

Existence of Countably Many Positive Solutions for Nonlinear Boundary Value Problems on Time Scales

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Abstract: In this paper, we consider the existence of countably many positive solutions for nonlinear singular boundary value problem on time scales. By using the fixed-point index theory and a new fixed-point theorem in cones, the sufficient conditions for the existence of countably many positive solutions are established.

Keywords: Time scales, Boundary value problem, Singularity, Positive solutions, Fixed point theorem

1 Introduction

The theory of time scales, which has recently received a lot of attention, was introduced and developed by Aulbach and Hilger [11] in 1988. It has been created in order to unify continuous and discrete analysis, and it allows a simultaneous treatment of differential and difference equations, extending those theories to so-called dynamic equations. Further, the study of time scales has led to several important applications, e.g., in the study of insect population models, heat transfer, neural networks, phytoremediation of metals, wound healing, and epidemic models.

In this paper, we are interested in the existence of countably many positive solutions for singular multipoint boundary value problem on time scales,

$$(\phi(u^\Delta(t)))^\nabla + a(t)f(u(t)) = 0, \quad t \in [0, T]_{\mathbb{T}}, \quad (1)$$

subject to following boundary condition

$$u^\Delta(0) = 0, \quad u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i), \quad (2)$$

where $\phi : R \rightarrow R$ is an increasing homeomorphism and positive homomorphism and $\phi(0) = 0$. $\xi_i \in [0, T]_{\mathbb{T}}$ with $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < T$, and b_i satisfies $b_i \in [0, T]_{\mathbb{T}}$, $0 < \sum_{i=1}^{m-2} b_i < 1$, $a(t) : [0, T]_{\mathbb{T}} \rightarrow [0, +\infty]$ and has countably many singularities in $[0, T]_{\mathbb{T}}$. The usual notation and terminology for time scales as can be found in [2, 3], will be used here.

A projection $\phi : R \rightarrow R$ is called an increasing homeomorphism and positive homomorphism if the following conditions are satisfied:

- (i) if $x \leq y$, then $\phi(x) \leq \phi(y)$, $\forall x, y \in R$;
- (ii) ϕ is a continuous bijection and its inverse mapping is also continuous;
- (iii) $\phi(xy) = \phi(x)\phi(y)$, $\forall x, y \in R$.

If the above conditions hold, then it implies that ϕ is homogeneous and generates a p -Laplacian operator, i.e., $\phi(u) = |u|^{p-2}u$, for some $p > 1$.

In recent years, there is much attention focused on the existence of positive solutions of boundary value problems on time scales, some authors have found many results; for details, see [1, 5, 6, 7, 8, 9, 10, 13, 19, 20, 21, 22, 23, 24, 25] and the references therein. But for the existence of countable many positive solutions for boundary value problem on time scales, few works were done as far as we know [14, 16].

We would like to mention the results of Ma et al. [17], Liang and Zhang [15] and Ji et al. [12].

Ma et al. [17] studied the existence of monotone positive solutions for the BVP

$$(\phi_p(u'))' + q(t)f(t, u) = 0, \quad t \in (0, 1),$$

$$u'(0) = \sum_{i=1}^n \alpha_i u'(\xi_i), \quad u(1) = \sum_{i=1}^n \beta_i u(\xi_i),$$

where $\xi_i \in (0, 1)$ and $0 \leq \alpha_i, \beta_i < 1$ satisfy $0 \leq \sum_{i=1}^n \alpha_i, \sum_{i=1}^n \beta_i < 1$. The main tool is the monotone iterative technique.

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Liang and Zhang [15] considered the existence of countably many positive solutions for singular BVP

$$(\phi_p(u'))' + a(t)f(u(t)) = 0, \quad t \in (0, 1),$$

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad \phi(u'(1)) = \sum_{i=1}^{m-2} \beta_i \phi(u'(\xi_i)),$$

where $\phi : R \rightarrow R$ is a increasing homeomorphism and positive homomorphism and $\phi(0) = 0$, $\xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $\alpha_i, \beta_i \in [0, +\infty)$, $0 < \sum_{i=1}^{m-2} \alpha_i < 1$, $0 < \sum_{i=1}^{m-2} \beta_i < 1$. They showed that there exist countably many positive solutions by using the fixed-point index theory and a new fixed-point theorem in cones.

Ji et al. [12] found the existence of countably many positive solutions for a singular multipoint BVP

$$(\phi_p(u'))'(t) + a(t)f(u(t)) = 0, \quad t \in (0, 1),$$

$$u'(0) - \sum_{i=1}^{m-2} \alpha_i u(\xi_i) = 0, \quad u'(1) + \sum_{i=1}^{m-2} \alpha_i u(\eta_i) = 0,$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$ and $\frac{1}{p} + \frac{1}{q} = 1$. They provided sufficient conditions for the existence of countably many positive solutions by using fixed-point index theory and the Leggett-Williams' fixed point theorem.

However, to the best of our knowledge, no work has been done for BVP (1) and (2). The aim of this paper is to fill the gap in the relevant literature.

Throughout the paper, we will suppose that the following conditions are satisfied:

- (H1) $f : [0, +\infty) \rightarrow [0, +\infty)$ is continuous;
- (H2) There exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that $0 < t_{i+1} < t_i < \frac{T}{2}$, $\lim_{i \rightarrow \infty} t_i = t_0 < \frac{T}{2}$, and $t_0 \in [0, T]_{\mathbb{T}}$. $\lim_{t \rightarrow t_i} = \infty$, $i = 1, 2, \dots$, and

$$0 < \int_0^T a(s) \nabla s < +\infty.$$

Moreover, $a(t)$ does not vanish identically on any subinterval of $[0, T]_{\mathbb{T}}$.

The rest of paper is arranged as follows. In Section 2, we state some definitions, notations, lemmas and prove several preliminary results. Section 3 is devoted to the presentation and proof of our main results. In last section 4, we present an example of a family of functions $a(t)$ that satisfy condition (H2).

2 Preliminaries

In this section, we provide some background materials from theory of cones in Banach spaces.

Definition 2.1. Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is a cone if it satisfies the following two conditions:

- (i) $x \in P, \lambda \geq 0$ imply $\lambda x \in P$;
- (ii) $x \in P, -x \in P$ imply $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\alpha : P \rightarrow [0, \infty)$ is continuous, and

$$\alpha(tx + (1-t)y) \leq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Definition 2.3. Given a nonnegative continuous functional γ on a cone P of E , for each $d > 0$ we define the set

$$P(\gamma, d) = \{x \in P : \gamma(x) < d\}.$$

The following fixed point theorems will play an important role in the proofs of our main results.

Theorem 2.1. ([4]). Let E be a Banach space and $P \subset E$ be a cone in E . For $r > 0$, define $\Omega_r = \{u \in P : \|u\| < r\}$. Assume that $T : P \cap \overline{\Omega_r} \rightarrow P$ is a completely continuous operator such that $Tu \neq u$ for $u \in \partial\Omega_r$;

- (a) If $\|Tu\| \leq \|u\|$ for $u \in \partial\Omega_r$, then $i(T, \Omega_r, P) = 1$;
- (b) If $\|Tu\| \geq \|u\|$ for $u \in \partial\Omega_r$, then $i(T, \Omega_r, P) = 0$.

Theorem 2.2. ([18]). Let P be a cone in a Banach space E . Let α, β and γ be three increasing, nonnegative and continuous functionals on P , satisfying for some $c > 0$ and $M > 0$ such that

$$\gamma(u) \leq \beta(u) \leq \alpha(u), \quad \|u\| \leq M\gamma(u)$$

for all $u \in \overline{P(\gamma, c)}$. Suppose there exists a completely continuous operator $T : \overline{P(\gamma, c)} \rightarrow P$ and $0 < a < b < c$ such that

- (S1) $\gamma(Tu) < c$, for all $u \in \partial P(\gamma, c)$;
- (S2) $\beta(Tu) > b$, for all $u \in \partial P(\beta, b)$;
- (S3) $P(\alpha, a) \neq \emptyset$, and $\alpha(Tu) < a$, for all $u \in \partial P(\alpha, a)$.

Then T has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma, c)}$ such that

$$0 \leq \alpha(u_1) < a < \alpha(u_2), \quad \beta(u_2) < b < \beta(u_3), \quad \gamma(u_3) < c.$$

Lemma 2.1. If $\sum_{i=1}^{m-2} b_i \neq 1$, then for $h \in C_{ld}[0, T]_{\mathbb{T}}$ and $h \geq 0$,

$$\phi(u^\Delta(t))^\nabla + h(t) = 0, \quad t \in [0, T], \quad (3)$$

$$u^\Delta(0) = 0, \quad u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i) \quad (4)$$

has the unique solution

$$u(t) = \int_0^t \phi^{-1} \left(\int_s^T h(r) \nabla r + A \right) \Delta s + B, \quad (5)$$

where

$$A = - \int_0^T h(r) \nabla r,$$

$$B = \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left[\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1} \left(\int_s^T h(r) \nabla r + A \right) \Delta s - \int_0^T \phi^{-1} \left(\int_s^T h(r) \nabla r + A \right) \Delta s \right].$$

Proof. Let u be as in (5), taking the delta derivative of (5), we have

$$u^\Delta(t) = \phi^{-1} \left(\int_t^T h(r) \nabla r + A \right),$$

moreover, we get

$$\phi(u^\Delta(t)) = \int_t^T h(r) \nabla r + A,$$

taking the nabla derivative of this expression yields

$$(\phi(u^\Delta(t)))^\nabla = -h(t).$$

Routine calculations verify that u satisfies the boundary value conditions in (4), so that u given in (5) is a solution of (3) and (4). It is easy to see that BVP $(\phi(u^\Delta))^\nabla = 0$, $u^\Delta(0) = 0$, $u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i)$ has only the trivial solution. Thus u in (5) is the unique solution of (3) and (4). The proof is complete. \square

Lemma 2.2. The solution of BVP (3) and (4) satisfies $u(t) \geq 0$, for $t \in [0, T]_{\mathbb{T}}$.

Proof. Let

$$\varphi_0(s) = \phi^{-1} \left(\int_s^T h(r) \nabla r - \int_0^T h(r) \nabla r \right).$$

Since

$$\int_s^T h(r) \nabla r \leq \int_0^T h(r) \nabla r,$$

it follows that $\varphi_0(s) \leq 0$. According to Lemma 2.1, we get

$$\begin{aligned} u(0) &= B \\ &= \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s) \Delta s - \int_0^T \varphi_0(s) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &= \frac{\sum_{i=1}^{m-2} b_i \left(\int_0^T \varphi_0(s) \Delta s - \int_{\xi_i}^T \varphi_0(s) \Delta s \right) - \int_0^T \varphi_0(s) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &= - \left[\int_0^T \varphi_0(s) \Delta s + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T \varphi_0(s) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \right] \geq 0 \end{aligned}$$

and

$$\begin{aligned} u(T) &= \int_0^T \varphi_0(s) \Delta s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s) \Delta s - \int_0^T \varphi_0(s) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &= \frac{-\sum_{i=1}^{m-2} b_i \int_0^T \varphi_0(s) \Delta s + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &= \frac{- \left[\sum_{i=1}^{m-2} b_i \left(\int_0^T \varphi_0(s) \Delta s - \int_0^{\xi_i} \varphi_0(s) \Delta s \right) \right]}{1 - \sum_{i=1}^{m-2} b_i} \\ &= \frac{-\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T \varphi_0(s) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \geq 0. \end{aligned}$$

If $t \in (0, T)$, we have

$$\begin{aligned} u(t) &= \int_0^t \varphi_0(s) \Delta s + \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left[\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s) \Delta s - \int_0^T \varphi_0(s) \Delta s \right] \\ &= \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left[\int_0^t \varphi_0(s) \Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s) \Delta s + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s) \Delta s - \int_0^T \varphi_0(s) \Delta s \right] \\ &\geq \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left[\int_0^t \varphi_0(s) \Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s) \Delta s + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s) \Delta s - \int_0^T \varphi_0(s) \Delta s \right] \\ &= \frac{-\sum_{i=1}^{m-2} b_i \left(\int_0^T \varphi_0(s) \Delta s - \int_0^{\xi_i} \varphi_0(s) \Delta s \right)}{1 - \sum_{i=1}^{m-2} b_i} \\ &= \frac{-\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T \varphi_0(s) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \geq 0. \end{aligned}$$

So $u(t) \geq 0$, $t \in [0, T]$. The proof is complete.

Lemma 2.3. If $u \in P$, then

$$u(t) \geq \frac{\theta}{T} \|u\|, \quad t \in [\theta, T - \theta],$$

where

$$\|u\| = \sup_{t \in [0, T]_{\mathbb{T}}} |u(t)|.$$

Proof. Let

$$\tau = \inf \left\{ \xi \in [0, T] : \sup_{t \in [0, T]_{\mathbb{T}}} u(t) = u(\xi) \right\}.$$

Case (i). $\tau \in [0, \theta]$. It follows from the concavity of $u(t)$ that each point on chord between $(\tau, u(\tau))$ and $(T, u(T))$

is below the graph of $u(t)$. Thus,

$$u(t) \geq u(\tau) + \frac{u(T) - u(\tau)}{T - \tau}(t - \tau), \quad t \in [\theta, T - \theta].$$

Hence,

$$\begin{aligned} u(t) &\geq \min_{t \in [\theta, T - \theta]} \left[u(\tau) + \frac{u(T) - u(\tau)}{T - \tau}(t - \tau) \right] \\ &= u(\tau) + \frac{u(T) - u(\tau)}{T - \tau}(T - \theta - \tau) \\ &= \frac{T - \theta - \tau}{T - \tau}u(T) + \frac{\theta}{T - \tau}u(\tau) \geq \frac{\theta}{T}u(\tau), \end{aligned}$$

which implies that

$$u(t) \geq \frac{\theta}{T}\|u\|.$$

Case (ii). $\tau \in [\theta, T - \theta]$. If $t \in [\theta, \tau]$, similarly, we have

$$u(t) \geq u(\tau) + \frac{u(\tau) - u(0)}{\tau}(t - \tau), \quad t \in [\theta, \tau].$$

Thus,

$$\begin{aligned} u(t) &\geq \min_{t \in [\theta, \tau]} \left[u(\tau) + \frac{u(\tau) - u(0)}{\tau}(t - \tau) \right] \\ &= u(\tau) + \frac{u(\tau) - u(0)}{\tau}(\theta - \tau) \\ &= \frac{\theta}{\tau}u(\tau) + \left(1 - \frac{\theta}{\tau}\right)u(0) \geq \frac{\theta}{T}u(\tau). \end{aligned}$$

If $t \in [\tau, T - \theta]$, similarly,

$$u(t) \geq u(\tau) + \frac{u(T) - u(\tau)}{T - \tau}(t - \tau), \quad t \in [\tau, T - \theta].$$

Thus,

$$\begin{aligned} u(t) &\geq \min_{t \in [\tau, T - \theta]} \left[u(\tau) + \frac{u(T) - u(\tau)}{T - \tau}(t - \tau) \right] \\ &= u(\tau) + \frac{u(T) - u(\tau)}{T - \tau}(T - \theta - \tau) \\ &= \frac{\theta}{T - \tau}u(\tau) + \frac{T - \theta - \tau}{T - \tau}u(T) \geq \frac{\theta}{T}u(\tau). \end{aligned}$$

Therefore, we find

$$u(t) \geq \frac{\theta}{T}\|u\|, \quad t \in [\theta, T - \theta].$$

Case (iii). $\tau \in [T - \theta, T]$. Similarly, we have

$$u(t) \geq u(\tau) + \frac{u(T) - u(0)}{\tau}(t - \tau), \quad t \in [\theta, T - \theta].$$

Thus,

$$\begin{aligned} u(t) &\geq \min_{t \in [\theta, T - \theta]} \left[u(\tau) + \frac{u(\tau) - u(0)}{\tau}(t - \tau) \right] \\ &= u(\tau) + \frac{u(\tau) - u(0)}{\tau}(\theta - \tau) \\ &= \frac{\theta}{\tau}u(\tau) + \left(1 - \frac{\theta}{\tau}\right)u(0) \geq \frac{\theta}{T}u(\tau), \end{aligned}$$

which yields

$$u(t) \geq \frac{\theta}{T}\|u\|, \quad t \in [\theta, T - \theta].$$

This completes the proof.

3 Existence of positive solutions

Let the Banach space $E = C_{ld}([0, T]_{\mathbb{T}}, \mathbb{R})$ with norm $\|u\| = \sup_{[0, T]_{\mathbb{T}}} |u(t)|$ and define the cone $P \subset E$ by

$P = \{u \in E : u(t) \text{ is a nondecreasing concave and nonnegative function on } [0, T]_{\mathbb{T}}\}$.

Let $\theta_k < r_k < T - \theta_k$, we define the nonnegative, increasing, continuous functional γ_k, β_k , and α_k by

$$\gamma_k(u) = \max_{t \in [\theta_k, r_k]_{\mathbb{T}}} u(t) = u(r_k),$$

$$\beta_k(u) = \min_{t \in [r_k, T - \theta_k]_{\mathbb{T}}} u(t) = u(r_k),$$

$$\alpha_k(u) = \max_{t \in [\theta_k, T - \theta_k]_{\mathbb{T}}} u(t) = u(T - \theta_k).$$

It is easy to see that for each $u \in P$,

$$\gamma_k(u) \leq \beta_k(u) \leq \alpha_k(u).$$

Moreover, by Lemma 2.3, for each $u \in P$,

$$\gamma_k(u) = u(r_k) \geq \frac{\theta_k}{T}\|u\|.$$

Define the operator $T : P \rightarrow E$ by

$$\begin{aligned} (Tu)(t) &= \int_0^t \phi^{-1} \left(\int_s^T a(r)f(u(r))\nabla r \right. \\ &\quad \left. - \int_0^T a(r)f(u(r))\nabla r \right) \Delta s \\ &\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1} \left(\int_s^T a(r)f(u(r))\nabla r \right. \\ &\quad \left. - \int_0^T a(r)f(u(r))\nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\quad - \frac{\int_0^T a(r)f(u(r))\nabla r}{1 - \sum_{i=1}^{m-2} b_i} \Delta s \\ &\quad - \frac{\int_0^T \phi^{-1} \left(\int_s^T a(r)f(u(r))\nabla r \right. \\ &\quad \left. - \int_0^T a(r)f(u(r))\nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\quad - \frac{\int_0^T a(r)f(u(r))\nabla r}{1 - \sum_{i=1}^{m-2} b_i} \Delta s. \end{aligned}$$

It is easy to see that $(Tu)(t) \geq 0, (Tu)^\Delta(0) = 0, (Tu)(T) = \sum_{i=1}^{m-2} b_i(Tu)(\xi_i)$ and

$[\phi(Tu)^\Delta(t)]^\nabla = -a(t)f(u(t)) \leq 0$. This shows $TP \subset P$. One may show that $T : P \rightarrow P$ is completely continuous.

Lemma 3.1. Assume condition (H2) holds. Then there exists a constant $\theta \in \max\{t \in T : 0 < t < \frac{T}{2}\}$ satisfies

$$0 < \int_{\theta}^{T-\theta} a(s) \nabla s < +\infty.$$

Furthermore, the function

$$A(t) = \int_t^{T-t_1} \phi^{-1} \left(\int_s^{T-t_1} a(r) \nabla r - \int_0^{T-t_1} a(r) \nabla r \right) \Delta s + \frac{\sum_{i=1}^{m-2} b_i \int_{t_1}^t \phi^{-1} \left(\int_s^t a(r) \nabla r - \int_0^t a(r) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} - \frac{\int_{t_1}^t \phi^{-1} \left(\int_s^t a(r) \nabla r - \int_0^t a(r) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i}$$

is a positive continuous function on $[t_1, T - t_1]$ and has a minimum on $[t_1, T - t_1]$, therefore there exist $L > 0$ such that $A(t) \geq 0, t \in [t_1, T - t_1]$.

Proof. At first, it is easily seen that $A(t)$ is continuous on $[t_1, T - t_1]$. Next, let

$$A_1(t) = \int_t^{T-t_1} \phi^{-1} \left(\int_s^{T-t_1} a(r) \nabla r - \int_0^{T-t_1} a(r) \nabla r \right) \Delta s, A_2(t) = \frac{\sum_{i=1}^{m-2} b_i \int_{t_1}^t \phi^{-1} \left(\int_s^t a(r) \nabla r - \int_0^t a(r) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} - \frac{\int_{t_1}^t \phi^{-1} \left(\int_s^t a(r) \nabla r - \int_0^t a(r) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i}.$$

Then, from condition (H2), we have the function $A_1(t)$ is strictly monotone decreasing on $[t_1, T - t_1]$ and $A_1(T - t_1) = 0$, the function $A_2(t)$ is strictly monotone increasing on $[t_1, T - t_1]$ and $A_2(t_1) = 0$. Because $A_1(t)$ and $A_2(t)$ are not equal to zero at the same time. Therefore the function $A(t) = A_1(t) + A_2(t)$ is positive on $[t_1, T - t_1]$, which implies $L = \min_{t \in [t_1, T - t_1]} A(t) > 0$. The proof is complete. \square

For convenience, we denote by

$$\lambda_1 = 1/L, \lambda_2 = 1 / \left\{ \int_0^T \phi^{-1} \left(\int_0^T a(r) \nabla r \right) \Delta s + \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left[\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1} \left(\int_0^T a(r) \nabla r \right) \Delta s - \int_0^T \phi^{-1} \left(\int_s^T a(r) \nabla r \right) \Delta s \right] \right\}.$$

Theorem 3.1. Assume that (H1) and (H2) hold. Let $\{\theta_k\}_{k=1}^\infty$ be such that $\theta_k \in (t_{k+1}, t_k), (k = 1, 2, \dots)$. Let $\{r_k\}_{k=1}^\infty$ and $\{R_k\}_{k=1}^\infty$ be such that

$$R_{k+1} < \frac{\theta_k}{T} r_k < r_k < mr_k < R_k, mr_k \leq MR_k, k = 1, 2, \dots$$

Furthermore for each natural number k , we assume that f satisfies:

- (H3) $f(u) \geq \phi(mr_k)$, for all $u \in [\frac{\theta_k}{T} r_k, r_k]$,
- (H4) $f(u) \leq \phi(MR_k)$, for all $u \in [0, R_k]$,

where $m \in (\lambda_1, +\infty), M \in (0, \lambda_2)$. Then BVP (1) and (2) has infinitely many solutions $\{u_k\}_{k=1}^\infty$ such that

$$r_k \leq \|u_k\| \leq R_k, k = 1, 2, \dots$$

Proof. Because $0 < t_0 \leq t_{k+1} < \theta_k < t_k < T, k = 1, 2, \dots$, then for any $k \in N, u \in P$, by Lemma 2.3, we get

$$u(t) \geq \frac{\theta_k}{T} \|u\|, t \in [\theta_k, T - \theta_k]. \tag{6}$$

We consider the sequence $\{\Omega_{1,k}\}_{k=1}^\infty$ and $\{\Omega_{2,k}\}_{k=1}^\infty$ of open subsets of E defined by

$$\Omega_{1,k} = \{u \in P : \|u\| < r_k\}, k = 1, 2, \dots,$$

$$\Omega_{2,k} = \{u \in P : \|u\| < R_k\}, k = 1, 2, \dots$$

For a fixed k and $u \in \partial\Omega_{1,k}$, by (6) we have

$$r_k = \|u\| = \sup_{0 \leq t \leq T} |u(t)| \geq \sup_{\theta_k \leq t \leq T - \theta_k} u(t) \geq \frac{\theta_k}{T} \|u\| = \frac{\theta_k}{T} r_k, \text{ for all } t \in [\theta_k, T - \theta_k].$$

By condition (H3), we have

$$f(u) \geq \phi(mr_k), \text{ for all } t \in [\theta_k, T - \theta_k].$$

Since $(t_1, T - t_1) \subset [\theta_k, T - \theta_k]$, if (H2) holds, we consider three possibilities:

- (i) If $\xi_1 \in [t_1, T - t_1]$, then for $u \in \partial\Omega_{1,k}$, by (H3) and

Lemma 3.1, we have

$$\begin{aligned} \|Tu\| &= \int_0^T \phi^{-1} \left(\int_s^T a(r)f(u(r))\nabla r \right. \\ &\quad \left. - \int_0^T a(r)f(u(r))\nabla r \right) \Delta s \\ &\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1} \left(\int_s^T a(r)f(u(r))\nabla r \right. \\ &\quad \left. - \int_0^T a(r)f(u(r))\nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\quad - \frac{\int_0^T a(r)f(u(r))\nabla r}{1 - \sum_{i=1}^{m-2} b_i} \\ &\quad - \frac{\int_0^T \phi^{-1} \left(\int_s^T a(r)f(u(r))\nabla r \right. \\ &\quad \left. - \int_0^T a(r)f(u(r))\nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\geq \int_{\xi_1}^{T-t_1} \phi^{-1} \left(\int_s^{T-t_1} a(r)f(u(r))\nabla r \right. \\ &\quad \left. - \int_0^{T-t_1} a(r)f(u(r))\nabla r \right) \Delta s \\ &\quad + \frac{\sum_{i=1}^{m-2} b_i \int_{t_1}^t \phi^{-1} \left(\int_s^t a(r)f(u(r))\nabla r \right. \\ &\quad \left. - \int_0^t a(r)f(u(r))\nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\quad - \frac{\int_0^t a(r)f(u(r))\nabla r}{1 - \sum_{i=1}^{m-2} b_i} \\ &\quad - \frac{\int_{t_1}^t \phi^{-1} \left(\int_s^t a(r)f(u(r))\nabla r \right. \\ &\quad \left. - \int_0^t a(r)f(u(r))\nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\geq (mr_k) \left[\int_{\xi_1}^{T-t_1} \phi^{-1} \left(\int_s^{T-t_1} a(r)\nabla r \right. \right. \\ &\quad \left. \left. - \int_0^{T-t_1} a(r)\nabla r \right) \Delta s \right. \\ &\quad \left. + \frac{\sum_{i=1}^{m-2} b_i \int_{t_1}^{\xi_1} \phi^{-1} \left(\int_s^{\xi_1} a(r)\nabla r \right. \right. \right. \\ &\quad \left. \left. + \frac{\sum_{i=1}^{m-2} b_i \int_{t_1}^{\xi_1} \phi^{-1} \left(\int_s^{\xi_1} a(r)\nabla r \right. \right. \right. \end{aligned}$$

$$\begin{aligned} &\quad \left. \left. - \int_0^{\xi_1} a(r)\nabla r \right) \Delta s \right] \\ &= mr_k A(\xi_1) \geq mr_k L > r_k = \|u\|. \end{aligned}$$

(ii) If $\xi_1 \in (T - t_1, T)$, then for $u \in \partial\Omega_{1,k}$, by (H3) and Lemma 3.1, we have

$$\begin{aligned} \|Tu\| &\geq \frac{\sum_{i=1}^{m-2} b_i \int_{t_1}^{T-t_1} \phi^{-1} \left(\int_s^{T-t_1} a(r)f(u(r))\nabla r \right. \\ &\quad \left. - \int_0^{T-t_1} a(r)f(u(r))\nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\quad - \frac{\int_0^{T-t_1} a(r)f(u(r))\nabla r}{1 - \sum_{i=1}^{m-2} b_i} \\ &\quad - \frac{\int_{t_1}^{T-t_1} \phi^{-1} \left(\int_s^{T-t_1} a(r)f(u(r))\nabla r \right. \\ &\quad \left. - \int_0^{T-t_1} a(r)f(u(r))\nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\quad - \frac{\int_0^{T-t_1} a(r)f(u(r))\nabla r}{1 - \sum_{i=1}^{m-2} b_i} \\ &\geq mr_k A(T - t_1) > mr_k L > r_k = \|u\|. \end{aligned}$$

(iii) If $\xi_1 \in (0, t_1)$, then for $u \in \partial\Omega_{1,k}$, by (H3) and Lemma 3.1, we have

$$\begin{aligned} \|Tu\| &\geq \int_0^T \phi^{-1} \left(\int_s^T a(r)f(u(r))\nabla r \right. \\ &\quad \left. - \int_0^T a(r)f(u(r))\nabla r \right) \Delta s \\ &\geq \int_{t_1}^{T-t_1} \phi^{-1} \left(\int_s^{T-t_1} a(r)f(u(r))\nabla r \right. \\ &\quad \left. - \int_0^{T-t_1} a(r)f(u(r))\nabla r \right) \Delta s \\ &\geq mr_k \int_{t_1}^{T-t_1} \phi^{-1} \left(\int_s^{T-t_1} a(r)\nabla r \right. \\ &\quad \left. - \int_0^{T-t_1} a(r)\nabla r \right) \Delta s \\ &= mr_k A(t_1) \geq mr_k L > r_k = \|u\|. \end{aligned}$$

Thus, in all cases, an application of Theorem 2.1 implies that

$$i(T, \Omega_{1,k}, P) = 0. \tag{7}$$

On the other hand, let $u \in \partial\Omega_{2,k}$, we have $u(t) \leq \|u\| = R_k$, by (H4) we have

$$f(u(t)) \leq \phi(MR_k), \text{ for all } t \in [0, T].$$

Therefore

$$\begin{aligned} \|Tu\| &= \int_0^T \phi^{-1} \left(\int_s^T a(r) f(u(r)) \nabla r \right. \\ &\quad \left. - \int_0^T a(r) f(u(r)) \nabla r \right) \Delta s \\ &\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1} \left(\int_s^T a(r) f(u(r)) \nabla r \right. \\ &\quad \left. - \int_0^T a(r) f(u(r)) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\quad - \frac{\int_0^T a(r) f(u(r)) \nabla r}{1 - \sum_{i=1}^{m-2} b_i} \Delta s \\ &\quad - \frac{\int_0^T \phi^{-1} \left(\int_s^T a(r) f(u(r)) \nabla r \right. \\ &\quad \left. - \int_0^T a(r) f(u(r)) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\leq \int_0^T \phi^{-1} \left(\int_0^T a(r) f(u(r)) \nabla r \right) \Delta s \\ &\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1} \left(\int_0^T a(r) f(u(r)) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\quad - \frac{\int_0^T \phi^{-1} \left(\int_s^T a(r) f(u(r)) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\leq MR_k \left[\int_0^T \phi^{-1} \left(\int_0^T a(r) \nabla r \right) \Delta s \right. \\ &\quad \left. + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1} \left(\int_0^T a(r) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \right. \\ &\quad \left. - \frac{\int_0^T \phi^{-1} \left(\int_s^T a(r) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \right] \\ &\leq R_k = \|u\|. \end{aligned}$$

Hence Theorem 2.1 implies that

$$i(T, \Omega_{2,k}, P) = 1. \tag{8}$$

Because $r_k < R_k$ for $k \in N$, (7) and (8), it follows from the additivity of the fixed point index that

$$i(T, \Omega_{2,k} \setminus \overline{\Omega}_{1,k}, P) = 1, \text{ for } k \in N.$$

Thus, T has a fixed point in $\Omega_{2,k} \setminus \overline{\Omega}_{1,k}$ such that $r_k \leq \|u_k\| \leq R_k$. Since $k \in N$ was arbitrary, the proof is completed. \square

For notational convenience, we denote ρ_k and η_k by

$$\begin{aligned} \rho_k &= \int_0^T \phi^{-1} \left(\int_0^T a(r) \nabla r \right) \Delta s \\ &\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1} \left(\int_0^T a(r) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\quad - \frac{\int_0^T \phi^{-1} \left(\int_s^T a(r) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i}, \\ \eta_k &= \int_{\theta_k}^{r_k} \phi^{-1} \left(\int_s^{T-\theta_k} a(r) \nabla r - \int_0^{T-\theta_k} a(r) \nabla r \right) \Delta s. \end{aligned}$$

Theorem 3.2. Suppose that (H1) and (H2) hold, and let $\{\theta_k\}_{k=1}^\infty$ be such that $\theta_k \in (t_{k+1}, t_k)$, ($k = 1, 2, \dots$). Let $\{a_k\}_{k=1}^\infty$, $\{b_k\}_{k=1}^\infty$ and $\{c_k\}_{k=1}^\infty$ be such that

$$c_{k+1} < a_k < \frac{\theta_k}{T} b_k < b_k < c_k,$$

and

$$\rho_k b_k < \eta_k c_k, \text{ for } k = 1, 2, \dots$$

Furthermore for each natural number k we assume that f satisfies:

- (H5) $f(u) < \phi\left(\frac{c_k}{\rho_k}\right)$, for all $0 \leq u(t) \leq \frac{T}{\theta_k} c_k$;
- (H6) $f(u) > \phi\left(\frac{b_k}{\eta_k}\right)$, for all $b_k \leq u(t) \leq \frac{T}{\theta_k} b_k$;
- (H7) $f(u) < \phi\left(\frac{a_k}{\rho_k}\right)$, for all $0 \leq u(t) \leq \frac{T}{\theta_k} a_k$.

Then the BVP (1) and (2) has three infinite families of solutions $\{u_{1k}\}_{k=1}^\infty$, $\{u_{2k}\}_{k=1}^\infty$ and $\{u_{3k}\}_{k=1}^\infty$ satisfying

$$\begin{aligned} 0 &\leq \alpha_k(u_{1k}) < a_k < \alpha_k(u_{2k}), \\ \beta_k(u_{2k}) &< b_k < \beta_k(u_{3k}), \\ \gamma(u_{3k}) &< c_k, \text{ for } k \in N. \end{aligned}$$

Proof. By the definition of the completely continuous operator T , it is easy to check that $T : P(\gamma_k, c_k) \rightarrow P$, for $k \in N$.

We prove that all the conditions of Theorem 2.2 are satisfied. In order to make use of property (i) of Theorem 2.2, we chose $u \in \partial P(\gamma_k, c_k)$. Then $\gamma_k(u) = \max_{t \in [\theta_k, r_k]_{\mathbb{T}}} u(t) = u(r_k) = c_k$, this implies $0 \leq u(t) \leq c_k$ for $t \in [0, r_k]_{\mathbb{T}}$. If we recall that $\|u\| \leq \frac{T}{\theta_k} \gamma_k(u) = \frac{T}{\theta_k} c_k$. Therefore we get

$$0 \leq u(t) \leq \frac{T}{\gamma_k} c_k, \text{ } t \in [0, T]_{\mathbb{T}}.$$

Then assumption (H5) implies

$$f(u) < \phi\left(\frac{c_k}{\rho_k}\right), \text{ } t \in [0, T]_{\mathbb{T}}.$$

So

$$\begin{aligned}
 \gamma_k(Tu) &= \max_{t \in [\gamma_k, r_k]_{\mathbb{T}}} (Tu)(t) = (Tu)(r_k) \\
 &= \int_0^{r_k} \phi^{-1} \left(\int_s^T a(r)f(u(r))\nabla r \right. \\
 &\quad \left. - \int_0^T a(r)f(u(r))\nabla r \right) \Delta s \\
 &\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1} \left(\int_s^T a(r)f(u(r))\nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\
 &\quad - \frac{\int_0^T a(r)f(u(r))\nabla r \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\
 &\quad - \frac{\int_0^T \phi^{-1} \left(\int_s^T a(r)f(u(r))\nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\
 &\quad - \frac{\int_0^T a(r)f(u(r))\nabla r \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\
 &\leq \int_0^T \phi^{-1} \left(\int_0^T a(r)f(u(r))\nabla r \right) \Delta s \\
 &\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1} \left(\int_0^T a(r)f(u(r))\nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\
 &\quad - \frac{\int_0^T \phi^{-1} \left(\int_s^T a(r)f(u(r))\nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\
 &< \frac{c_k}{\rho_k} \left[\int_0^T \phi^{-1} \left(\int_0^T a(r)\nabla r \right) \Delta s \right. \\
 &\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1} \left(\int_0^T a(r)\nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\
 &\quad \left. - \frac{\int_0^T \phi^{-1} \left(\int_s^T a(r)\nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \right] \\
 &= c_k.
 \end{aligned}$$

Consequently, condition (i) is satisfied.

Secondly, we show that (ii) of Theorem 2.2 is fulfilled. For this we choose $u \in \partial P(\beta_k, b_k)$. Then $\beta_k(u) = \min_{t \in [r_k, T - \theta_k]_{\mathbb{T}}} u(t) = u(r_k) = b_k$, this means $u(t) \geq b_k$, for $t \in [r_k, T - \theta_k]_{\mathbb{T}}$. Therefore we have $\|u\| \geq b_k$, for $t \in [r_k, T - \theta_k]_{\mathbb{T}}$. Noticing that

$$\|u\| \leq \frac{T}{\theta_k} \gamma_k(u) \leq \frac{T}{\theta_k} \beta_k(u) = \frac{T}{\theta_k} b_k, \text{ we get}$$

$$b_k \leq u(t) \leq \frac{T}{\theta_k} b_k \text{ for } t \in [r_k, T - \theta_k]_{\mathbb{T}}.$$

By (H6) we get

$$f(u) > \phi\left(\frac{b_k}{\eta_k}\right), \text{ for } t \in [r_k, T - \theta_k]_{\mathbb{T}}.$$

So

$$\begin{aligned}
 \beta_k(Tu) &= \min_{t \in [r_k, T - \theta_k]_{\mathbb{T}}} (Tu)(t) = (Tu)(r_k) \\
 &= \int_0^{r_k} \phi^{-1} \left(\int_s^T a(r)f(u(r))\nabla r \right. \\
 &\quad \left. - \int_0^T a(r)f(u(r))\nabla r \right) \Delta s \\
 &\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1} \left(\int_s^T a(r)f(u(r))\nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\
 &\quad - \frac{\int_0^T a(r)f(u(r))\nabla r \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\
 &\quad - \frac{\int_0^T \phi^{-1} \left(\int_s^T a(r)f(u(r))\nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\
 &> \int_{\theta_k}^{r_k} \phi^{-1} \left(\int_s^{T-\theta_k} a(r)f(u(r))\nabla r \right. \\
 &\quad \left. - \int_0^{T-\theta_k} a(r)f(u(r))\nabla r \right) \Delta s \\
 &\geq \frac{b_k}{\eta_k} \int_{\theta_k}^{r_k} \phi^{-1} \left(\int_s^{T-\theta_k} a(r)\nabla r \right. \\
 &\quad \left. - \int_0^{T-\theta_k} a(r)\nabla r \right) \Delta s \\
 &= b_k.
 \end{aligned}$$

Thus, condition (ii) is satisfied.

Lastly we verify that (iii) of Theorem (2.2) is also satisfied. We note that $u(t) \equiv \frac{a_k}{4}$, $0 \leq t \leq T$ is a member of $P(\alpha_k, a_k)$ and $\alpha_k(u) = \frac{a_k}{4} < a_k$. Therefore $P(\alpha_k, a_k) \neq \emptyset$. Now let $u \in \partial P(\alpha_k, a_k)$. Then $\alpha_k(u) = \max_{t \in [\theta_k, T - \theta_k]_{\mathbb{T}}} u(t) = u(T - \theta_k) = a_k$. This implies that $0 \leq u(t) \leq a_k$ for $t \in [\theta_k, T - \theta_k]_{\mathbb{T}}$.

Together with $\|u\| \leq \frac{T}{\theta_k} \gamma_k(u) \leq \frac{T}{\theta_k} \alpha_k(u) = \frac{T}{\theta_k} a_k$. Then we have

$$0 \leq u(t) \leq \frac{T}{\theta_k} a_k, \quad t \in [0, T]_{\mathbb{T}}.$$

By (H7) we have

$$f(u) < \phi\left(\frac{a_k}{\rho_k}\right), \quad t \in [0, T]_{\mathbb{T}}.$$

Therefore

$$\begin{aligned} \alpha_k(Tu) &= \max_{t \in [\theta_k, T - \theta_k]_{\mathbb{T}}} (Tu)(t) = (Tu)(T - \theta_k) \\ &= \int_0^{T - \theta_k} \phi^{-1} \left(\int_s^T a(r) f(u(r)) \nabla r \right. \\ &\quad \left. - \int_0^T a(r) f(u(r)) \nabla r \right) \Delta s \\ &\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1} \left(\int_s^T a(r) f(u(r)) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\quad - \frac{\int_0^T a(r) f(u(r)) \nabla r \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\quad - \frac{\int_0^T \phi^{-1} \left(\int_s^T a(r) f(u(r)) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\quad - \frac{\int_0^T a(r) f(u(r)) \nabla r \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\leq \int_0^T \phi^{-1} \left(\int_0^T a(r) f(u(r)) \nabla r \right) \Delta s \\ &\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1} \left(\int_0^T a(r) f(u(r)) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\quad - \frac{\int_0^T \phi^{-1} \left(\int_s^T a(r) f(u(r)) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \\ &< \frac{a_k}{\rho_k} \left[\int_0^T \phi^{-1} \left(\int_0^T a(r) \nabla r \right) \Delta s \right. \\ &\quad \left. + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1} \left(\int_0^T a(r) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \right. \\ &\quad \left. - \frac{\int_0^T \phi^{-1} \left(\int_s^T a(r) \nabla r \right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i} \right] \\ &= a_k. \end{aligned}$$

Hence condition (iii) of Theorem 2.2 is satisfied. Because all hypotheses of Theorem 2.2 are satisfied, claim follows. \square

If we add the condition of $a(t)f(t, 0, 0) \neq 0, t \in [0, T]_{\mathbb{T}}$, to Theorem 3.2 we can get three infinite families of positive solutions $\{u_{1k}\}_{k=1}^{\infty}, \{u_{2k}\}_{k=1}^{\infty}$ and $\{u_{3k}\}_{k=1}^{\infty}$ satisfying

$$0 < \alpha_k(u_{1k}) < a_k < \alpha_k(u_{2k}),$$

$$\beta_k(u_{2k}) < b_k < \beta_k(u_{3k}),$$

$$\gamma(u_{3k}) < c_k, \text{ for } k \in N.$$

4 Example

There exists the function $a(t)$ satisfying condition (H2). Let $\mathbb{T} \equiv 1$ and $\delta = \sqrt{2}(\pi^2/3 - 9/4)$,

$$t_0 = \frac{5}{16}, \quad t_n = t_0 - \sum_{i=0}^{n-1} \frac{1}{(i+2)^4}, \quad n = 1, 2, \dots$$

We consider the function $a(t) : [0, 1] \rightarrow (0, +\infty)$ is given by $a(t) = \sum_{n=1}^{\infty} a_n(t), t \in [0, 1]$, where

$$a_n(t) = \begin{cases} \frac{2}{n(n+1)(t_{n+1}+t_n)}, & 0 \leq t < \frac{t_{n+1}+t_n}{2}, \\ \frac{1}{\delta(t_n-t)^{1/2}}, & \frac{t_{n+1}+t_n}{2} \leq t < t_n, \\ \frac{1}{\delta(t-t_n)^{1/2}}, & t_n < t \leq \frac{t_n+t_{n-1}}{2}, \\ \frac{2}{n(n+1)(2-t_n-t_{n-1})}, & \frac{t_n+t_{n-1}}{2} < t \leq 1. \end{cases}$$

At first, it is easily seen that $t_1 = 1/4 < 1/2, t_n - t_{n+1} = \frac{1}{(n+2)^4}, n = 1, 2, \dots$ and (note that $\sum_{i=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$),

$$\begin{aligned} t^* &= \lim_{n \rightarrow \infty} t_n = \frac{5}{16} - \sum_{i=0}^{\infty} \frac{1}{(i+2)^4} \\ &= \frac{5}{16} - \left(\frac{\pi^4}{90} - 1\right) = \frac{21}{16} - \frac{\pi^4}{90} > \frac{1}{5}. \end{aligned}$$

Next, since $\sum_{i=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \int_0^1 a_n(t) \nabla t &= \sum_{n=1}^{\infty} \frac{2}{n(n+1)} \\ &+ \frac{1}{\delta} \sum_{n=1}^{\infty} \left[\int_{(t_{n+1}+t_n)/2}^{t_n} \frac{1}{(t_n-t)^{1/2}} \nabla t \right. \\ &\left. + \int_{t_n}^{(t_n+t_{n-1})/2} \frac{1}{(t-t_n)^{1/2}} \nabla t \right] \\ &= 2 + \frac{\sqrt{2}}{\delta} \sum_{n=1}^{\infty} \left[(t_n-t_{n+1})^{1/2} \right. \\ &\left. + (t_{n-1}-t_n)^{1/2} \right] \\ &= 2 + \frac{\sqrt{2}}{\delta} \sum_{n=1}^{\infty} \left[\frac{1}{(n+2)^2} + \frac{1}{(n+1)^2} \right] \\ &= 2 + \frac{\sqrt{2}}{\delta} \left[\left(\frac{\pi^2}{6} - \frac{5}{4} \right) + \left(\frac{\pi^2}{6} - 1 \right) \right] \\ &= 2 + \frac{\sqrt{2}}{\delta} \left[\frac{\pi^2}{3} - \frac{9}{4} \right] = 3. \end{aligned}$$

Hence,

$$\int_0^1 a(t) \nabla t = \int_0^1 \sum_{n=1}^{\infty} a_n(t) \nabla t = \sum_{n=1}^{\infty} \int_0^1 a_n(t) \nabla t < \infty.$$

This implies that condition (H2) holds.

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