Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.18576/msl/060102

The Adomian Decomposition Method for Boundary Value Problems with Discontinuities

Waleed Al-Hayani[1,*] *and Luis Casasús*[2]

[1] Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, Iraq.
[2] Departamento de Matemática Aplicada, Escuela Técnica Superior de Ingenieros Industriales, Universidad Politécnica de Madrid, C / José Gutiérrez Abascal, 2, 28006 Madrid, Spain.

Received: 7 Jan. 2015, Revised: 20 Apr. 2016, Accepted: 26 Apr. 2016
Published online: 1 Jan. 2017

Abstract: In this paper, the Adomian Decomposition Method is applied to the Boundary Value Problem (BVP) for linear and nonlinear ordinary differential equations with discontinuities. We analyze two classes of problems: with discontinuities in the driving term and with discontinuous coefficients.

Keywords: Adomian Decomposition Method; Adomian's polynomials; Boundary Value Problems (BVP's); Heaviside function; Dirac delta function.

1 Introduction

The so called Adomian Decomposition Method (ADM) is an analytic approximation to the solution of linear and non-linear problems which does not require linearization or perturbation [1,2,3,4]. The main objective of this paper is to explore the possibilities of this method in Boundary Value Problems with discontinuous coefficients and/or driving terms. This is a field of growing interest in the theory of differential equations and in many areas of applications [5,6,7,8] In recent papers [9,10,11,12] the ADM has been applied to a broad range of BVP's, but the case of discontinuous coefficients or discontinuous solutions has not been investigated. In [13] an application of ADM is given to impulsive Initial Value Problems. In [14] Casasus and Al-Hayani applied ADM to Initial Value Problems with discontinuities. The existence of solutions of boundary problems with impulse was studied in [15] and [16]. Classical techniques are affected near discontinuities, but ADM is well suited to deal with these situations. In this paper we analyze the behavior of ADM in the presence of discontinuous coefficients and/or driving terms like Heaviside or Dirac delta functions.

Let us consider the general functional equation

$$y - \mathrm{N}\,y = f, \tag{1}$$

where N is a nonlinear operator, f is a known function, and we are seeking the solution y satisfying (1). We assume that for every f, Eq. (1) has one and only one solution.

The Adomian technique consists of approximating the solution of (1) as an infinite series

$$y = \sum_{n=0}^{\infty} y_n, \tag{2}$$

and decomposing the nonlinear operator N as

$$\mathrm{N}\,y = \sum_{n=0}^{\infty} A_n, \tag{3}$$

where A_n are polynomials (called Adomian polynomials) of $y_0, \ldots, y_n$ [1,2,3,4] given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\mathrm{N}\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}, \; n = 0, 1, 2, \ldots .$$

The proofs of the convergence of the series $\sum_{n=0}^{\infty} y_n$ and $\sum_{n=0}^{\infty} A_n$ are given in [3,17,18,19,20,21]. Substituting (2) and (3) into (1) yields

$$\sum_{n=0}^{\infty} y_n - \sum_{n=0}^{\infty} A_n = f.$$

* Corresponding author e-mail: waleedalhayani@yahoo.es

Thus, we can identify

$$y_0 = f,$$

$$y_{n+1} = A_n(y_0, \dots, y_n), \quad n = 0, 1, 2, \dots .$$

Thus all components of y can be calculated once the A_n are given. We then define the n-term approximant to the solution y by $\phi_n[y] = \sum_{i=0}^{n-1} y_i$ or equivalently $\phi_{n+1}[y] = N(y_0 + \phi_n[y])$ with $\lim_{n\to\infty} \phi_n[y] = y$.

2 ADM applied to a BVP

Consider the general BVP:

$$y'' + 2hg\,(y, y') + k^2 f_1\,(x)\,y = \lambda f_2\,(x)\,, 0 \le x \le 2 \tag{4}$$

$$y(0) = \alpha,\; y(2) = \beta \tag{5}$$

where h, k, λ, α and β are real constants, g is a (possibly) nonlinear function of y, y' and f_1, f_2 are functions with some discontinuity.

Applying the decomposition method as in [1,2,3,4] Eq. (4) can be written as

$$\mathrm{L}\,y = \lambda f_2\,(x) - 2h\,\mathrm{N}\,y - k^2 f_1\,(x)\,y, \tag{6}$$

where $\mathrm{L} = \dfrac{d^2}{dx^2}$ is the linear operator and $\mathrm{N}\,y = g\,(y, y')$ is the nonlinear operator. Operating on both sides of Eq. (6) with the inverse operator of L (namely $\mathrm{L}^{-1}\,[\cdot] = \int_0^x \int_0^x\,[\cdot]\,dxdx$) yields

$$y(x) = c_1 + c_2 x + \lambda\,\mathrm{L}^{-1}\,f_2\,(x) - 2h\,\mathrm{L}^{-1}\,\mathrm{N}\,y - k^2\,\mathrm{L}^{-1}\,f_1\,(x)\,y,$$

where c_1, c_2 are constants of integration evaluated from the given conditions (5).

Upon using (2) and (3) it follows that

$$\sum_{n=0}^{\infty} y_n = c_1 + c_2 x + \lambda\,\mathrm{L}^{-1}\,f_2\,(x) - 2h\,\mathrm{L}^{-1} \sum_{n=0}^{\infty} A_n - k^2\,\mathrm{L}^{-1}\,f_1\,(x) \sum_{n=0}^{\infty} y_n. \tag{7}$$

From Eq. (7), the iterates defined using the Standard Adomian Method are determined in the following recursive way:

$$y_0 = c_1 + c_2 x + \lambda\,\mathrm{L}^{-1}\,f_2\,(x)\,,$$

$$y_{n+1} = -2h\,\mathrm{L}^{-1}\,A_n - k^2\,\mathrm{L}^{-1}\,f_1\,(x)\,y_n, \quad n = 0, 1, 2, \dots .$$

and the iterates defined using the Modified Technique [22] are determined in the following recursive way:

$$y_0 = c_1 + c_2 x,$$

$$y_1 = \lambda\,\mathrm{L}^{-1}\,f_2\,(x) - 2h\,\mathrm{L}^{-1}\,A_0 - k^2\,\mathrm{L}^{-1}\,f_1\,(x)\,y_0,$$

$$y_{n+2} = -2h\,\mathrm{L}^{-1}\,A_{n+1} - k^2\,\mathrm{L}^{-1}\,f_1\,(x)\,y_{n+1}, \quad n = 0, 1, 2, \dots .$$

To give a sufficient condition for convergence, we derive an extension of the fixed point theorems used in [23] and [24]. To this end, we reformulate (1) as follows:

$$\mathrm{L}\,y = \mathrm{N}\,y + \mathrm{R}\,y + f \tag{8}$$

where L is the second derivative operator, R a linear (possibly) discontinuous operator and N the nonlinear operator. Applying the inverse operator L^{-1} to both sides of (8)

$$y = \theta + \mathrm{L}^{-1}\,\mathrm{N}\,y + \mathrm{L}^{-1}\,\mathrm{R}\,y + \mathrm{L}^{-1}\,f \tag{9}$$

where θ is determined by the boundary conditions.

ADM defines the solution by $y = \lim_{n\to\infty} \phi_n[y]$, where $\phi_n[y] = \sum_{i=0}^{n-1} y_i$ and $y_0 = \theta + \mathrm{L}^{-1}\,f$.

Theorem 1. *Let* N *be an operator from a Hilbert space* H *to itself. Let the problem defined by (8) and (9) have a unique solution* y*. Then, if there is a real constant* $0 \le \alpha < 1$ *such that* $\|y_{n+1}\| \le \alpha\,\|y_n\|$ *for* $n = 0, 1, 2, \dots,$ *then* $\lim_{n\to\infty} \phi_n[y] = \phi$.

Proof. It is enough to show that $\{\phi_n[y]\}_{n=0}^{\infty}$ is a Cauchy sequence in H. We have

$$\|\phi_{n+1}[y] - \phi_n[y]\| = \|y_{n+1}\| \le \alpha\,\|y_n\| \le \alpha^2\,\|y_{n-1}\| \le \dots \le \alpha^{n+1}\,\|y_0\|$$

On the other hand, $\forall\ m, n \in \mathbb{N}$, $m \geqslant n$ we have

$$\begin{aligned}
&\|\phi_m[y] - \phi_n[y]\| = \\
&\left\|(\phi_m[y] - \phi_{m-1}[y]) + (\phi_{m-1}[y] - \phi_{m-2}[y]) + \dots + (\phi_{n+1}[y] - \phi_n[y])\right\| \\
&\le \left\|\phi_m[y] - \phi_{m-1}[y]\right\| + \left\|\phi_{m-1}[y] - \phi_{m-2}[y]\right\| + \dots + \left\|\phi_{n+1}[y] - \phi_n[y]\right\| \\
&\le \alpha^m\,\|y_0\| + \alpha^{m-1}\,\|y_0\| + \dots + \alpha^{n+1}\,\|y_0\| \\
&\le \frac{\alpha^{n+1}}{1-\alpha}\,\|y_0\|\,.
\end{aligned}$$

Therefore, $\lim_{m,n\to\infty} \|\phi_m[y] - \phi_n[y]\| = 0$ and $\{\phi_n[y]\}_{n=0}^{\infty}$ is a Cauchy sequence which is equivalent in H to converge to a limit ϕ : $\lim_{n\to\infty} \phi_n[y] = \phi \in \mathrm{H}$. Therefore $\phi = \sum_{i=0}^{\infty} y_i$.

The nonlinear operator is continuous, so we can use the iteration $\phi_{n+1}[y] = N(y_0 + \phi_n[y])$ defined by ADM to write

$$\begin{aligned}
N(y_0 + \phi) &= N\left(y_0 + \lim_{n\to\infty} (\phi_n[y])\right) \\
&= \lim_{n\to\infty} N\,(y_0 + \phi_n[y]) \\
&= \lim_{n\to\infty} \phi_{n+1}[y] = \phi
\end{aligned}$$

i.e., ϕ is a solution of (8). $\square$

According to this result, it is enough to compute the quotients

$$\alpha_n = \frac{\|y_{n+1}\|}{\|y_n\|},\; n = 0, 1, 2, \dots \tag{10}$$

to have guaranteed the convergence of the method if there is a $\alpha = \max\,\{\|\phi_n[y]\|\}_{n=0}^{\infty}$, $\alpha < 1$.

Table 1

h	3	1	0	0	-1	-1	-4	-4
k	1	9	1	13	1	12	1	8
n	35	34	4	35	11	30	34	16

Table 2

h	-1	-2	-2	-3	-3	-4
k	13	11	13	11	13	10
n_0	27	17	21	15	19	15

2.1 Linear Problems

In the case of a linear problem we take in (4) $g(y, y') = y'$. In our computations we choose $\lambda = 1$ and $\alpha = \beta = 0$.

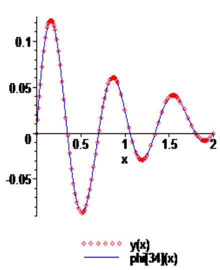
2.1.1 Example 1

In this case we take in (4) $f_1(x) = 1$ and $f_2(x) = \mathrm{H}(x-1)$, the Heaviside function with jump at $x = 1$.

The exact solution $y(x)$ can be computed by Laplace Transform techniques. In Table 1 we represent the minimum order n of the approximation for which the norm of the error is smaller than 10^{-3} for some values of (h, k).

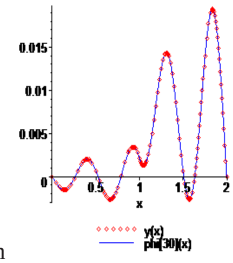
Numerical instabilities can arise for some pairs (h, k) when k is relatively large. In these cases, the divergence of the method is manifested by the following behavior. The approximants $\phi_n(x)$ approach the solution for n smaller than a certain n_0, then for $n \geq n_0$ $\phi_n(x)$ has oscillations at the right side of the discontinuity $(x > 1)$. Some of these pairs (h, k) are shown in Table 2 and the numerical instability is represented in Figures 4, 5. These oscillations dissapear if the computations are carried out with more precision, e.g. 20 or more digits.

In the following Figures 1-6, we represent the exact solution $y(x)$ with the symbol $\diamond$ and our approximations $\phi_n(x) \equiv phi[n](x)$ with a continuous line.



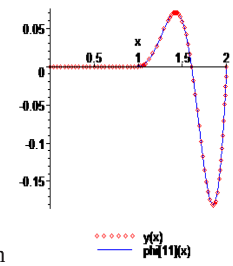
5cm

Fig. 1: h = 1, k= 9



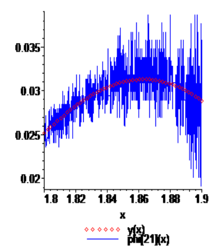
5cm

Fig. 2: h = -1, k = 12



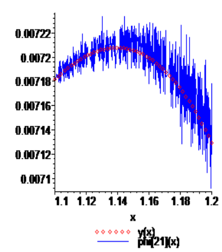
5cm

Fig. 3: h =-3, k = 8



5cm

Fig. 4: h =-2, k = 12



5cm

Fig. 5: Spurious oscillations for h =-4, k = 12

The exact solution $y(x)$ for $h = k = 1$, obtainable by Laplace Transform techniques is

$$y(x) = -\frac{1}{2} x\, \mathrm{e}^{2-x} + x\, \mathrm{e}^{1-x} + \left(1 - x\, \mathrm{e}^{1-x}\right) \mathrm{H}(x-1).$$

In order to analyze the error near the discontinuity, we use the estimated Local Order of Convergence, which is defined as follows:

Definition 1.*Let $\phi_n(x)$, $n = 1, 2, \ldots$ be the successive approximations to the solution $y(x)$ of a problem. If the positive constants K, p exist such that*

$$K = \lim_{n\to\infty} \frac{|\phi_{n+1}(x_i) - y(x_i)|}{|\phi_n(x_i) - y(x_i)|^p},$$

then we call p the (estimated) Local Order of Convergence at the point x_i. The constant K is called Convergence Factor at x_i.

In this case, the value of p is 1.174 at $x = 0.9$ and 1.169 at $x = 1.1$. So, they have essentially the same value on both sides of the discontinuity. Both the exact solution and $\phi_{12}(x)$ are represented in Fig. 6.

5cm

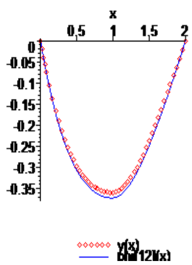


Fig. 6: h=k=1

2.1.2 Example 2

Again, this is a linear case where we take in (4)

$$f_1(x) = \sum_{n=1}^{3} \delta\left(x - \frac{n}{2}\right),\ f_2(x) = \mathrm{H}(x-1)$$

where δ is the Dirac delta function and $\mathrm{H}(x-1)$ is the Heaviside function with jump at $x = 1$. In this case, the exact solution is not available. In Fig. 7 we show two successive approximations $\phi_{27}(x)$ and $\phi_{26}(x)$.

2.1.3 Example 3

Now we take in (4)

$$f_1(x) = \sum_{n=1}^{3} \delta\left(x - \frac{n}{2}\right),\ f_2(x) = x + 1.$$

In Fig. 8 we show two successive approximations $\phi_{28}(x)$ and $\phi_{27}(x)$.

5cm

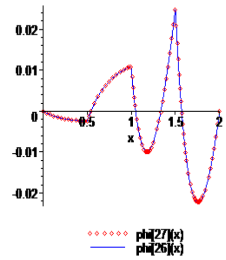


Fig. 7: h = 2, k = 5

5cm

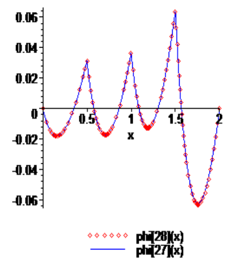


Fig. 8: h = 2, k = 5

2.1.4 Example 4

In this case we take in (4)

$$f_1(x) = \sum_{n=1}^{3} \delta\left(x - \frac{n}{2}\right),\ f_2(x) = x^3$$

Again, we compare in Fig. 9 two of our approximations $\phi_{26}(x)$ and $\phi_{25}(x)$.

5cm

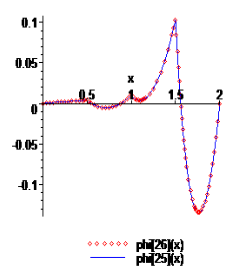


Fig. 9: h = 2, k= 5

2.2 Nonlinear Problems

Here we consider two different nonlinearities in (4): $g(y, y') = yy'$ and $g(y, y') = y^2$, with $k = 1$ and $\alpha = 1$, $\beta = 0$.

In the first case, the nonlinear term is

$$\mathrm{N}\, y = g\left(y, y'\right) = yy' = \sum_{n=0}^{\infty} A_n,$$

and the corresponding Adomian polynomials are [25]:

$$A_n = \sum_{i=0}^{n} y_{n-i} y_i',\ n \geq i,\ n = 0, 1, 2, \ldots\ .$$

For the second case, the nonlinear term is

$$\mathrm{N}\, y = g\left(y, y'\right) = y^2 = \sum_{n=0}^{\infty} A_n,$$

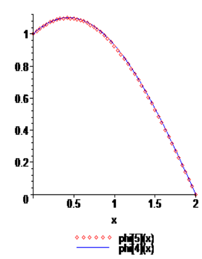
and the corresponding Adomian polynomials are [25]:

$$A_n = \sum_{i=0}^{n} y_{n-i} y_i,\ n \geq i,\ n = 0, 1, 2, \ldots\ .$$

Now we consider two examples with small and moderate values of the coefficient h in the nonlinear term $2hg\left(y, y'\right)$, where $g\left(y, y'\right) = yy'$.

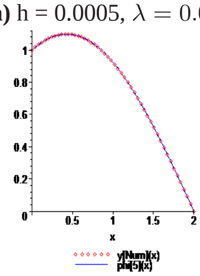
2.2.1 Example 1

In (4) we take $f_1\left(x\right) = 1$ and $f_2(x) = \mathrm{H}\left(x - 1.7\right)$, the Heaviside function with jump at $x = 1.7$.



5cm

(a) h = 0.0005, $\lambda = 0.01$,



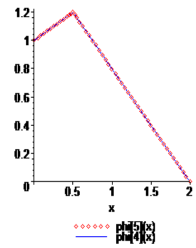
(b) h = 0.0005, $\lambda = 0.01$

Fig. 10

where $y\left[Num\right]\left(x\right)$ is the numerical solution with a finite difference method.

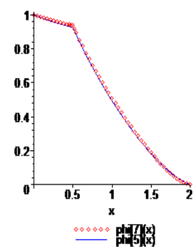
2.2.2 Example 2

In (4) we take $f_1\left(x\right) = \delta\left(x - \frac{1}{2}\right)$ and $f_2(x) = \mathrm{H}\left(x - 1\right)$.



5cm

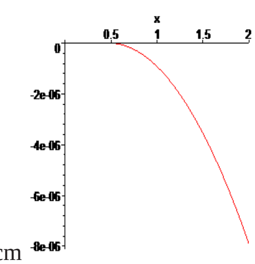
Fig. 11: h=0.0005, $\lambda = 0.01$



5cm

Fig. 12: h=0.5, $\lambda = 0$

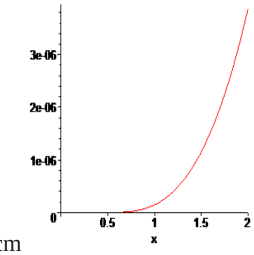
In the following Figures 13 and 14, we represent the residual error for two consecutive low order approximations.



5cm

Fig. 13: $\phi_4\left(x\right),\ h = 0.0005,\ \lambda = 0.01$

Our following nonlinear examples 3 and 4 are intended to show some limitations of the ADM in the nonlinear discontinuous problems.



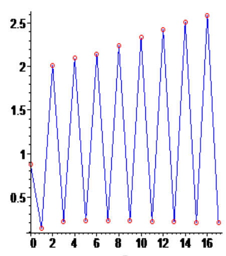
5cm

Fig. 14: $\phi_5(x)\,,\ h = 0.0005,\ \lambda = 0.01$

2.2.3 Example 3

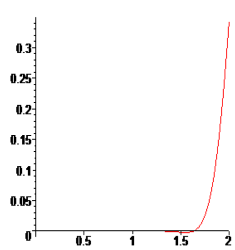
We consider some values of the parameters h and λ in the case $g(y, y') = yy'$, with $f_1(x) = 1$ and $f_2(x) = \mathrm{H}(x-1)$. The following Table 3 summarizes the results of applying Theorem 1 to a range of values of h and λ.

Figures 15 and 16 show clearly a case of divergence, according to (10), for $h = 1$ and $\lambda = 1$.



5cm

Fig. 15: Plot of $\dfrac{\|y_{n+1}\|_2}{\|y_n\|_2}$ for $n = 0, 1, \ldots, 18$



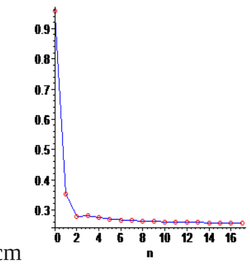
5cm

Fig. 16: Residual Error $\phi_{19}(x)$

2.2.4 Example 4

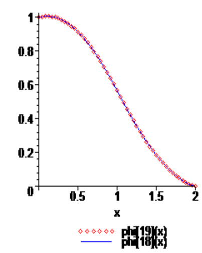
The results are very similar when we move to $g(y, y') = y^2$, leaving the rest as in Example 3 a can be shown in Table 4.

Figures 17 and 18 show a case of convergence, according to (10), for $h = 10^{-1}$ and $\lambda = 1$.



5cm

Fig. 17: Plot of $\dfrac{\|y_{n+1}\|_2}{\|y_n\|_2}$ for $n = 0, 1, \ldots, 18$



5cm

Fig. 18

3 Conclusions

To our best knowledge this is the first result on the application of Adomian Method to BVP's with these classes of discontinuities. For some of the Examples considered in this work it would be difficult to find an approximate analytical solution with the existing methods.

Taking $\mathrm{L} = \dfrac{d^2}{dx^2}$ in (6) with the Green function as the inverse operator, the convergence of the method is worse. For other choices like $\mathrm{L} = \dfrac{d^2}{dx^2} + 2h\dfrac{d}{dx}$ and $\mathrm{L} = \dfrac{d^2}{dx^2} + k^2$ in (6) with the Green function as the inverse operator, we find difficulties in carrying out the integrations, which makes the method impracticable. In all previous cases, the results with the Modified

Table 3

h	1	10^{-1}	10^{-2}	1	10^{-1}	10^{-2}	1	10^{-1}	10^{-2}
λ	1	1	1	10^{-1}	10^{-1}	10^{-1}	10^{-2}	10^{-2}	10^{-2}
ADM	Diverg.	Diverg.	Conv.	Diverg.	Diverg.	Conv.	Diverg.	Conv.	Conv.

Table 4

h	1	10^{-1}	10^{-2}	1	10^{-1}	10^{-2}	1	10^{-1}	10^{-2}
λ	1	1	1	10^{-1}	10^{-1}	10^{-1}	10^{-2}	10^{-2}	10^{-2}
ADM	Diverg.	Conv.	Conv.	Diverg.	Diverg.	Conv.	Diverg.	Conv.	Conv.

Technique did not differ significantly from those obtained previously with the Standard Adomian Method.

Moreover, we give a simple result on the convergence of the method, allowing to simultaneously checking if convergence holds. As an application, we show some cases of severe nonlinearity where the method is not convergent. For these cases we are presently developing a new and promising form of ADM, with continuation techniques to be presented in a forthcoming paper. Preliminary experiments with some Partial Differential Equations with discontinuous terms confirm that our results can be extended to this area.

Acknowledgements

We would like to thank the referees for their careful review of our manuscript.

References

[1] G. Adomian, Stochastic Systems, Academic Press, New York, 1983.
[2] G. Adomian, Nonlinear Stochastic Operator Equations, Academic Press, New York, 1986.
[3] G. Adomian, Nonlinear Stochastic Systems Theory and Applications to Physics, Kluwer Academic Publishers, Dordrecht, 1989.
[4] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Dordrecht, 1994.
[5] A. Cabada, R. L. Pouso, E. Liz, Appl. Math. Comput. 114, 135-148 (2000).
[6] A. Cabada, E. Liz, Nonlinear Anal. 28, 1491-1497 (1997).
[7] S. Heikkilä, A. Cabada, Nonlinear World 3, 487-503 (1996).
[8] S. Heikkilä, V. Lakshmikantham, Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations, Marcel Dekker, New York, 1994.
[9] M. Inc, D. J. Evans, Int. J. Comput. Math. 80 (7), 869-882 (2003).
[10] Y. Q. Hasan, Liu Ming Zhu, Comm. Nonlinear Sci. Numer. Simulat., 14 (6), 2592-2596 (2009).
[11] Bongsoo Jang, J. Comput. Appl. Math. 219, 253-262 (2008).
[12] Hossein Jafari, Varsha Daftardar-Gejji, Appl. Math. Comput. 180, 700-706 (2006).
[13] H. Hossainzadeh, G. A. Afrouzi, A. Yazdani, J. Math. Comput. Sci. 2 (4), 672-681 (2011).
[14] Luis Casasús, Waleed Al-Hayani, Appl. Math. Comput. 131, 245-251 (2002).
[15] P. W. Eloe, J. Henderson, Dyn. Contin. Discret. Imp. Syst. 4, 285-294 (1998).
[16] P. W. Eloe, M. Sokol, Dyn. Syst. Appl. 7, 441-449 (1998).
[17] K. Abbaoui, Y. Cherruault, Math. Comput. Modelling 28 (5), 103-109 (1994).
[18] K. Abbaoui, Y. Cherruault, Comput. Math. Appl. 29 (7), 103-108 (1995).
[19] K. Abbaoui, Y. Cherruault, Math. Comput. Modelling 20 (9), 60-73 (1994).
[20] Y. Cherruault, G. Adomian, Math. Comput. Modelling 18 (12), 103-106 (1993).
[21] S. Guellal, Y. Cherruault, Int. J. Biomed Comput. 36, 223-228 (1994).
[22] A. M. Wazwaz, Appl. Math. Comput. 102, 77-86 (1999).
[23] T. L. Saaty, J. Bram, Nonlinear Mathematics, Mc Graw-Hill, New York, 1964.
[24] M. M. Hosseini, H. Nasabzadeh, Appl. Math. Comput. 182, 536-543 (2006).
[25] A. M. Wazwaz, Appl. Math. Comput. 111, 53-69 (2000).

Waleed Al-Hayaniis an Assistant Professor in the Department of Mathematics, College of Computer Science and Mathematics, Mosul University, Iraq. He was an Assistant Professor, Departamento de Matemáticas, Escuela Politécnica Superior, Universidad Carlos III de Madrid, Spain (2007-2012). He received his Ph.D. degree from the Universidad Politécnica de Madrid, Spain. His fields of research include Adomian Decomposition Method, Homotopy Perturbation Method, Homotopy Analysis Method,Variational Iteration Method, Laplace Transform related to before topics and Fractional Calculus related to befor topics.