t^2 w_x^2 w_{xt}) \cdot f^{(5)} \\
&+(v w_t^2 w_x^2 - \frac{1}{2} p v \alpha w_t^2 w_x^2 - \alpha w_t w_x w_{xt}^2 + \frac{1}{2} p\alpha^2 w_t w_x w_{xt}^2) \cdot f^{(4)} \\
&+(-v_t w_t^3 - v_x w_t^3 + v w_t^2 w_{tt} + 4 v w_t^2 w_{xt} - 3 p v \alpha w_t^2 w_{xt} \\
&-\alpha w_{xt}^3 + \frac{1}{2} p\alpha^2 w_{xt}^3 + v w_t^2 w_{xx} - \frac{1}{2}\alpha w_{tt} w_{xx} w_{xt} \\
&+\frac{1}{2}\alpha w_t w_{xx} w_{xtt} + \frac{1}{2}\alpha w_t w_{tt} w_{xxt} - \frac{1}{2}\alpha w_t^2 w_{xxtt}) \cdot f^{(3)} \\
&+(3 p v^2 w_t^2 + v_{xx} w_t^2 - v_x w_t w_{xx} - 2 v_t w_t w_{xt} - 2 v_x w_t w_{xt} \\
&+2 v w_{xt}^2 - 3 p v \alpha w_{xt}^2 + 2 v w_t w_{xxt} - v_t w_t w_{xx} \\
&+v w_{tt} w_{xx} + 2 v w_x w_{xtt} + \alpha w_{xtt} w_{xxt} - \alpha w_{xt} w_{xxtt}) \cdot f'' \\
&+(3 p v^2 w_{xt} + v_{xt} w_{xt} - v_x w_{xtt} - v_t w_{xxt} + v w_{xxtt}) \cdot f' \\
&+(v v_{xx} - v_t v_x + p v^3 + q) \cdot 1 = 0. \quad (9)
\end{aligned}$$

We can find that the coefficient of $f^{(6)}, f^{(5)}, f^{(4)}$ are just zeros. Then setting the coefficients of $f^{(3)}, f'', f', 1$ to zero yields a set of PDEs for $w = w(x,t)$ and $v = v(x,t)$:

$$\begin{aligned}
&-v_t w_t^3 - v_x w_t^3 + v w_t^2 w_{tt} + 4 v w_t^2 w_{xt} - 3 p v \alpha w_t^2 w_{xt} \\
&-\alpha w_{xt}^3 + \frac{1}{2} p\alpha^2 w_{xt}^3 + v w_t^2 w_{xx} - \frac{1}{2}\alpha w_{tt} w_{xx} w_{xt} \\
&+\frac{1}{2}\alpha w_t w_{xx} w_{xtt} + \frac{1}{2}\alpha w_t w_{tt} w_{xxt} - \frac{1}{2}\alpha w_t^2 w_{xxtt} = 0, \\
&3 p v^2 w_t^2 + v_{xx} w_t^2 - v_x w_t w_{xx} - 2 v_t w_t w_{xt} - 2 v_x w_t w_{xt} \\
&+2 v w_{xt}^2 - 3 p v \alpha w_{xt}^2 + 2 v w_t w_{xxt} - v_t w_t w_{xx} \\
&+v w_{tt} w_{xx} + 2 v w_x w_{xtt} + \alpha w_{xtt} w_{xxt} - \alpha w_{xt} w_{xxtt} = 0, \\
&3 p v^2 w_{xt} + v_{xt} w_{xt} - v_x w_{xtt} - v_t w_{xxt} + v w_{xxtt} = 0, \\
&v v_{xx} - v_t v_x + p v^3 + q = 0. \quad (10)
\end{aligned}$$

From (4) and (7), we can obtain the Bäcklund transformation of Eq.(3)

$$u = \frac{2}{p} \frac{\partial^2}{\partial x \partial t} \ln w + v. \quad (11)$$

where w satisfies (10) and v is given solution to Eq.(3).

3 Exact solutions for the E-DBM equation

In this section we use the Bäcklund transformation to obtain some exact solutions for E-DBM equation (3). If

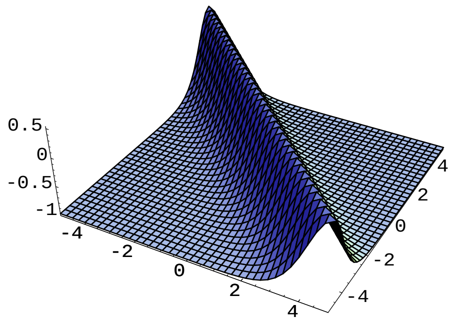

Fig 1(a)

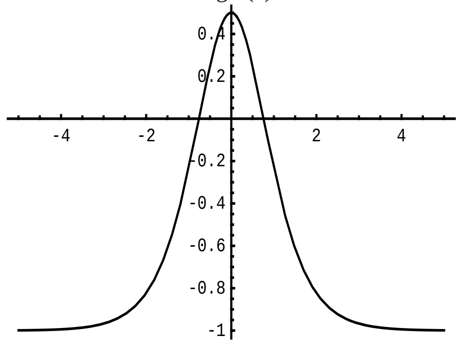

Fig 1(b)

Fig. 1: The Bell-shaped solitary waves with $c_0 = c_1 = p = q = \lambda = 1$

we take the given solution $v =$ Constant which is a special solution for eq.(3), then (10) will be reduced to

$$\begin{aligned} &vw_t^2 w_{tt} + 4vw_t^2 w_{xt} - 3pv\alpha w_t^2 w_{xt} - \alpha w_{xt}^3 + \frac{1}{2}p\alpha^2 w_{xt}^3 \\ &+vw_t^2 w_{xx} - \frac{1}{2}\alpha w_{tt} w_{xx} w_{xt} + \frac{1}{2}\alpha w_t w_{xx} w_{xtt} \\ &+\frac{1}{2}\alpha w_t w_{tt} w_{xxt} - \frac{1}{2}\alpha w_t^2 w_{xxtt} = 0, \\ &3pv^2 w_t^2 + 2vw_{xt}^2 - 3pv\alpha w_{xt}^2 + 2vw_t w_{xxt} + vw_{tt} w_{xx} \\ &+2vw_x w_{xtt} + \alpha w_{xtt} w_{xxt} - \alpha w_{xt} w_{xxtt} = 0, \\ &3pv^2 w_{xt} + vw_{xxtt} = 0, \\ &pv^3 + q = 0. \end{aligned} \quad (12)$$

Now we assume that the form of $w(x,t)$ is as follows:

$$w = c_0 + c_1 \exp k(x - \lambda t), \quad (13)$$

Where $c_0 \neq 0, c_1 \neq 0$ and k, λ are constants to be determined later. Substituting (13) into (12), we find that (12) satisfies (13) in the following conditions:

$$v = -\frac{q^{1/3}}{p^{1/3}}, \qquad k = \pm\frac{\sqrt{3}p^{1/3}q^{1/6}}{\sqrt{\lambda}}, \quad (14)$$

From (11), (13) and (14), which demands $\lambda > 0$, we can obtain some exact solutions, specially a new peakon solitary solution to Eq.(3) given by

$$u = -\frac{q^{1/3}}{p^{1/3}}\Big(1 + \frac{6c_1^2 \exp[-2|k(x+\lambda t)|]}{(c_0 + c_1 \exp[-|k(x+\lambda t)|])^2} - \frac{6c_1 \exp[-|k(x+\lambda t)|]}{c_0 + c_1 \exp[-|k(x+\lambda t)|]}\Big). \quad (15)$$

which has abundant solitary wave solutions as follows:

(i). When $c_0 = c_1, c_0c_1 > 0$, (15) describes the Bell-shaped solitary waves. It is shown in Fig 1(a) with $c_0 = c_1 = 1, p = q = 1, \lambda = 1$. And Fig 1(b) shows the graph in the plane with $t = 0$;

(ii). When $c_0 \neq c_1, c_0c_1 > 0$, (15) describes the double solitary waves with peakon [10]. It is shown in Fig 2(a) with $c_0 = 1, c_1 = 6, p = q = 1, \lambda = 0.5$. And Fig 2(b) shows the graph in the plane with $t = 0$;

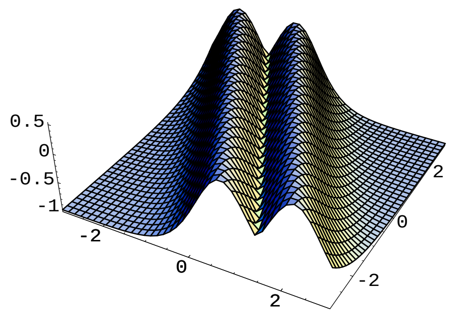

Fig 2(a)

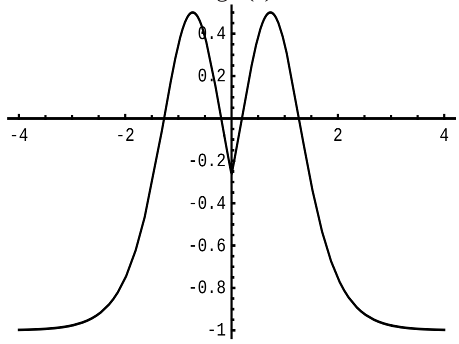

Fig 2(b)

Fig. 2: The peakon solitary waves with $c_0 = 1, c_1 = 6, p = q = 1, \lambda = 0.5$

(iii).When $|c_0| = |c_1|, c_0c_1 < 0$, (15) describes the singular solitary waves. It is shown in Fig 3(a) with $c_0 = -1, c_1 = 1, p = q = 1, \lambda = 1$.And Fig 3(b) shows the graph in the plane with $t = 0$;

(iv).When $|c_0| \neq |c_1|, c_0c_1 < 0$, (15) describes the double singular solitary waves with peakon [10]. It is shown in Fig 4(a) with $c_0 = -1, c_1 = 6, p = q = 1,$

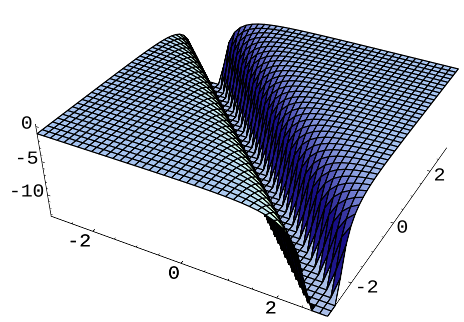

Fig 3(a)

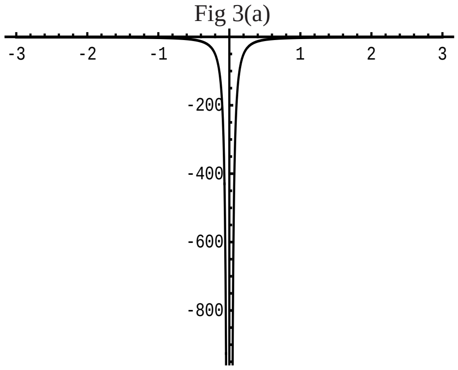

Fig 3(b)

Fig. 3: The singular solitary waves with $c_0 = -1, c_1 = 1, p = q = 1, \lambda = 1$

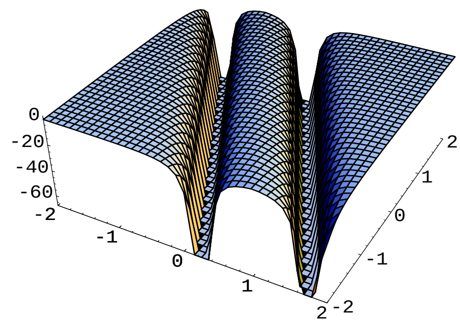

Fig 4(a)

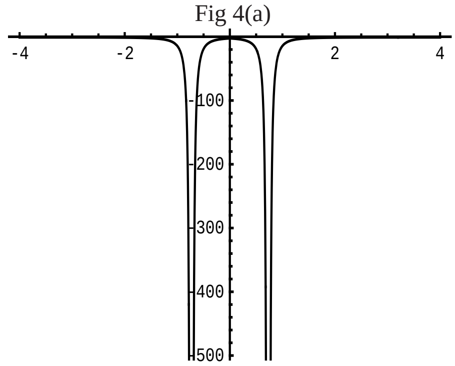

Fig 4(b)

Fig. 4: The peakon singular solitary waves with $c_0 = -1, c_1 = 6, p = q = 1, \lambda = 0.5$

$\lambda = 0.5$. And Fig 4(b) shows the graph in the plane with $t = 0$.

In the paper [1], the following exact solution of the E-DBM equation (3) (when $p = q = 1$) was obtained by the tanh method

$$u = \frac{1}{2}(1 - \tanh[\frac{1}{2}\sqrt{\frac{3}{\lambda}}(x + \lambda t)]), \tag{16}$$

In the paper [2], some following exact explicit traveling wave solutions for the E-DBM equation (3) were build using the extended tanh method

$$u_1 = -c_2\lambda sech^2[\frac{\sqrt{c_2}}{2}(x + \lambda t)], c_2 > 0,$$

$$u_2 = -c_2\lambda \sec^2[\frac{\sqrt{-c_2}}{2}(x + \lambda t)], c_2 < 0,$$

$$u_3 = -\frac{\lambda}{(x + \lambda t)^2},$$

$$u_4 = \wp(\frac{1}{2}\sqrt{-\frac{a}{\lambda}}(x + \lambda t), g_2, g_3). \tag{17}$$

where $\wp$ be Weierstrass elliptic function, $g_2 = 0, g_3 = 4c_0/c_1$ (see [2] p42).

To compare solution expression Eq.(15) with Eq.(16) and (17), we can find that these solutions of E-DBM equation are difference and Eq.(15) includes more abundant solutions.

4 conclusions

The HB method is a primary and concise method to seek for exact solutions of NPDEs [6]-[11].In fact, there are other powerful approaches to construct exact solutions such as the transformed rational function method [13] and F-expansion method [14]. In this paper we used the HB method to obtain Bäcklund transformation of E-DBM equation (3) and new exact solutions, including peakon solitary wave solutions and peakon singular solitary wave solutions. These solutions may be helpful to explain certain physical phenomena because a singular solitary solutions is a disjoint union of manifolds [15,16,17,18,19,20]. This method is algebraic rather than analytic. As well known, the use of different solving solution methods may get different exact solutions. So, in our future work we should consider other method to find more solutions of E-DBM equation and its N-peakon solitary wave solution if exists.

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References

[1] A.M. Wazwaz,Tanh method:solitons and periodic solutions for the Dodd-Bullough-Mikhailov and Tzitzeica-Dodd-Bullough equations. Chaos,Solitons and Fractals, **25**, 55-63 (2005).
[2] E.G. Fan, Integrable System and Computer Algebra, China. Sci. Publ, (2004).
[3] B.Tian, Y.T.Gao, Truncated Painleve expansion and a wide-ranging type of generalized variable-coefficient Kadomstsev-Petviashvili equations, Phys.Lett.A, **209**, 297-304 (1995).
[4] Z.Y.Yan, Painleve analysis,auto-Backlund transformations and exact solutions for a simplified model forreacting mixtures, Physica A, **326**, 344-359 (2003).
[5] D.S.Wang, H.Q.Zhang, Auto-Backlund transformation and new exact solutions of (2+1)-dimensional Nizhnik-Novikov-Veselov Equation, Int. J. Mod. Phys. C, **16**, 393-412 (2005).
[6] E.G.Fan, H.Q.Zhang, New exact solutions to a system of coupled KdV equations, Phys.Lett.A, **245**, 389-392 (1998).
[7] E.G.Fan,H.Q.Zhang, A note on the homogeneous balance method, Phys.Lett. A, **246**, 403-406 (1998).
[8] M.L.Wang, Application of homogeneous balance method to exact solutions of nonlinear equation in mathematical physics, Phys. Lett. A , **216**, 67-75 (1996).
[9] M.L.Wang, Y.M.Wang, A new Backlund transformation and multi-soliton solutions to the KdV model with general variable coefficients, Phys. Lett. A, **287**, 211-216 (2001).
[10] L.X.Tian, J.L.Yin, Multi-compacton and double symmetric peakon for generalized Ostrovsky equation, Chaos Solitons and Fractls, **35**, 991-995 (2008).
[11] K.Mohammed, Exact traveling wave solutions of the Boussinesq-Burgers equation, Math. Comput. Model, **49**, 666-671 (2009).
[12] W.X.Ma and B.Fuchssteiner, Explicit and exact solutions to a Kolmogorov-Petrovskii- Piskunov equation, International Journal of Non-Linear Mechanics, **31**, 329-338 (1996).
[13] W. X. Ma and J.-H. Lee, A transformed rational function method and exact solutions to the 3+1dimensional Jimbo-Miwa equation, Chaos,Solitons and Fractals, **42**, 1356-1363 (2009).
[14] D.S Wang and H.Q. Zhang, Further improved F-expansion method and new exact solutions of Konopelchenko-Dubrovsky equation, Chaos, Solitons and Fractals, **25**, 601–610 (2005).
[15] A.Nakamura, R.Hirota, A new example of explode-decay solitary waves in one-dimension. J. Phys. Soc. Jpn, **54**, 491-499 (1985).
[16] R.L.Sachs, On the integrable variant of the Boussinesq system: Painlev property, rational solutions, a related many-body system, and equivalence with the AKNS hierarchy, Physica. D, **30**, 1-27 (1988).
[17] H. Eleuch and R. Bennaceur An Optical Soliton Pair Among Absorbing Three-Level Atoms, J. Opt. A : Pure Appl. Opt., **5**, 528–533 (2003).
[18] N. Boutabba, H. Eleuch and H. Bouchriha, Thermal bath effect on soliton propagation in three-level atomic system, Synthetic Metals, **159**, 1239–1243 (2009).
[19] H. Eleuch, D. Elser and R. Bennaceur, Soliton propagation in an absorbing three-level atomic system, Laser Phys. Lett., **1**, 391–396 (2004).
[20] N. Boutabba and H. Eleuch, Soliton-pair Propagation under Thermal Bath Effect, MATHEMATICAL Modelling of Natural Phenomena, **7**, 32–37 (2012).

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