Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.12785/msl/040110

Numerical Solution of Linear System of Integro-Differential Equations by using Chebyshev Wavelet Method

Hossein Aminikhah and Sahar Hosseini*

Department of Applied Mathematics, School of Mathematical Sciences, University of Guilan, P.O. Box 41635-1914, P.C. 4193833697, Rasht, Iran

Received: 7 Jun. 2014, Revised: 21 Sep. 2014, Accepted: 23 Sep. 2014
Published online: 1 Jan. 2015

Abstract: In this paper, a numerical method for solving linear system of integro-differential equations has been proposed. The method is based on Chebyshev wavelets approximations. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique and the results have been compared with the exact solution. It is shown that the numerical results are in good agreement with the exact solutions for each problem.

Keywords: system of integro-differential equations, Chebyshev wavelets, operational matrix integration

1 Introduction

The concept of a system of integro-differential equations has motivated a huge amount of research work in recent years. These equations have been found to describe various kind of phenomena such as wind ripple in the desert, nono-hydrodynamics, dropwise consideration and glass-forming process [1,2,3,4]. Several numerical methods have been used, such as the rationalized Haar functions method [5], Galerkin methods with hybrid functions [6], the tau method [7], the differential transform method [8], Runge-Kutta methods [9], the spline approximation method [10], the block pulse functions method [11], the spectral method [12], the finite difference approximation method [13], new homotopy perturbation method [14] and etc. Chebyshev wavelets have been used by many authors for solving various functional. The main idea of using Chebyshev basis is that the problem under study reduces to a system of linear or nonlinear algebraic equations. This can be done by truncated series of orthogonal basis functions for the solution of problem and using the operational matrices. An extension of Chebyshev wavelets method for solving linear systems of integro-differential equations is the novelty of this paper.

2 Chebyshev wavelets preliminaries

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet [15,16,17]. When the dilation parameter, a and the translation parameter,b,vary continuously we have the following family of continuous wavelets as

$$\psi_{a,b}(x)=|a|^{-\frac{1}{2}}\psi(\frac{x-b}{a}),a,b\in\mathbb{R},a\neq 0. \tag{1}$$

we take dilation and translation parameters a^{-k} ,and nba^{-k} ,respectively where $a>1,b>0,n$ and k positive integers, then we have the following family of discrete wavelets

$$\psi_{k,n}(x)=|a|^{\frac{k}{2}}\psi(a^k x-nb). \tag{2}$$

These functions are a wavelet basis for $L^2(\mathbb{R})$ and in special case $a=2$, and $b=1$,the functions $\psi_{k,n}(x)$ are an orthonormal basis.
Chebyshev wavelets $\psi_{n,m}(x)=\psi(k,n,m,x)$ have four arguments, $n=1,2,...,2^{k-1}$, k is an arbitrary positive integer and m is the order of Chebyshev polynomials of the first kind. They are defined on the interval $[0,1]$, as follows:

* Corresponding author e-mail: hossein.aminikhah@gmail.com, aminikhah@guilan.ac.ir

$$\psi_{n,m}(x) = \psi(k,n,m,x) = \begin{cases} 2^{\frac{k}{2}}\widetilde{T}_m(2^kx-2n+1), \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}} \\ 0, otherwise \end{cases} \tag{3}$$

where

$$\widetilde{T}_m(x) = \begin{cases} \frac{1}{\sqrt{\pi}}, m=0, \\ \sqrt{\frac{2}{\pi}}T_m(x), m>0. \end{cases} \tag{4}$$

and $m = 0,1,..,M-1$ and $n = 1,2,...,2^{k-1}$. $T_m(x)$ are the famous Chebyshev polynomials of the first kind of degree m which are orthogonal with respect to the weight function $W(x) = \frac{1}{\sqrt{1-x^2}}$, on the interval $[-1,1]$ and satisfy the following recursive formula

$$\begin{aligned} &T_0(x) = 1, T_1(x) = x, \\ &T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x), m = 1,2,... \end{aligned} \tag{5}$$

The set of Chebyshev wavelets is an orthogonal set with respect to the weight function $W_n(x) = W(2^kx-2n+1)$.
A function $f(x)$ defined on the interval $[0,1]$ may be presented as

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x). \tag{6}$$

The series representation of $f(x)$ in (6) called a wavelet series and the wavelet coefficients c_{nm} are given by $c_{nm} = (f(x), \psi_{nm}(x))_{W_n(x)}$.
The convergence of the series (6), in $L^2[0,1]$, means that

$$\lim_{s1,s2 \longrightarrow \infty} \| f(x) - \sum_{n=1}^{s1}\sum_{m=0}^{s2} c_{nm}\psi_{nm}(x)\| = 0. \tag{7}$$

Therefore one can consider the following truncated series for series (6)

$$f(x) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm} = C^T\psi(x), \tag{8}$$

where C and $\psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$\begin{aligned} C = &[c_{1,0}, c_{1,1}, ..., c_{1,M-1}, c_{2,0}, c_{2,1}, ..., c_{2,M-1}, \\ &..., c_{2^{k-1},0}, ..., c_{2^{k-1},M-1}]^T = \\ &[c_1, c_2, ..., c_M, c_{M+1}, ..., c_{2^{k-1}M}]^T \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\psi(x) = [\psi_{1,0}(x), \psi_{1,1}(x), ..., \psi_{1,M-1}(x), \psi_{2,0}(x), \\ &\psi_{2,1}(x), ..., \psi_{2,M-1}(x), ..., \psi_{2^{k-1},0}(x), ..., \psi_{2^{k-1},M-1}(x)]^T = \\ &[\psi_1(x), \psi_2(x), ..., \psi_M(x), \psi_{M+1}(x), ..., \psi_{2^{k-1}M}(x)]^T \end{aligned} \tag{10}$$

The integration of the product of two Chebyshev wavelets vector functions with respect to the weight function $W_n(x)$, is derived as

$$\int_0^1 W_n(x)\psi(x)\psi^T(x)dx = I \tag{11}$$

where I is an identity matrix.
A function $f(x,y)$ defined on $[0,1] \times [0,1]$ can be approximated as the following

$$f(x,y) \simeq \psi^T(x)K\psi(y). \tag{12}$$

Here the entries of matrix $K = [k_{ij}]_{2^{k-1}M \times 2^{k-1}M}$ will be obtain by

$$\begin{aligned} &k_{ij} = (\psi_i(x), (f(x,y), \psi_j(y))W_n(y))W_n(x), \\ &\quad i,j = 1,2,...,2^{k-1}M. \end{aligned} \tag{13}$$

The integration of the vector $\psi(x)$, defined in (10) can be achieved as

$$\int_0^x \psi(t)dt = P\psi(x) \tag{14}$$

where P is the $2^{k-1}M$ operational matrix of integration [18, 19]. This matrix is determined as follows

$$P = \frac{1}{2^k}\begin{bmatrix} L & F & F & \dots & F \\ O & L & F & \ddots & \vdots \\ O & O & L & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & F \\ O & \cdots & O & O & L \end{bmatrix} \tag{15}$$

Where L, F and O are $M \times M$ matrices given by

$$L = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \cdots & 0 \\ -\frac{\sqrt{2}}{4} & 0 & \frac{1}{4} & 0 & 0 & \cdots & 0 \\ -\frac{\sqrt{2}}{3} & -\frac{1}{2} & 0 & \frac{1}{6} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}(-1)^r(\frac{1}{r-2}-\frac{1}{r}) & \cdots & -\frac{1}{2(r-2)} & 0 & \frac{1}{2r} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}(-1)^M(\frac{1}{M-2}-\frac{1}{M}) & 0 & 0 & 0 & \cdots & -\frac{1}{2(M-2)} & 0 \end{bmatrix} \tag{16}$$

$$F = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ -\frac{2\sqrt{2}}{3} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}(\frac{1-(-1)^r}{r}-\frac{1-(-1)^{r-2}}{r-2}) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}(\frac{1-(-1)^M}{M}-\frac{1-(-1)^{M-2}}{M-2}) & 0 & \cdots & 0 \end{bmatrix} \tag{17}$$

$$O = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{18}$$

The property of the product of two Chebyshev wavelets vector functions will be as follows

$$\psi(x)\psi^T(x)Y \approx \widetilde{Y}\psi(x) \tag{19}$$

where Y is a given vector and $\widetilde{Y}$ is a $2^{k-1}M \times 2^{k-1}M$ matrix. This matrix is called the operational matrix of product.

3 Chebyshev wavelets applied to systems of integro-differential equations

A system of integro-differential can be considered in general as follows:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(t,\mathbf{x}(t)) + \int_0^t \mathbf{K}(t,s,\mathbf{x}(s))ds, \tag{20}$$

where

$\mathbf{x}(t) = (x_1(t), x_2(t), ..., x_n(t))^T,\ \mathbf{K}(t,s,\mathbf{x}(t)) = (k_1(t,s,\mathbf{x}(s)), k_2(t,s,\mathbf{x}(s)), ..., k_n(t,s,\mathbf{x}(s)))^T.$

If $\mathbf{K}(t,s,\mathbf{x}(s))$ and $\mathbf{F}(t,\mathbf{x}(t))$ be linear, the system (20) can be represented as the following simple form

$$\frac{dx_i(t)}{dt} = f_i(t) + \sum_{j=1}^{m}(w_{i,j}(t)x_j(t) + \int_0^t k_{i,j}(t,s)x_j(s)ds),$$
$$x_i(0) = \alpha_i,\ i = 1,2,...,n,\ m = 1,2,... \tag{21}$$

For solving system (21), by Chebyshev wavelets method, consider the following approximations:

$$\frac{dx_i(t)}{dt} = C_i^T\psi(t),\ i = 1,2,...,n \tag{22}$$

Where $C_i,\ i = 1,2,...,n$ are $2^{k-1}M \times 1$ matrices given by

$$C = [c^i_{1,0}, c^i_{1,1}, ..., c^i_{1,M-1}, c^i_{2,0}, c^i_{2,1}, ..., c^i_{2,M-1}, ..., c^i_{2^{k-1},0}, ..., c^i_{2^{k-1},M-1}]^T = [c_{i,1}, c_{i,2}, ..., c_{i,M}, c_{i,M+1}, ..., c_{i,2^{k-1}M}]^T \tag{23}$$

and $\psi(t)$ is defined in (10). Also consider the following approximations

$$x_i(t) \simeq C_i^T P\psi(t) + D_i^T\psi(t),\ f_i(t) \simeq F_i^T\psi(t),$$
$$w_{i,j}(t)x_j(t) \simeq Y_{ij}^T\psi(t),\ i = 1,2,...,n,\ j = 1,2,...m,$$
$$k_{i,j}(t,s) \simeq \psi^T(t)K_{ij}\psi(s). \tag{24}$$

where K_{ij} are the $2^{k-1}M \times 2^{k-1}M$ matrices, F_i are the $2^{k-1}M \times 1$ matrices, and Y_{ij} are column vectors with the entries of the vectors C_i for $i = 1,2,...,n,\ j = 1,2,...,m.$ substitution of approximations (22) and (24) into the system (21), will be resulted to:

$$C_i^T\psi(t) \simeq F_i^T\psi(t) + \sum_{j=1}^{m} Y_{ij}^T\psi(t) +$$
$$\sum_{j=1}^{m}\int_0^t \psi^T(t)K_{ij}\psi(s)(C_i^T P\psi(s) + D_i^T\psi(s))ds =$$
$$F_i^T\psi(t) + \sum_{j=1}^{m} Y_{ij}^T\psi(t) +$$
$$\sum_{j=1}^{m}\psi^T(t)K_{ij}(\int_0^t \psi(s)(C_i^T P + D_i^T)\psi(s)ds =$$
$$F_i^T\psi(t) + \sum_{j=1}^{m} Y_{ij}^T\psi(t) + \sum_{j=1}^{m}\psi^T(t)K_{ij}\widetilde{Y}_i P\psi(t),$$
$$i = 1,2,...,n,\ m = 1,2,... \tag{25}$$

where $\widetilde{Y}_i$ are $2^{k-1}M \times 1$ matrices and P is the $2^{k-1}M \times 2^{k-1}M$ operational matrix of integration.
According to the Galerkin method by multiplying $W_n(t)\psi^T(t)$, in both sides of the system (25) and then applying $\int_0^1(.)dt$, linear system in terms of the entries of $C_i, i = 1,2,...,n$, will be obtained. The elements of vector functions $C_i, i = 1,2,...,n$ can be computed by solving this system.

4 Examples

In this section we present two examples. These examples are considered to illustrate the Chebyshev wavelets approach for systems of integro-differential equations.

Example 1. [14]: Consider the following system of integro-differential equations with the exact solutions $x_1(t) = e^t$ and $x_2(t) = e^{-t}$,

$$\frac{dx_1(t)}{dt} = t^4 - t^3 - 2t^2 - 6 + (3t^2 - 6t + 7)x_1(t) +$$
$$2t^2(t+1)x_2(t) + \int_0^1((s^3 - t^3)x_1(s) + t^2(s^2 - t^2)x_2(s))ds,$$
$$x_1(0) = 1,$$
$$\frac{dx_2(t)}{dt} = -t^4 - 3t^2 + 2 + 2(t-1)x_1(t) +$$
$$(2t^4 + 2t^3 + 2t^2 - 1)x_2(t) +$$
$$\int_0^1((s^2 - t^2)x_1(s) + t^2(s^2 + t^2)x_2(s))ds,$$
$$x_2(0) = 1. \tag{26}$$

Lets

$$\frac{dx_1(t)}{dt} \simeq C_1^T \psi(t),$$
$$x_1(t) \simeq C_1^T P\psi(t) + x_1(0) \simeq C_1^T P\psi(t) + D_1^T \psi(t),$$
$$\frac{dx_2(t)}{dt} \simeq C_2^T \psi(t),$$
$$x_2(t) \simeq C_2^T P\psi(t) + x_2(0) \simeq C_2^T P\psi(t) + D_2^T \psi(t),$$
$$t^4 - t^3 - 2t^2 - 6 \simeq F_1^T \psi(t),\ -t^4 - 3t^2 + 2 \simeq F_2^T \psi(t),$$
$$(3t^2 - 6t + 7)x_1(t) \simeq Y_1^T \psi(t),\ 2t^2(t+1)x_2(t) \simeq Y_2^T \psi(t),$$
$$2(t-1)x_1(t) \simeq Y_3^T \psi(t),$$
$$(2t^4 + 2t^3 + 2t^2 - 1)x_2(t) \simeq Y_4^T \psi(t),$$
$$(s^3 - t^3) \simeq \psi^T(t)K_1\psi(s),\ t^2(s^2 - t^2) \simeq \psi^T(t)K_2\psi(s),$$
$$(s^2 - t^2) \simeq \psi^T(t)K_3\psi(s),\ t^2(s^2 + t^2) \simeq \psi^T(t)K_4\psi(s) \quad (27)$$

Substitution into the system (26), leads to the following system

$$C_1^T \psi(t) = F_1^T \psi(t) + Y_2^T \psi(t) +$$
$$\psi^T(t)K_1 \int_0^t \psi(s)(C_1^T P + D_1^T)\psi(s)ds +$$
$$\psi^T(t)K_2 \int_0^t \psi(s)(C_2^T P + D_2^T)\psi(s)ds =$$
$$F_1^T \psi(t) + Y_1^T \psi(t) + Y_2^T \psi(t) + \psi^T(t)K_1\widetilde{Y}_1 P\psi(t) +$$
$$\psi^T(t)K_2\widetilde{Y}_2 P\psi(t),$$
$$C_2^T \psi(t) = F_2^T \psi(t) + Y_3^T \psi(t) + Y_4^T \psi(t) +$$
$$\psi^T(t)K_3 \int_0^t \psi(s)(C_1^T P + D_1^T)\psi(s)ds +$$
$$\psi^T(t)K_4 \int_0^t \psi(s)(C_2^T P + D_2^T)\psi(s)ds =$$
$$F_2^T \psi(t) + Y_3^T \psi(t) + Y_4^T \psi(t) + \psi^T(t)K_3\widetilde{Y}_1 P\psi(t) +$$
$$\psi^T(t)K_4\widetilde{Y}_2 P\psi(t), \quad (28)$$

Multiply $W_n(t)\psi^T(t)$, on both sides of the system(28), applying $\int_0^1 (.)dt$, and then solve the system.
The elements of vector functions C_1 and C_2 can be obtained as follows

$$C_1 = [2.197072263, 0.7532217714, 0.09314234913, 0.007745016935, 0.0005007730657, 0.00003345920292, 0.0.000005002300867, 0.000001789657758]^T,$$

$$C_2 = [-0.8082321301, 0.2774598271, -0.03418569156, 0.002900083585, -0.0001498235218, 0.00002099796862, 0.000004561044304, 0.000001798662095]^T.$$

Therefore, the following solutions will result.

$x_1(t) \simeq C_1^T P\psi(t) + x_1(0) \simeq C_1^T P\psi(t) + D_1^T(t) = 0.001651417113x^7 - 0.002730551472x^6 + 0.01312030509x^5 + 0.03868281601x^4 +$

Table 1: Numerical results of Example 1.

	$x_1(exact)$	$x_1(CWM)$	$Error(x_1(t))$
0.0	1.0000000000	0.9999999367	0.0000000633
0.1	1.1051709180	1.105168372	0.000002546
0.2	1.2214027580	1.221395345	0.000007413
0.3	1.3498588080	1.349843148	0.000015660
0.4	1.4918246980	1.491795531	0.000029167
0.5	1.6487212710	1.648671304	0.000049967
0.6	1.8221188000	1.822038466	0.000080334
0.7	2.0137527070	2.013629713	0.000122994
0.8	2.2255409280	2.225360121	0.000180807
0.9	2.4596031110	2.459347874	0.000255237
1.0	2.7182818280	2.717938847	0.000342981

$x_2(exact)$	$x_2(CWM)$	$Error(x_2(t))$
1.0000000000	0.9999999364	0.0000000636
0.9048374180	0.9048383793	0.0000009613
0.8187307531	0.8187330857	0.0000023326
0.7408182207	0.7408227857	0.0000045650
0.6703200460	0.6703277642	0.0000077182
0.6065306597	0.6065429205	0.0000122608
0.5488116361	0.5488306415	0.0000190054
0.4965853038	0.4966142473	0.0000289435
0.4493289641	0.4493727673	0.0000438032
0.4065696597	0.4066378071	0.0000681474
0.3678794412	0.3679932628	0.0001138216

$0.1674220136x^3 + 0.4998030468x^2 + 0.9999898624x + 0.9999999367,$
$x_2(t) \simeq C_2^T P\psi(t) + x_2(0) \simeq C_2^T P\psi(t) + D_2^T(t) = 0.001505744420x^7 - 0.003421435813x^6 - 0.002759244740x^5 + 0.03843721505x^4 - 0.1656548697x^3 + 0.4998700759x^2 - 0.9999841587x + 0.9999999364.$

Table 1 shows some values of the solutions and absolute errors at some and plots of the exact and approximate solutions are shown in Figure 1.

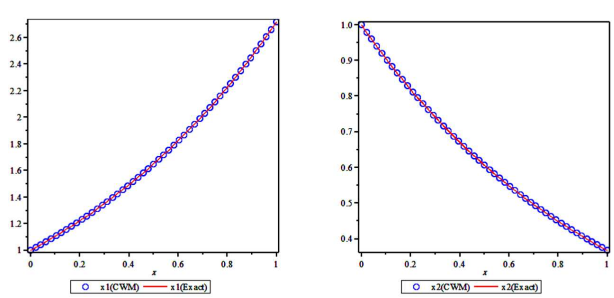

Fig. 1: Comparison of the exact and approximate solutions of Example 1.

Example 2. [14]: Consider the following system of integro-differential equations with the exact solutions $x_1(t) = cosh(t)$ and $x_2(t) = sinh(t)$.

$$\frac{dx_1(t)}{dt} = -t^3 - 6t - 1 + x_1(t) + (7-2t)x^2(t) + \int_0^t ((s+t)x_1(s) + (s-t)^3 x_2(s))ds, \; x_1(0) = 1,$$
$$\frac{dx_2(t)}{dt} = -3t^2 + t - 6 + (7-2t)x_1(t) + x_2(t) + \int_0^t ((s-t)^3 x_1(s) + (s+t)x_2(s))ds, \; x_2(0) = 0. \quad (29)$$

By applying the Chebyshev wavelets method and solving the resulted linear system, the following results would be achieved.

$$C_1 = [0.6913440397, 0.5116838793, 0.02727964186, 0.004340530682, -0.0001604934677, -0.00006543048876, -0.00001670580723, -0.000003004951770]^T,$$

$$C_2 = [1.499774940, 0.23444339640, 0.06157643030, 0.001498845797, 0.00001601905546, -0.000074065054542, -0.000016397378437, -0.000003021159578]^T.$$

Therefore, we have the following approximate solutions

$x_1(t) = -0.005515113281x^7 + 0.01329204418x^6 - 0.01266244896x^5 + 0.04796640532x^4 - 0.001878448025x^3 + 0.5001752830x^2 - 0.00001296863640x + 1.000000106.$
$x_2(t) = -0.005413291216x^7 + 0.01210581944x^6 - 0.004592966988x^5 + 0.006532724767x^4 + 0.1646412918x^3 + 0.0002430555256x^2 + 0.9999694387x + 1.065316727x \times 10^{-7}.$
Some values of exact, approximate solutions and absolute errors are presented in Table 2 and the plots of exact and approximate solutions are shown in Figure 2.

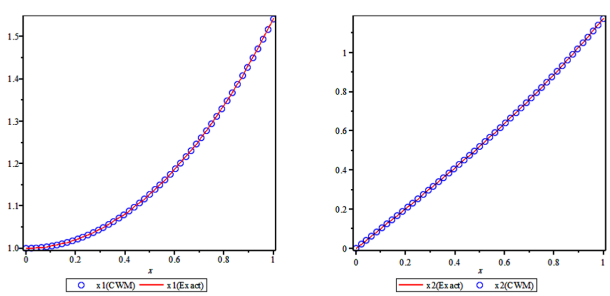

Fig. 2: Comparison of the exact and approximate solutions of Example 2.

Table 2: Numerical results of Example 2.

	$x_1(exact)$	$x_1(CWM)$	$Error(x_1(t))$
0.0	1.0000000000	1.000000106	0.000000106
0.1	1.005004168	1.005003366	0.000000802
0.2	1.020066756	1.020062970	0.000003786
0.3	1.045338514	1.045327515	0.000010999
0.4	1.081072372	1.081046428	0.000025944
0.5	1.127625965	1.127569437	0.000056528
0.6	1.185165218	1.185347262	0.000117956
0.7	1.255169006	1.254930768	0.000238238
0.8	1.337434946	1.336965782	0.000469164
0.9	1.433086385	1.132180810	0.000905575
1.0	1.543080635	1.541364860	0.001715775

$x_2(exact)$	$x_2(CWM)$	$Error(x_2(t))$
0.0000000000	0.0000001065	0.0000001065
0.4001667500	0.4001647411	0.0000020089
0.2013360025	0.2013305348	0.0000054677
0.3045202934	0.3045075234	0.0000127700
0.4107523258	0.4107247356	0.0000275902
0.5210953055	0.5210373784	0.0000579271
0.6366535821	0.6365315523	0.0001190298
0.7585837018	0.7583417709	0.0002389309
0.8881059822	0.8876365536	0.0004694286
1.026516726	1.025611367	0.0009053590
1.175201194	1.173486179	0.0017150150

5 Conclusion

In this research, we have presented the Chebyshev wavelet method for solving system of integro-differential equations. The Chebyshev operational matrix of integration is used to solve these systems. The present method reduces system of integro-differential equations into a set of algebraic equations. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique and the results have been compared with the exact solution.

References

[1] T.L. Bo, L. Xie, X.J. Zheng, Numerical approach to wind ripple in desert, International Journal of Nonlinear Sciences and Numerical Simulation 8 (2) (2007) 223-228.
[2] F.Z. Sun, M. Gao, S.H. Lei, et al., The fractal dimension of the fractal model of drop-wise condensation and its experimental study, International Journal of Nonlinear Sciences and Numerical Simulation 8 (2) (2007) 211-222.
[3] H. Wang, H.M. Fu, H.F. Zhang, et al., A practical thermodynamic method to calculate the best glass-forming composition for bulk metallic glasses, International Journal of Nonlinear Sciences and Numerical Simulation 8 (2) (2007) 171-178.
[4] L. Xu, J.H. He, Y. Liu, Electrospun nano-porous spheres with Chinese drug, International Journal of Nonlinear Sciences and Numerical Simulation 8 (2) (2007) 199-202.

[5] K. Maleknejad, F. Mirzae, S. Abbasbandy, Solving linear integro-differential equations system by using rationalized Haar functions method, Appl. Math. Comput. 155 (2004) 317-328.
[6] K. Maleknejad, M. Tavassoli Kajani, Solving linear integro-differential equation system by Galerkin methods with hybrid functions, Appl. Math. Comput. 159 (2004) 603-612.
[7] J. Pour-Mahmoud, M.Y. Rahimi-Ardabili, S. Shahmorad, Numerical solution of the system of Fredholm integro-differential equations by the Tau method, Appl. Math. Comput. 168 (2005) 465-478.
[8] A. Arikoglu, I. Ozkol, Solutions of integral and integro-differential equation systems by using differential transform method, Comput. Math. Appl. 56(2008) 2411-2417.
[9] C.T.H. Baker, A. Tang, Stability analysis of continuous implicit RungeKutta methods for Volterra integro-differential systems with unbounded delays, Appl. Numer. Math. 24 (1997) 153-173.
[10] M. Ganesh, I.H. Sloan, Optimal order spline methods for nonlinear differential and integro-differential equations, Appl. Numer. Math. 29 (1999) 445-478.
[11] K. Maleknejad, H. Safdari, M. Nouri, Numerical solution of an integral equations system of the first kind by using an operational matrix with block pulse functions, Internat. J. Systems Sci. 42 (1) (2011) 195-199.
[12] Omer M.A. Al-Faour, Rostam K. Saeed, Solution of a system of linear Volterra integral and integro-differential equations by spectral method, Al-Nahrain University Journal for Science 6 (2) (2006) 30-46.
[13] T. Jangveladze, Z. Kiguradze, B. Neta, Finite difference approximation of a nonlinear integro-differential system, Appl. Math. Comput. 215 (2009) 615-628.
[14] Hossein Aminikhah, A new analytical method for solving systems of linear integro-differential equations, Journal of King Saud University - Science (2011) 23, 349-353.
[15] I. Daubeches, Ten lectures on wavelets, CBMS-NSF, 1992.
[16] Ole Christensen, Khadjia L. Christensen, Approximation Theory: from Taylor Polynomial to Wavelets, Birkhauser, Boston, 2004.
[17] D. Gottlieb, S.A. Orszag, Numerical Analysis of Spectral Methods, SIAM, Philadelphia, PA, 1997.
[18] E. Babolian, F. Fattahzadeh, Numerical computation method in solving integral equations by using Chebyshev wavelet operational matrix of integration, Applied Mathematics and Computations 188 (2007) 1016-1022.
[19] E. Babolian, F. Fattahzadeh, Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration, Applied Mathematics and Computations 188 (2007) 417-426.

Hossein Aminikhah was born in Iran, in 1979. He received a Ph.D. degree in Applied Mathematics (Numerical Analysis) from the University of Guilan, Rasht, Iran, in 2008, where he is currently Assistant Professor in the Department of Applied Mathematics. His research interests include numerical methods for functional differential equations and numerical linear algebra.

Sahar Hosseini was born in Iran, in 1985. She received a Master degree in Applied Mathematics (Numerical Analysis) from the University of Guilan, Rasht, Iran, in 2013.