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Cyclic Contractions and Related Fixed Point Theorems on G-Metric Spaces

N. Bilgili[1,2], *İ. M. Erhan*[3,*], *E. Karapınar*[3] *and D. Türkoğlu*[1,2]

[1] Department of Mathematics, Faculty of Science, Gazi University, Ankara, Turkey
[2] Department of Mathematics, Faculty of Science and Arts, Amasya University, Amasya, Turkey
[3] Department of Mathematics, Faculty of Science and Arts, Atılım University, Ankara, Turkey

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Abstract: Very recently, Jleli and Samet [53] and Samet et. al. [52] reported that some fixed point result in G-metric spaces can be derived from the fixed point theorems in the setting of usual metric space. In this paper, we prove the existence and uniqueness of fixed points of certain cyclic mappings in the context of G-metric spaces that can not be obtained by usual fixed point results via techniques used in [53,52]. We also give an example to illustrate our statements.

Keywords: fixed point,G-metric space,cyclic maps,cyclic contractions

1 Introduction

The wide application potential of fixed point theory is the main motivation of research activities in this field. The theoretical studies are advancing in two main directions: one of them is related with the attempts to generalize the contractive conditions on the maps and thus, weaken them; the other with the attempts to generalize the space on which these contractions are defined. Among the results in the first direction one can mention cyclic contractions, almost contractions, non-expansive and expansive maps [3,4,28,29,30,31,32,33,34,37,50]. In the second direction some of the most extensively studied fields are the cone metric spaces, partial metric spaces and G-metric spaces [1,2,6,7,8,9,10,12,15,16,17,20,21,39,40]. There is also a rapidly growing interest in studies combining the two directions [11,13,18,19,22,23,24,25,26,27,36,51].

The concepts of G-metric spaces and cyclic contractions, more specifically, various types of cyclic contractions on G-metric spaces have been investigated in the past few years [25,36]. On the other hand, recently, Jleli and Samet [53] and Samet et. al. [52] proved that some fixed point result in G-metric spaces can be easily deduced from their analogs in usual metric spaces. However, this is not possible in general, that is, not all the results in G-metric spaces can be derived from those in usual metric spaces. In this paper, we prove the existence and uniqueness of fixed points of certain cyclic mappings in the context of G-metric spaces which cannot be obtained from usual fixed point results via the techniques used in [53,52]. We also improve some existing statements regarding these two topics. For the sake of completeness, we will state the basic definitions and crucial results that we need throughout the paper.

Cyclic maps have been first introduced by Kirk-Srinavasan-Veeramani [29] in 2003 together with the concept of best proximity points. The main advantage of cyclic maps is that they do not need be continuous. After this first article, best proximity theorems and, in particular, the fixed point theorems in the context of cyclic mapping have been studied extensively (see, e.g., [30,31,32,33,34,35,36,37,38,41,42,43,44,45].

Definition 1.*Let X be a nonempty set and let $Y = \cup_{j=1}^{m} A_j$ where $\{A_j\}_{j=1}^{m}$ is a family of nonempty subsets of X. A map $T : Y \to Y$ is called cyclic map if*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1, \dots m, \text{ where } A_{m+1} = A_1. \qquad (1)$$

The concept G-metric spaces introduced by Mustafa and Sims [6] is actually an improvement of the concept of D-metric spaces defined in [39],[40]. G-metrics and G-metric spaces have been thoroughly studied so far. Basic

* Corresponding author e-mail: ierhan@atilim.edu.tr

notions including the definition and properties of G-metric are listed below.

Definition 2.(See [6]). *Let X be a non-empty set, $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties :*
(G1) $G(x,y,z) = 0$ *if* $x = y = z$,
(G2) $0 < G(x,x,y)$ *for all* $x,y \in X$ *with* $x \neq y$,
(G3) $G(x,x,y) \leq G(x,y,z)$ *for all* $x,y,z \in X$ *with* $y \neq z$,
(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ *(symmetry in all three variables),*
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ *(rectangle inequality) for all* $x,y,z,a \in X$.

Then the function G is called a generalized metric, or a G-metric on X, and the pair (X,G) is called a G-metric space.

It can be easily shown that every G-metric on X induces a metric d_G on X defined by

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \text{ for all } x,y \in X. \tag{2}$$

The following trivial examples give a better idea about the notion of G-metrics:

*Example 1.*Let (X,d) be a metric space. The function $G : X \times X \times X \to [0,+\infty)$, defined by

$$G(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\},$$

for all $x,y,z \in X$, is a G-metric on X.

Example 2.(See e.g. [6]) Let $X = [0,\infty)$. The function $G : X \times X \times X \to [0,+\infty)$, defined by

$$G(x,y,z) = |x-y| + |y-z| + |z-x|,$$

for all $x,y,z \in X$, is a G-metric on X.

The following basic topological concepts on G-metric spaces have also been defined by Mustafa and Sims [6].

Definition 3.(See [6]). *Let (X,G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. The sequence $\{x_n\}$ is said to be G-convergent to $x \in X$ if*

$$\lim_{n,m\to+\infty} G(x,x_n,x_m) = 0,$$

that is, if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \varepsilon$, for all $n,m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n\to+\infty} x_n = x$.

Proposition 1.(See [6]). *Let (X,G) be a G-metric space. The following statements are equivalent:*
(1) $\{x_n\}$ *is G-convergent to x,*
(2) $G(x_n,x_n,x) \to 0$ *as* $n \to +\infty$,
(3) $G(x_n,x,x) \to 0$ *as* $n \to +\infty$,
(4) $G(x_n,x_m,x) \to 0$ *as* $n,m \to +\infty$.

Definition 4.(See [6]). *Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called a G-Cauchy sequence if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \varepsilon$ for all $m,n,l \geq N$, that is, $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to +\infty$.*

Proposition 2.(See [6]). *Let (X,G) be a G-metric space. The following statements are equivalent:*
(1) *the sequence $\{x_n\}$ is G-Cauchy,*
(2) *for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \varepsilon$, for all $m,n \geq N$.*

Definition 5.(See [6]). *A G-metric space (X,G) is called G-complete if every G-Cauchy sequence is G-convergent in (X,G).*

Definition 6.*Let (X,G) be a G-metric space. A mapping $F : X \times X \times X \to X$ is said to be continuous if for any three G-convergent sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converging to x, y and z respectively, $\{F(x_n,y_n,z_n)\}$ is G-convergent to $F(x,y,z)$.*

Every G-metric on X generates a topology τ_G on X with base a family of open G-balls $\{B_G(x,\varepsilon), x \in X, \varepsilon > 0\}$, where $B_G(x,\varepsilon) = \{y \in X, G(x,y,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. A non-empty set $A \subset X$ is G- closed in the G-metric space (X,G) if $\overline{A} = A$ where

$$x \in \overline{A} \Longleftrightarrow B_G(x,\varepsilon) \cap A \neq \emptyset,$$

for all $\varepsilon > 0$.

Proposition 3.*(See e.g. [36]) Let (X,G) be a G-metric space and A be a nonempty subset of X. The set A is G-closed if for any G-convergent sequence $\{x_n\}$ in A with limit x, we have $x \in A$.*

The celebrated Banach Contraction Principle (see [5]) in the context of G-metric spaces has been stated in [9] as follows:

Theorem 1.*(See [9]) Let (X,G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying the following condition for all $x,y,z \in X$:*

$$G(Tx,Ty,Tz) \leq kG(x,y,z), \tag{3}$$

where $k \in [0,1)$. Then T has a unique fixed point.

A particular case of Theorem 1 is given below.

Theorem 2.*(See [9]) Let (X,G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying the following condition for all $x,y \in X$:*

$$G(Tx,Ty,Ty) \leq kG(x,y,y), \tag{4}$$

where $k \in [0,1)$. Then T has a unique fixed point.

*Remark.*Observe that the condition (3) implies the condition (4). However, the converse is not true unless $k \in [0,\frac{1}{2})$ (see [10] for details).

Lemma 1.*[9] For a G-metric G defined on a set X, the following inequality holds*

$$G(x,y,y) = G(y,y,x) \leq G(y,x,x) + G(x,y,x) = 2G(y,x,x), \tag{5}$$

for all $x,y \in X$.

One of the attempts to improve the contractive condition on a map is the so-called weak ϕ- contraction introduced by Alber and Guerre-Delabriere [47].

A map $T: X \to X$ on a metric space (X,d) is called a weak ϕ-contraction if there exists a strictly increasing function $\phi: [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$d(Tx,Ty) \le d(x,y) - \phi(d(x,y)),$$

for all $x,y \in X$. It is worth to mention that these types of contractions have also been a subject of considerable interest(see e.g. [41,48,49,50]).

Some very recent results regarding cyclic maps on G-metric spaces are given in [36,54]. In [36], the authors discussed two types cyclic contractions: cyclic type Banach contractions and cyclic weak ϕ-contractions. Their main results are listed below.

Denote the set of continuous functions $\phi: [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$ by Ψ.

Theorem 3.*Let (X,G) be a G-complete G-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X with $Y = \cup_{j=1}^m A_j$. Let $T: Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1,\dots m, \text{ where } A_{m+1} = A_1. \tag{6}$$

Suppose that there exists a function $\phi \in \Psi$ such that the map T satisfies the inequality

$$G(Tx,Ty,Tz) \le M(x,y,z) - \phi(M(x,y,z)) \tag{7}$$

for all $x \in A_j$ and $y,z \in A_{j+1}$, $j = 1,\dots m$ where

$$M(x,y,z) = \max\{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz)\}. \tag{8}$$

Then T has a unique fixed point in $\cap_{j=1}^m A_j$.

As a particular case of Theorem 3, authors presented the following result [36].

Theorem 4.*(See [36]) Let (X,G) be a G-complete G-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X. Let $Y = \cup_{j=1}^m A_j$ and $T: Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1,\dots m, \text{ where } A_{m+1} = A_1. \tag{9}$$

If there exists $k \in (0,1)$ such that

$$G(Tx,Ty,Tz) \le kG(x,y,z) \tag{10}$$

holds for all $x \in A_j$ and $y,z \in A_{j+1}$, $j = 1,\dots m$ then, T has a unique fixed point in $\cap_{j=1}^m A_j$.

On the other hand, Bilgili and Karapınar [54] proved a more general version of the contractive condition given in [36]. We next give their result.

Theorem 5.*Let (X,G) be a G-complete G-metric spaces and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X with $Y = \bigcup_{j=1}^m A_j$. Let $T: Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1,2,...,m, \text{ where } A_{m+1} = A_1. \tag{11}$$

Suppose that there exist functions ϕ and ψ satisfying

$$\psi,\phi: [0,\infty] \to [0,\infty], \quad \psi(t) = \phi(t) = 0 \Longleftrightarrow t = 0,$$

ψ is continuous and nondecreasing, ϕ is lower semi-continuous for which the map T satisfies the inequality

$$\psi(G(Tx,Ty,Ty)) \le \psi(M(x,y,y)) - \phi(M(x,y,y)) \tag{12}$$

for all $x \in A_j$ and $y \in A_{j+1}$, $j = 1,2,...,m$ where

$$\begin{aligned} M(x,y,y) = \max\Big\{ & G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty), \\ & G(x,y,Tx), \tfrac{1}{3}[2G(x,Ty,Ty) + G(y,Tx,Tx)], \\ & \tfrac{1}{3}[G(x,Ty,Ty) + 2G(y,Tx,Tx)] \Big\}. \end{aligned} \tag{13}$$

Then T has a unique fixed point in $\cap_{j=1}^m A_j$.

The aim of this paper is to generalize the results regarding cyclic contractions on G-metric spaces reported so far.

2 Main Results

In our discussion we will need some sets of auxiliary functions which are defined below.

Let $\mathscr{F}$ denote all functions $f: [0,\infty) \to [0,\infty)$ such that $f(t) = 0$ if and only if $t = 0$. Let Ψ and Φ be the subsets of $\mathscr{F}$ such that

$$\Psi = \{\psi \in \mathscr{F} : \psi \text{ is continuous and nondecreasing}\},$$
$$\Phi = \{\phi \in \mathscr{F} : \phi \text{ is lower semi-continuous}\}.$$

The following results are needed in the proof of the main theorem.

Lemma 2.*Let (X,G) be a G-complete G-metric spaces and $\{x_n\}$ be a sequence in X such that $G(x_n,x_{n+1},x_{n+1})$ is nonincreasing and*

$$\lim_{n\to\infty} G(x_n,x_{n+1},x_{n+1}) = 0. \tag{14}$$

If $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon > 0$ and two sequences $\{n_k\}$ and $\{\ell_k\}$ of positive integers such that the following sequences have the same limit ε as $k \to \infty$:

$$\begin{aligned} &\{G(x_{\ell(k)},x_{n(k)},x_{n(k)})\},\ \{G(x_{n(k)},x_{\ell(k)},x_{\ell(k)})\}, \\ &\{G(x_{\ell(k)},x_{n(k)+1},x_{n(k)+1})\},\ \{G(x_{n(k)+1},x_{\ell(k)},x_{\ell(k)})\}, \\ &\{G(x_{\ell(k)-1},x_{n(k)},x_{n(k)})\},\ \{G(x_{n(k)},x_{\ell(k)-1},x_{\ell(k)-1})\}, \\ &\{G(x_{\ell(k)-1},x_{n(k)+1},x_{n(k)+1})\},\ \{G(x_{n(k)+1},x_{\ell(k)-1},x_{\ell(k)-1})\}, \\ &\{G(x_{n(k)+1},x_{\ell(k)+2},x_{\ell(k)+2})\},\ \{G(x_{\ell(k)+2},x_{n(k)+1},x_{n(k)+1})\}. \end{aligned} \tag{15}$$

*Proof.*Since $\{x_n\}$ is not G-Cauchy then, according to Proposition 2 there exists $\varepsilon > 0$ and subsequences $\{n(k)\}$ and $\{\ell(k)\}$ of $\mathbb{N}$ such that $n(k) \geq \ell(k) > k$ for which

$$G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) \geq \varepsilon \text{ and } G(x_{n(k)}, x_{\ell}(k), x_{\ell}(k)) \geq \varepsilon, \tag{16}$$

where $n(k) \geq \ell(k)$ are chosen as the smallest integers satisfying (16), that is,

$$\begin{aligned} &G(x_{\ell(k)}, x_{n(k)-1}, x_{n(k)-1}) < \varepsilon \text{ and} \\ &G(x_{n(k)}, x_{\ell(k)-1}, x_{\ell(k)-1}) < \varepsilon. \end{aligned} \tag{17}$$

From the rectangle inequality $(G5)$ and (16),(17) we have

$$\begin{aligned} \varepsilon \leq G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) &\leq G(x_{\ell(k)}, x_{n(k)-1}, x_{n(k)-1}) \\ &\quad + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) \\ &< \varepsilon + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}), \end{aligned} \tag{18}$$

and

$$\begin{aligned} \varepsilon \leq G(x_{n(k)}, x_{\ell(k)}, x_{\ell(k)}) &\leq G(x_{n(k)}, x_{\ell(k)-1}, x_{\ell(k)-1}) \\ &\quad + G(x_{\ell(k)-1}, x_{\ell(k)}, x_{\ell(k)}) \\ &< \varepsilon + G(x_{\ell(k)-1}, x_{\ell(k)}, x_{\ell(k)}). \end{aligned} \tag{19}$$

Letting $k \to \infty$ in (18) ,(19) and making use of (14) we get

$$\lim_{k\to\infty} G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon. \tag{20}$$

and

$$\lim_{k\to\infty} G(x_{n(k)}, x_{\ell(k)}, x_{\ell(k)}) = \varepsilon. \tag{21}$$

We next notice that

$$\begin{aligned} G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) &\leq G(x_{\ell(k)}, x_{n(k)+1}, x_{n(k)+1}) \\ &\quad + G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}), \end{aligned} \tag{22}$$

and

$$\begin{aligned} G(x_{\ell(k)}, x_{n(k)+1}, x_{n(k)+1}) &\leq G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) \\ &\quad + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}). \end{aligned} \tag{23}$$

Taking limit as $k \to \infty$ and using (14) and (20), we obtain

$$\lim_{k\to\infty} G(x_{\ell(k)}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{24}$$

By similar arguments we have

$$\begin{aligned} G(x_{n(k)}, x_{\ell(k)}, x_{\ell(k)}) &\leq G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) \\ &\quad + G(x_{n(k)+1}, x_{\ell(k)}, x_{\ell(k)}), \end{aligned} \tag{25}$$

and

$$\begin{aligned} G(x_{n(k)+1}, x_{\ell(k)}, x_{\ell(k)}) &\leq G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}) \\ &\quad + G(x_{n(k)}, x_{\ell(k)}, x_{\ell(k)}). \end{aligned} \tag{26}$$

Taking limit as $k \to \infty$ and using (14) and (21), we deduce

$$\lim_{k\to\infty} G(x_{n(k)+1}, x_{\ell(k)}, x_{\ell(k)}) = \varepsilon. \tag{27}$$

On the other hand, the inequalities

$$\begin{aligned} G(x_{\ell(k)-1}, x_{n(k)}, x_{n(k)}) &\leq G(x_{\ell(k)-1}, x_{\ell(k)}, x_{\ell(k)}) \\ &\quad + G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}), \end{aligned} \tag{28}$$

and

$$\begin{aligned} G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) &\leq G(x_{\ell(k)}, x_{\ell(k)-1}, x_{\ell(k)-1}) \\ &\quad + G(x_{\ell(k)-1}, x_{n(k)}, x_{n(k)}), \end{aligned} \tag{29}$$

will give

$$\lim_{k\to\infty} G(x_{\ell(k)-1}, x_{n(k)}, x_{n(k)}) = \varepsilon. \tag{30}$$

upon letting $k \to \infty$ and using (14) and (20). By using rectangle inequality again we observe that

$$\begin{aligned} G(x_{n(k)}, x_{\ell(k)-1}, x_{\ell(k)-1}) &\leq G(x_{n(k)}, x_{\ell(k)}, x_{\ell(k)}) \\ &\quad + G(x_{\ell(k)}, x_{\ell(k)-1}, x_{\ell(k)-1}), \end{aligned} \tag{31}$$

and

$$\begin{aligned} G(x_{n(k)}, x_{\ell(k)}, x_{\ell(k)}) &\leq G(x_{n(k)}, x_{\ell(k)-1}, x_{\ell(k)-1}) \\ &\quad + G(x_{\ell(k)-1}, x_{\ell(k)}, x_{\ell(k)}). \end{aligned} \tag{32}$$

Therefore

$$\lim_{k\to\infty} G(x_{n(k)}, x_{\ell(k)-1}, x_{\ell(k)-1}) = \varepsilon \tag{33}$$

follows from (14) and (21). Repeated application of $(G5)$ results in

$$\begin{aligned} G(x_{\ell(k-1)}, x_{n(k)+1}, x_{n(k)+1}) &\leq G(x_{\ell(k-1)}, x_{\ell(k)}, x_{\ell(k)}) \\ &\quad + G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) \\ &\quad + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}), \end{aligned} \tag{34}$$

and

$$\begin{aligned} G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) &\leq G(x_{\ell(k)}, x_{\ell(k-1)}, x_{\ell(k-1)}) \\ &\quad + G(x_{\ell(k-1)}, x_{n(k)+1}, x_{n(k)+1}) \\ &\quad + G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}). \end{aligned} \tag{35}$$

As $k \to \infty$ we have

$$\lim_{k\to\infty} G(x_{\ell(k)-1}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon \tag{36}$$

due to (14) and (20). Next, observe that

$$\begin{aligned} G(x_{n(k)+1)}, x_{\ell(k)-1}, x_{\ell(k)-1}) &\leq G(x_{n(k)+1)}, x_{n(k)}, x_{n(k)}) \\ &\quad + G(x_{n(k)}, x_{\ell(k)}, x_{\ell(k)}) \\ &\quad + G(x_{\ell(k)}, x_{\ell(k)-1}, x_{\ell(k)-1}), \end{aligned} \tag{37}$$

and

$$\begin{aligned} G(x_{n(k)}, x_{\ell(k)}, x_{\ell(k)}) &\leq G(x_{n(k)}, x_{n(k)+1)}, x_{n(k)+1)}) \\ &\quad + G(x_{n(k)+1)}, x_{\ell(k)+1}, x_{\ell(k)+1}) \\ &\quad + G(x_{\ell(k)+1}, x_{\ell(k)}, x_{\ell(k)}). \end{aligned} \tag{38}$$

Letting $k \to \infty$ and using (14) and (21), we obtain

$$\lim_{k\to\infty} G(x_{n(k)+1}, x_{\ell(k)-1}, x_{\ell(k)-1}) = \varepsilon. \tag{39}$$

Now we consider the inequalities

$$\begin{aligned} G(x_{n(k)+1},x_{\ell(k)-1},x_{\ell(k)-1}) \le\ & G(x_{n(k)+1},x_{\ell(k)+2},x_{\ell(k)+2}) \\ &+ G(x_{\ell(k)+2},x_{\ell(k)+1},x_{\ell(k)+1}) \\ &+ G(x_{\ell(k)+1},x_{\ell(k)},x_{\ell(k)}) \\ &+ G(x_{\ell(k)},x_{\ell(k)-1},x_{\ell(k)-1}), \end{aligned} \tag{40}$$

and

$$\begin{aligned} G(x_{n(k)+1},x_{\ell(k)+2},x_{\ell(k)+2}) \le\ & G(x_{n(k)+1},x_{\ell(k)-1},x_{\ell(k)-1}) \\ &+ G(x_{\ell(k-1)},x_{\ell(k)},x_{\ell(k)}) \\ &+ G(x_{\ell(k)},x_{\ell(k)+1},x_{\ell(k)+1}) \\ &+ G(x_{\ell(k)+1},x_{\ell(k)+1},x_{\ell(k)+2}). \end{aligned} \tag{41}$$

which together with (14) and (20) imply that

$$\lim_{k\to\infty} G(x_{n(k)+1},x_{\ell(k)+2},x_{\ell(k)+2}) = \varepsilon. \tag{42}$$

Finally, from

$$\begin{aligned} G(x_{\ell(k)-1},x_{n(k)+1},x_{n(k)+1}) \le\ & G(x_{\ell(k)-1},x_{\ell(k)},x_{\ell(k)}) \\ &+ G(x_{\ell(k)},x_{\ell(k)+1},x_{\ell(k)+1}) \\ &+ G(x_{\ell(k)+1},x_{\ell(k)+2},x_{\ell(k)+2}) \\ &+ G(x_{\ell(k)+2},x_{n(k)+1},x_{n(k)+1}), \end{aligned} \tag{43}$$

and

$$\begin{aligned} G(x_{\ell(k)+2},x_{n(k)+1},x_{n(k)+1}) \le\ & G(x_{\ell(k)+2},x_{\ell(k)+1},x_{\ell(k)+1}) \\ &+ G(x_{\ell(k)+1},x_{\ell(k)},x_{\ell(k)}) \\ &+ G(x_{\ell(k)},x_{\ell(k)-1},x_{\ell(k)-1}) \\ &+ G(x_{\ell(k)-1},x_{n(k)+1},x_{n(k)+1}), \end{aligned} \tag{44}$$

we conclude by using (14) and (21) that

$$\lim_{k\to\infty} G(x_{\ell(k)+2},x_{n(k)+1},x_{n(k)+1}) = \varepsilon. \tag{45}$$

This comlpetes the proof of the Lemma.

The main result is stated next.

Theorem 6.*Let (X,G) be a G-complete G-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X with $Y = \bigcup_{j=1}^m A_j$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1,2,\dots,m, \text{ where } A_{m+1} = A_1. \tag{46}$$

Suppose that there exist functions $\phi \in \Phi$ and $\psi \in \Psi$ such that the map T satisfies the inequality

$$\psi(G(Tx,T^2x,Ty)) \le \psi(M(x,y,y)) - \phi(M(x,y,y)) \tag{47}$$

for all $x \in A_j$ and $y \in A_{j+1}$, $j = 1,2,\dots,m$ where

$$\begin{aligned} M(x,y,y) = \max\Big\{ & G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty), \\ & G(x,Tx,y), \tfrac{1}{2}G(x,T^2x,Ty), \tfrac{1}{2}G(y,Ty,Tx), \\ & \tfrac{1}{2}G(y,T^2x,Ty), \tfrac{1}{2}[G(x,Ty,Ty)+G(y,Tx,Tx)], \\ & \tfrac{1}{2}[G(x,T^2x,Ty)+G(y,Tx,Tx)] \Big\}. \end{aligned} \tag{48}$$

Then T has a unique fixed point in $\bigcap_{j=1}^m A_j$.

*Proof.*First we consider the existence part. To show the existence of a fixed point of the map T we pick an arbitrary $x_0 \in A_1$ and construct the sequence $\{x_n\}$ as follows:

$$x_n = Tx_{n-1}, \quad n = 1,2,3,\cdots. \tag{49}$$

Since T is cyclic, we have $x_0 \in A_1, x_1 = Tx_0 \in A_2, x_2 = Tx_1 \in A_3, \cdots$. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}$, then clearly x_{n_0} is the fixed point of T. Assume that $x_{n+1} \ne x_n$ for all $n \in \mathbb{N}$. Set $x = x_n$ and $y = x_{n+1}$ in the inequality (47) to obtain

$$\begin{aligned} \psi(G(Tx_n,T^2x_n,Tx_{n+1})) &= \psi(G(x_{n+1},x_{n+2},x_{n+2})) \\ &\le \psi(M(x_n,x_{n+1},x_{n+1})) \\ &\quad - \phi(M(x_n,x_{n+1},x_{n+1})), \end{aligned} \tag{50}$$

where

$$\begin{aligned} & M(x_n,x_{n+1},x_{n+1}) = \max\{G(x_n,x_{n+1},x_{n+1}), G(x_n,Tx_n,Tx_n), \\ & G(x_{n+1},Tx_{n+1},Tx_{n+1}), G(x_n,Tx_n,x_{n+1}), \tfrac{1}{2}G(x_n,T^2x_n,Tx_{n+1}), \\ & \tfrac{1}{2}G(x_{n+1},Tx_{n+1},Tx_n), \tfrac{1}{2}G(x_{n+1},T^2x_n,Tx_{n+1}), \\ & \tfrac{1}{2}[G(x_n,Tx_{n+1},Tx_{n+1})+G(x_{n+1},Tx_n,Tx_n)], \\ & \tfrac{1}{2}[G(x_n,T^2x_n,Tx_{n+1})+G(x_{n+1},Tx_n,Tx_n)]\} \\[1em] & = \max\{G(x_n,x_{n+1},x_{n+1}), G(x_n,x_{n+1},x_{n+1}), \\ & G(x_{n+1},x_{n+2},x_{n+2}), G(x_n,x_{n+1},x_{n+1}), \tfrac{1}{2}G(x_n,x_{n+2},x_{n+2}), \\ & \tfrac{1}{2}G(x_{n+1},x_{n+2},x_{n+1}), \tfrac{1}{2}G(x_{n+1},x_{n+2},x_{n+2}), \\ & \tfrac{1}{2}[G(x_n,x_{n+2},x_{n+2})+G(x_{n+1},x_{n+1},x_{n+1})], \\ & \tfrac{1}{2}[G(x_n,x_{n+2},x_{n+2})+G(x_{n+1},x_{n+1},x_{n+1})]\} \\[1em] & = \max\{G(x_n,x_{n+1},x_{n+1}), G(x_{n+1},x_{n+2},x_{n+2})\}. \end{aligned} \tag{51}$$

If $M(x_n,x_{n+1},x_{n+1}) = G(x_{n+1},x_{n+2},x_{n+2})$, then (50) becomes

$$\begin{aligned} \psi(G(x_{n+1},x_{n+2},x_{n+2})) \le\ & \psi(G(x_{n+1},x_{n+2},x_{n+2})) \\ & - \phi(G(x_{n+1},x_{n+2},x_{n+2})). \end{aligned} \tag{52}$$

This yields $\phi(G(x_{n+1},x_{n+2},x_{n+2})) = 0$ and we conclude that

$$G(x_{n+1},x_{n+2},x_{n+2}) = 0,$$

which contradicts the assumption $x_n \ne x_{n+1}$ for all $n \in \mathbb{N}$. Hence, we should have

$$M(x_n,x_{n+1},x_{n+1}) = G(x_n,x_{n+1},x_{n+1}). \tag{53}$$

In this case the inequality (50) turns into

$$\begin{aligned} \psi(G(x_{n+1},x_{n+2},x_{n+2})) \le\ & \psi(G(x_n,x_{n+1},x_{n+1})) \\ & - \phi(G(x_n,x_{n+1},x_{n+1})) \\ \le\ & \psi(G(x_n,x_{n+1},x_{n+1})). \end{aligned} \tag{54}$$

Since $\psi \in \Psi$, then $\{G(x_n,x_{n+1},x_{n+1})\}$ is a nonnegative, non-increasing sequence that converges to some $L \ge 0$. To show that $L = 0$ we assume the contrary, that is, $L > 0$.

Taking $\limsup_{n\to+\infty}$ in (54) we obtain

$$\begin{aligned}&\limsup_{n\to+\infty}\psi(G(x_{n+1},x_{n+2},x_{n+2}))\\&\le\limsup_{n\to+\infty}\psi(G(x_n,x_{n+1},x_{n+1}))\\&-\liminf_{n\to+\infty}\phi(G(x_n,x_{n+1},x_{n+1}))\\&\le\limsup_{n\to+\infty}\psi(G(x_n,x_{n+1},x_{n+1})).\end{aligned}\tag{55}$$

Taking into account the continuity of ψ and the lower semi-continuity of ϕ we deduce

$$\psi(L)\le\psi(L)-\phi(L).\tag{56}$$

which implies $\phi(L)=0$, and hence, $L=0$. Thus,

$$\lim_{n\to\infty}G(x_n,x_{n+1},x_{n+1})=0.\tag{57}$$

Regarding Lemma 1 with $x=x_n$ and $y=x_{n-1}$ we note that

$$G(x_n,x_{n-1},x_{n-1})\le 2G(x_{n-1},x_n,x_n),\tag{58}$$

which gives

$$\lim_{n\to\infty}G(x_n,x_{n-1},x_{n-1})=0.\tag{59}$$

Next, we shall show that $\{x_n\}$ is a G-Cauchy sequence in (X,G). Assume that $\{x_n\}$ is not G-Cauchy. Then, by the Proposition 2 there exists $\varepsilon>0$ and corresponding subsequences $\{n(k)\}$ and $\{\ell(k)\}$ of $\mathbb{N}$ satisfying $n(k)>\ell(k)>k$ for which

$$G(x_{\ell(k)},x_{n(k)},x_{n(k)})\ge\varepsilon,\tag{60}$$

where $n(k)$ is chosen as the smallest integer satisfying (60), that is,

$$G(x_{\ell(k)},x_{n(k)-1},x_{n(k)-1})<\varepsilon.\tag{61}$$

By (60),(61) and the rectangle inequality $(G5)$, it is easy to see that

$$\begin{aligned}\varepsilon&\le G(x_{\ell(k)},x_{n(k)},x_{n(k)})\\&\le G(x_{\ell(k)},x_{n(k)-1},x_{n(k)-1})+G(x_{n(k)-1},x_{n(k)},x_{n(k)})\\&<\varepsilon+G(x_{n(k)-1},x_{n(k)},x_{n(k)}).\end{aligned}\tag{62}$$

Letting $k\to\infty$ in (62) and using (57) we get

$$\lim_{k\to\infty}G(x_{\ell(k)},x_{n(k)},x_{n(k)})=\varepsilon.\tag{63}$$

Observe that for every $k\in\mathbb{N}$ we can find $s(k)$ satisfying $0\le s(k)\le m$ such that

$$n(k)-\ell(k)+s(k)\equiv 1(m).\tag{64}$$

Then, for large enough values of k we have $r(k)=\ell(k)-s(k)>0$ and $x_{r(k)}$ and $x_{n(k)}$ lie in the adjacent sets A_j and A_{j+1} respectively for some $0\le j\le m$. If we set $x=x_{r(k)}$ and $y=x_{n(k)}$ in (47), we obtain

$$\begin{aligned}\psi(G(Tx_{r(k)},T^2x_{r(k)},Tx_{n(k)}))\le{}&\psi(M(x_{r(k)},x_{n(k)},x_{n(k)}))\\&-\phi(M(x_{r(k)},x_{n(k)},x_{n(k)})),\end{aligned}\tag{65}$$

where

$$\begin{aligned}&M(x_{r(k)},x_{n(k)},x_{n(k)})=\max\{G(x_{r(k)},x_{n(k)},x_{n(k)}),\\&G(x_{r(k)},Tx_{r(k)},Tx_{r(k)}),G(x_{n(k)},Tx_{n(k)},Tx_{n(k)}),\\&G(x_{r(k)},Tx_{r(k)},x_{n(k)}),\tfrac{1}{2}G(x_{r(k)},T^2x_{r(k)},Tx_{n(k)}),\\&\tfrac{1}{2}G(x_{n(k)},Tx_{n(k)},Tx_{r(k)}),\tfrac{1}{2}G(x_{n(k)},T^2x_{r(k)},Tx_{n(k)}),\\&\tfrac{1}{2}[G(x_{r(k)},Tx_{n(k)},Tx_{n(k)})+G(x_{n(k)},Tx_{r(k)},Tx_{r(k)})],\\&\tfrac{1}{2}[G(x_{r(k)},T^2x_{r(k)},Tx_{n(k)})+G(x_{n(k)},Tx_{r(k)},Tx_{r(k)})]\}\\[1ex]&=\max\{G(x_{r(k)},x_{n(k)},x_{n(k)}),G(x_{r(k)},x_{r(k)+1},x_{r(k)+1}),\\&G(x_{n(k)},x_{n(k)+1},x_{n(k)+1}),G(x_{r(k)},x_{r(k)+1},x_{n(k)}),\\&\tfrac{1}{2}G(x_{r(k)},x_{r(k)+2},x_{n(k)+1}),\\&\tfrac{1}{2}G(x_{n(k)},x_{n(k)+1},x_{r(k)+1}),\tfrac{1}{2}G(x_{n(k)},x_{r(k)+2},x_{n(k)+1}),\\&\tfrac{1}{2}[G(x_{r(k)},x_{n(k)+1},x_{n(k)+1})+G(x_{n(k)},x_{r(k)+1},x_{r(k)+1})],\\&\tfrac{1}{2}[G(x_{r(k)},x_{r(k)+2},x_{n(k)+1})+G(x_{n(k)},x_{r(k)+1},x_{r(k)+1})]\}.\end{aligned}\tag{66}$$

Repeated application of the rectangle inequality $(G5)$ results in

$$\lim_{k\to\infty}G(x_{r(k)},x_{n(k)},x_{n(k)})=\varepsilon.\tag{67}$$

On the other hand,

$$\begin{aligned}\lim_{k\to\infty}G(x_{r(k)},x_{r(k)+1},x_{r(k)+1})&=0,\\\lim_{k\to\infty}G(x_{n(k)},x_{n(k)+1},x_{n(k)+1})&=0.\end{aligned}\tag{68}$$

In addition, from the rectangle inequality $(G5)$ it follows that

$$\begin{aligned}
G(x_{r(k)},x_{r(k)+1},x_{n(k)}) &= G(x_{n(k)},x_{r(k)},x_{r(k)+1}) \\
&\leq G(x_{n(k)},x_{r(k)},x_{r(k)}) \\
&+ G(x_{r(k)},x_{r(k)},x_{r(k)+1}),
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}G(x_{r(k)},x_{r(k)+2},x_{n(k)+1}) &\leq \tfrac{1}{2}[G(x_{r(k)},x_{r(k)+1},x_{r(k)+1}) \\
&+ G(x_{r(k)+1},x_{r(k)+2},x_{r(k)+2}) \\
&+ G(x_{r(k)+2},x_{r(k)+2},x_{n(k)+1})],
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}G(x_{n(k)},x_{n(k)+1},x_{r(k)+1}) &= \tfrac{1}{2}G(x_{r(k)+1},x_{n(k)+1},x_{n(k)}) \\
&\leq \tfrac{1}{2}[G(x_{r(k)+1},x_{r(k)},x_{r(k)}) \\
&+ G(x_{r(k)},x_{n(k)+1},x_{n(k)+1}) \\
&+ G(x_{n(k)+1},x_{n(k)+1},x_{n(k)})],
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}G(x_{n(k)},x_{r(k)+2},x_{n(k)+1}) &\leq \tfrac{1}{2}[G(x_{n(k)},x_{r(k)},x_{r(k)}) \\
&+ G(x_{r(k)},x_{r(k)+1},x_{r(k)+1}) \\
&+ G(x_{r(k)+1},x_{r(k)+2},x_{r(k)+2}) \\
&+ G(x_{r(k)+2},x_{r(k)+2},x_{n(k)+1})],
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}[G(x_{r(k)},x_{n(k)+1},x_{n(k)+1}) &+ G(x_{n(k)},x_{r(k)+1},x_{r(k)+1})] \\
&\leq \tfrac{1}{2}[G(x_{r(k)},x_{n(k)+1},x_{n(k)+1}) \\
&+ G(x_{n(k)},x_{r(k)},x_{r(k)}) \\
&+ G(x_{r(k)},x_{r(k)+1},x_{r(k)+1})],
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}[G(x_{r(k)},x_{r(k)+2},x_{n(k)+1}) &+ G(x_{n(k)},x_{r(k)+1},x_{r(k)+1})] \\
&\leq \tfrac{1}{2}[G(x_{r(k)},x_{r(k)+1},x_{r(k)+1}) \\
&+ G(x_{r(k)+1},x_{r(k)+2},x_{r(k)+2}) \\
&+ G(x_{r(k)+2},x_{r(k)+2},x_{n(k)+1}) \\
&+ G(x_{n(k)},x_{r(k)},x_{r(k)}) \\
&+ G(x_{r(k)},x_{r(k)+1},x_{r(k)+1})].
\end{aligned} \tag{69}$$

Passing to the limit as $k\to\infty$ in (69) and using Lemma 2, as well as (67) and (68), we end up with $\lim_{k\to\infty} M(x_{r(k)},x_{n(k)},x_{n(k)}) = \varepsilon$. Thus we have

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon), \tag{70}$$

which implies $\phi(\varepsilon)=0$. We conclude that $\varepsilon=0$. However, this contradicts the assumption that $\{x_n\}$ is not G-Cauchy. Hence, the sequence $\{x_n\}$ is G-Cauchy. Since (X,G) is G-complete, it is G-convergent to a limit, say $w\in X$. It easy to see that $w\in\bigcap_{j=1}^{m} A_j$. Indeed, if $x_0\in A_1$, then the subsequence $\{x_{m(n-1)}\}_{n=1}^{\infty}\in A_1$, the subsequence $\{x_{m(n-1)+1}\}_{n=1}^{\infty}\in A_2$ and continuing in this way, the subsequence $\{x_{mn-1)}\}_{n=1}^{\infty}\in A_m$. All the above subsequences are G-convergent in the G-closed sets A_j and hence, they all converge to the same limit $w\in\bigcap_{j=1}^{m} A_j$. We next show that the limit w is the fixed point of T, that is, $w=Tw$. We employ again (47) with $x=x_n, y=w$. This results in

$$\psi(G(Tx_n,T^2x_n,Tw)) \leq \psi(M(x_n,w,w)) - \phi(M(x_n,w,w)), \tag{71}$$

where

$$\begin{aligned}
&M(x_n,w,w)= \\
&\max\{G(x_n,w,w),G(x_n,Tx_n,Tx_n),G(w,Tw,Tw), \\
&G(x_n,Tx_n,w),\tfrac{1}{2}G(x_n,T^2x_n,Tw), \\
&\tfrac{1}{2}G(w,Tw,Tx_n),\tfrac{1}{2}G(w,T^2x_n,Tw), \\
&\tfrac{1}{2}[G(x_n,Tw,Tw)+G(w,Tx_n,Tx_n)], \\
&\tfrac{1}{2}[G(x_n,T^2x_n,Tw)+G(w,Tx_n,Tx_n)]\}
\end{aligned} \tag{72}$$

Using Lemma 1 and taking lim sup as $n\to\infty$, we get

$$\psi(G(w,Tw,Tw)) \leq \psi(G(w,Tw,Tw)) - \phi(G(w,Tw,Tw)). \tag{73}$$

Then $\phi((G(w,Tw,Tw)))=0$ and hence, $G(w,Tw,Tw)=0$, that is, $w=Tw$.

We prove next the uniqueness part of the theorem. Let $v\in X$ be another fixed point of T such that $v\neq w$. Then, both v and w belong to $\bigcap_{j=1}^{m} A_j$. Setting $x=v$ and $y=w$ in (47) results in

$$\psi(G(Tv,T^2v,Tw)) \leq \psi(M(v,w,w)) - \phi(M(v,w,w)), \tag{74}$$

where

$$\begin{aligned}
&M(v,w,w)= \\
&\max\{G(v,w,w),G(v,Tv,Tv),G(w,Tw,Tw),G(v,Tv,w), \\
&\tfrac{1}{2}G(v,T^2v,Tw),\tfrac{1}{2}G(w,Tw,Tv),\tfrac{1}{2}G(w,T^2v,Tw), \\
&\tfrac{1}{2}[G(v,Tw,Tw)+G(w,Tv,Tv)], \\
&\tfrac{1}{2}[G(v,T^2v,Tw)+G(w,Tv,Tv)]\}
\end{aligned} \tag{75}$$

On the other hand, setting $x=w$ and $y=v$ in (47) gives

$$\psi(G(Tw,T^2w,Tv)) \leq \psi(M(w,v,v)) - \phi(M(w,v,v)), \tag{76}$$

where

$$\begin{aligned}
&M(w,v,v)= \\
&\max\{G(w,v,v),G(w,Tw,Tw),G(v,Tv,Tv),G(w,Tw,v), \\
&\tfrac{1}{2}G(w,T^2w,Tv),\tfrac{1}{2}G(v,Tv,Tw),\tfrac{1}{2}G(v,T^2w,Tv), \\
&\tfrac{1}{2}[G(w,Tv,Tv)+G(v,Tw,Tw)], \\
&\tfrac{1}{2}[G(w,T^2w,Tv)+G(v,Tw,Tw)]\}
\end{aligned} \tag{77}$$

Now, if $G(v,w,w)=G(w,v,v)$ then $v=w$. Note that the Definition 2 of the induced metric implies $d_G(v,w)=0$ and hence, $v=w$. If $G(v,w,w)>G(w,v,v)$ then by (75) $M(v,w,w)=G(v,w,w)$ and regarding (74) we get

$$\psi(G(v,w,w)) \leq \psi(G(v,w,w)) - \phi((G(v,w,w))), \tag{78}$$

so that $G(v,w,w)=0$. We conclude that $v=w$. On the other hand, if $G(w,v,v)>G(v,w,w)$ then by (77) $M(w,v,v)=G(w,v,v)$ and by (76),

$$\psi(G(w,v,v)) \leq \psi(G(w,v,v)) - \phi(G(w,v,v)), \tag{79}$$

from which we deduce $G(w,v,v)=0$. Hence, $v=w$. Therefore, the fixed point of T is unique.

The following example helps to illustrate Theorem 6.

Example 3. Let $X = [-1,1]$ and the function $G : X \times X \times X \to [0,\infty)$ defined as

$$G(x,y,z) = |x-y| + |y-z| + |z-x|, \tag{80}$$

be a G-metric on X. Let $T : X \to X$ be given as $Tx = \frac{-x}{8}$. Let $A = [-1,0]$ and $B = [0,1]$. Define also $\phi : [0,\infty) \to [0,\infty)$ as $\phi(t) = \frac{t}{8}$ and $\psi : [0,\infty) \to [0,\infty)$ as $\psi(t) = \frac{t}{2}$. Clearly, the map T has a unique fixed point $x = 0 \in A \cap B$. It is easy to see that the map T satisfies the condition (47). Indeed,

$$\begin{aligned} G(Tx,T^2x,Ty) &= |Tx - T^2x| + |T^2x - Ty| + |Ty - Tx| \\ &= \left|\frac{-x}{8} - \frac{-x}{64}\right| + \left|\frac{x}{64} + \frac{y}{8}\right| + \left|\frac{-y}{8} + \frac{x}{8}\right| \\ &= \frac{9|x| + |x+8y| + 8|x-y|}{64}, \end{aligned} \tag{81}$$

which yields

$$\psi(G(Tx,T^2x,Ty)) = \frac{9|x| + |x+8y| + 8|x-y|}{128}. \tag{82}$$

Note also that

$$\begin{aligned} &M(x,y,y) = \max\{|x-y| + |y-y| + |y-x|, \\ &|x-Tx| + |Tx-Tx| + |Tx-x|, |y-Ty| + |Ty-Ty| \\ &+|Ty-y|, |x-Tx| + |Tx-y| + |y-x|, \\ &\frac{1}{2}[|x-T^2x| + |T^2x-Ty| + |Ty-x|], \\ &\frac{1}{2}[|y-Ty| + |Ty-Tx| + |Tx-y|], \\ &\frac{1}{2}[|y-T^2x| + |T^2x-Ty| + |Ty-y|], \\ &\frac{1}{2}[|x-Ty| + |Ty-Ty| + |Ty-x| \\ &+|y-Tx| + |Tx-Tx| + |Tx-y|], \\ &\frac{1}{2}[|x-T^2x| + |T^2x-Ty| + |Ty-x| \\ &+|y-Tx| + |Tx-Tx| + |Tx-y|]\} \\ &= \max\{2|x-y|, \frac{9|x|}{4}, \frac{9|y|}{4}, \frac{9|x|}{8} + \frac{|-x-8y|}{8} + |y-x|, \\ &\frac{1}{2}\left[\frac{63|x|}{64} + \frac{|8y+x|}{64} + \frac{|-8x-y|}{8}\right], \\ &\frac{1}{2}\left[\frac{9|y|}{8} + \frac{|x-y|}{8} + \frac{|-8y-x|}{8}\right], \\ &\frac{1}{2}\left[\frac{|64y-x|}{64} + \frac{|8y+x|}{64} + \frac{9|y|}{8}\right], \left[\frac{|8x+y|}{8} + \frac{|8y+x|}{8}\right], \\ &\frac{1}{2}\left[\frac{|63x|}{64} + \frac{17|8y+x|}{64} + \frac{|-8x-y|}{8}\right]\}. \end{aligned} \tag{83}$$

From (83) we deduce that

$$\frac{|-9x| + |-x-8y| + 8|y-x|}{8} \leq M(x,y,y). \tag{84}$$

Notice also that the following inequality

$$\begin{aligned} &\psi(M(x,y,y)) - \phi(M(x,y,y)) \\ &= \frac{M(x,y,y)}{2} - \frac{M(x,y,y)}{8} \\ &= \frac{3M(x,y,y)}{8} \end{aligned} \tag{85}$$

holds for all $x \in A, y \in B$. Performing some easy calculations and using (85) and (84) we obtain

$$\begin{aligned} &\frac{3(|-9x| + |x+8y| + 8|-y+x|)}{64} \\ &\leq \frac{3M(x,y,y)}{8} = \psi(M(x,y,y)) - \phi(M(x,y,y)). \end{aligned} \tag{86}$$

Now, it follows from (82) and (85) that

$$\begin{aligned} \psi(G(Tx,T^2x,Ty)) &= \frac{9|x| + |x+8y| + 8|x-y|}{128} \\ &\leq \frac{3(|-9x| + |x+8y| + 8|-y+x|)}{64} \\ &\leq \frac{3M(x,y,y)}{8} \\ &= \psi(M(x,y,y)) - \phi(M(x,y,y)). \end{aligned} \tag{87}$$

Clearly, all conditions of Theorem 6 are satisfied. Thus, the map T has a unique fixed point in $A \cap B$ which is 0.

Some special cases of the Theorem 6 can be obtained by choosing the functions ϕ, ψ in a particular way.

Corollary 1. *Let (X,G) be a G-complete G-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X with $Y = \bigcup_{j=1}^m A_j$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1,2,\ldots,m, \text{ where } A_{m+1} = A_1. \tag{88}$$

Suppose that there exist a constant $k \in (0,1)$ such that the inequality

$$G(Tx,T^2x,Ty) \leq kM(x,y,y) \tag{89}$$

holds for all $x \in A_j$ and $y \in A_{j+1}, j = 1,2,\ldots,m$ where

$$\begin{aligned} &M(x,y,y) = \\ &\max\Big\{G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty), G(x,Tx,y), \\ &\frac{1}{2}G(x,T^2x,Ty), \frac{1}{2}G(y,Ty,Tx), \frac{1}{2}G(y,T^2x,Ty), \\ &\frac{1}{2}[G(x,Ty,Ty) + G(y,Tx,Tx)], \\ &\tfrac{1}{2}[G(x,T^2x,Ty) + G(y,Tx,Tx)]\Big\}. \end{aligned} \tag{90}$$

Then T has a unique fixed point in $\bigcap_{j=1}^m A_j$.

Proof. The proof is obvious by choosing the functions ϕ, ψ in Theorem 6 as $\phi(t) = (1-k)t$ and $\psi(t) = t$.

Corollary 2. *Let (X,G) be a G-complete G-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X with $Y=\bigcup_{j=1}^m A_j$. Let $T:Y\to Y$ satisfy*

$$T(A_j)\subseteq A_{j+1},\quad j=1,2,\ldots,m,\ \textit{where } A_{m+1}=A_1. \tag{91}$$

Suppose that there exist constants a,b,c,d,e,f,h,k and l such that $0<a+b+c+d+e+f+h+k+l<1$ and a function $\psi\in\Psi$ for which the map T satisfies the inequality

$$\begin{aligned}\psi(G(Tx,T^2x,Ty)) \le\ & aG(x,y,y)+bG(x,Tx,Tx)\\ &+cG(y,Ty,Ty)+dG(x,Tx,y)\\ &+\frac{e}{2}G(x,T^2x,Ty)+\frac{f}{2}G(y,Ty,Tx)\\ &+\frac{h}{2}G(y,T^2x,Ty)\\ &+\frac{k}{2}[G(x,Ty,Ty)+G(y,Tx,Tx)]\\ &+\frac{l}{2}[G(x,T^2x,Ty)+G(y,Tx,Tx)].\end{aligned} \tag{92}$$

for all $x\in A_j$ and $y\in A_{j+1}$, $j=1,2,\ldots,m$. Then, T has a unique fixed point in $\bigcap_{j=1}^m A_j$.

Proof. Clearly we have,

$$\begin{aligned}&aG(x,y,y)+bG(x,Tx,Tx)+cG(y,Ty,Ty)+dG(x,Tx,y)\\ &+\frac{e}{2}G(x,T^2x,Ty)+\frac{f}{2}G(y,Ty,Tx)+\frac{h}{2}G(y,T^2x,Ty)\\ &+\frac{k}{2}[G(x,Ty,Ty)+G(y,Tx,Tx)]\\ &+\frac{l}{2}[G(x,T^2x,Ty)+G(y,Tx,Tx)]\\ &\le(a+b+c+d+e+f+h+k+l)M(x,y,y),\end{aligned} \tag{93}$$

where

$$\begin{aligned}&M(x,y,y)=\\ &\max\Big\{G(x,y,y),G(x,Tx,Tx),G(y,Ty,Ty),G(x,Tx,y),\\ &\frac{1}{2}G(x,T^2x,Ty),\frac{1}{2}G(y,Ty,Tx),\\ &\frac{1}{2}G(y,T^2x,Ty),\frac{1}{2}[G(x,Ty,Ty)+G(y,Tx,Tx)],\\ &\frac{1}{2}[G(x,T^2x,Ty)+G(y,Tx,Tx)]\Big\}.\end{aligned} \tag{94}$$

By Corollary 1, the map T has a unique fixed point.

Integral type contractive conditions are particularly interesting applications in fixed point theory. We consider the following cyclic contractions of integral type and state the related fixed point results.

Corollary 3. *Let (X,G) be a G-complete G-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X with $Y=\bigcup_{j=1}^m A_j$. Let $T:Y\to Y$ be a map satisfying*

$$T(A_j)\subseteq A_{j+1},\quad j=1,2,\ldots,m,\ \textit{where } A_{m+1}=A_1. \tag{95}$$

Suppose also that there exist functions $\phi\in\Phi$ and $\psi\in\Psi$ such that the map T satisfies

$$\begin{aligned}\psi\left(\int_0^{G(Tx,T^2x,Ty)}ds\right)\le\ &\psi\left(\int_0^{M(x,y,y)}ds\right)\\ &-\phi\left(\int_0^{M(x,y,y)}ds\right),\end{aligned} \tag{96}$$

where

$$\begin{aligned}&M(x,y,y)=\\ &\max\Big\{G(x,y,y),G(x,Tx,Tx),G(y,Ty,Ty),G(x,Tx,y),\\ &\frac{1}{2}G(x,T^2x,Ty),\frac{1}{2}G(y,Ty,Tx),\frac{1}{2}G(y,T^2x,Ty),\\ &\frac{1}{2}[G(x,Ty,Ty)+G(y,Tx,Tx)],\\ &\frac{1}{2}[G(x,T^2x,Ty)+G(y,Tx,Tx)]\Big\}.\end{aligned} \tag{97}$$

for all $x\in A_j$ and $y\in A_{j+1}$, $j=1,2,\ldots,m$. Then T has a unique fixed point in $\bigcap_{j=1}^m A_j$.

Corollary 4. *Let (X,G) be a G-complete G-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X with $Y=\bigcup_{j=1}^m A_j$. Let $T:Y\to Y$ satisfy*

$$T(A_j)\subseteq A_{j+1},\quad j=1,2,\ldots,m,\ \textit{where } A_{m+1}=A_1. \tag{98}$$

Suppose also that

$$\int_0^{G(Tx,T^2x,Ty)}ds\le k\int_0^{M(x,y,y)}ds, \tag{99}$$

where $k\in(0,1)$ and

$$\begin{aligned}&M(x,y,y)=\\ &\max\Big\{G(x,y,y),G(x,Tx,Tx),G(y,Ty,Ty),G(x,Tx,y),\\ &\frac{1}{2}G(x,T^2x,Ty),\frac{1}{2}G(y,Ty,Tx),\frac{1}{2}G(y,T^2x,Ty),\\ &\frac{1}{2}[G(x,Ty,Ty)+G(y,Tx,Tx)],\\ &\frac{1}{2}[G(x,T^2x,Ty)+G(y,Tx,Tx)]\Big\}.\end{aligned} \tag{100}$$

for all $x\in A_j$ and $y\in A_{j+1}$, $j=1,2,\ldots,m$. Then T has a unique fixed point in $\bigcap_{j=1}^m A_j$.

Proof. The proof follows immediately by choosing the function ϕ,ψ in Corollary 3 as $\phi(t)=(1-k)t$ and $\psi(t)=t$.

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Nurcan Bilgili is a PhD student in the Department of Mathematics at Gazi University, Ankara, Turkey. Her research interests are in the areas of fixed point theory and its applications. She is an author of research articles in reputed international journals of applied mathematics.

İnci M. Erhan is an associate professor in the Department of Mathematics at the Atılım University. She earned an MS in Applied Mathematics in 1996 and Ph.D. in Applied Mathematics in 2003. She is author of more than 10 articles most of which are published in international ISI journals. Her research activity has been developed in the framework of ”Numerical Analysis”, and ”Fixed point theory and its applications”. Current research interests of İ. M. Erhan is centered on ”Fixed point theory and Applications”. She is a referee of many journals.

Erdal Karapınar is an associated professor in the Department of Mathematics at the Atılım University. He earned an MS in Mathematics (Point Set Topology) in 1998 and Ph.D. in Mathematics (Functional Analysis) in 2004. He is author of more than 80 articles most of which are published in international ISI journals. His research activity has been developed in the framework of ”linear topological invariants”, ”Orlicz spaces”, ”Fréchet spaces”, ”Köthe spaces”, and ”Fixed point theory and its applications”. Current research interests of E. Karapınar is centered on ” Fixed point theory and its applications”. He is an editor and referee of many journals.

Duran Türkoğlu is a Professor in the Department of Mathematics at Gazi University, Ankara, Turkey and a Dean of the Faculty of Science and Arts at Amasya University, Turkey. He earned his MS and Ph.D. degrees in Topology at İnönü University, Malatya, Turkey. He is author of more than 50 articles most of which are published in international ISI journals. Research interests of D. Türkoğlu is ”Fixed point theory and its applications” and ”Topology”.