

# Modified Analog-to-Information Conversion and Reconstruction for Approximately Sparse Signal

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**Abstract:** The framework for analog-to-information conversion (AIC) as an alternative to conventional ADC is inspired by the recent theory of Compressed Sensing (CS). Both of them require that the input signal has a sparse representation in some domain. But mostly, the signals in our daily lives are approximately sparse ones. The modified analog-to-information conversion (MAIC) and a new approximately sparse signal reconstruction algorithm are proposed. The contour of the input real-time streaming signal is pre-extracted and the details are compressed, and then adaptive piece-wise basis pursuit (APBP) algorithm is used to reconstruct the input precisely. The validity of the MAIC framework and APBP algorithm is demonstrated. As is shown in the simulation, the mean square error of APBP reconstruction in the MAIC framework is of the order of about  $10^{-14}$

**Keywords:** Analog-to-Information Conversion, Compressed Sensing, Basis Pursuit, Sparse Reconstruction, Convex Optimization

## 1 Introduction

Recent theory of compressed sensing [1]-[4] (CS) proposed by Donoho, Emmanuel J. Cands, etc. in 2004 is a brand-new sampling theory compared with traditional one (Shannon's sampling theorem).

CS is only oriented to data compression now. The sampling process often used in the CS defeats one of the primary purposes of it, which is avoiding high rate sampling. Whether we can just directly measure the information that will not end up being thrown away during analog to digital conversion is the research focus.

Another practical approach to CS, which avoids high rate sampling, has been presented [5,6], and the name analog-to-information conversion (AIC) has been proposed. Generalizing the CS theory to continuous-time sparse signals, AIC measures the analog compressible signal at a low rate, compresses and digitizes signal coincidentally. At the receiving end, a  $\ell_1$  norm minimization is solved to recover the original signal.

However, true signals in our daily lives are always not absolutely sparse. If the coefficients of the signal in some domain (e.g. wavelet) attenuate exponentially, the signal is compressible and approximately sparse. Because the whole CS theoretical fundament is based on the signal

absolute sparsity, in fact, the lack of sparsity results in the poor performance of signal recovery even complete failure.

To reconstruct approximately sparse signal in compressed sensing, the method of ASLO (smoothed  $\ell_0$  norm with analysis) is proposed based on analysis model and SLO [7]. Analysis approach can avoid the accumulative errors caused by synthesis approach, and the smoothed  $\ell_0$  norm makes the optimization process simpler. An analog signal measured by a number of parallel branches of mixers and integrators (BMIs) is compressed in the meantime [8]. The segmented compressed sampling (CS) method for AIC is proposed.

In this paper, a modified AIC (MAIC) structure and a new approximately sparse signal reconstruction algorithm is proposed. Firstly, the contour of the original signal is extracted at a low rate. Secondly, the detail of signal multiplies by transformation matrix and then is measured by conventional AIC. Last but not least, adaptive piece-wise basis pursuit (APBP) algorithm is used to reconstruct the detail information of the original signal. Contour pre-extraction, signal effectivity detection and some other techniques are introduced in recovery, and the simulation displays the excellent performance of MAIC-APBP reconstruction.

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## 2 Modified Analog-to-Information Conversion

The structure of the MAIC and its reconstruction is shown in Fig.1. Firstly, the contour of the original signal  $a[n]$  is extracted by low rate ADC, and compressing was applied to measuring only fine scale properties (detail information) of the signal  $d[m]$ . Secondly, the detail of signal multiplies by transformation matrix  $\Psi$  and then is measured by conventional AIC. The detail measurements and the contour are encoded for next transmitting or store, and the process of data acquisition is complete. Last but not least, APBP algorithm is used to reconstruct the detail information of the original signal. Adding by signal contour, reconstruction signal  $\hat{x}(t)$  is acquired. Because of the contour sampling rate is different from the detail one, the interpolator is supposed to be used to match them to each other.

As is shown in the structure of the MAIC,  $\Psi$  and  $\Psi^T$  are some orthogonal sparse transform (e.g. Fourier, DCT, etc.) matrix and the inverse one. Assuming that  $T$  is the transform operator,  $\Psi = T(\mathbf{E})$  where  $\mathbf{E}$  is a  $N \times N$  unit matrix and  $N$  is the length of each processing frame. The components in each row of  $\Psi$  are multiplied by the input signal which share the clock with the PN series, and the product is held until next rising edge of the clock. The circuit design of the multiplier and integrator of this kind is discussed in [9].

In the structure of MAIC, the input is an analog signal and the matrix  $\Psi$  is pre-stored in the memory. Once the clock rising edge comes, the sum of products made by multiplying the elements in each row of  $\Psi(\psi_{i,\cdot})$  and the signal value in that time is modulated by PN series. Every frame consists of  $N$  points and the time interval of each point is  $T_1$ . It can be expressed as

$$\beta(\tau)|_{iT} = \beta_i = \sum_{n=1}^N \psi_i(n) \cdot d(t)|_{t=nT_1} \quad (1)$$

the detail information measurements in one frame

$$\begin{aligned} d[m] &= \langle \beta(\tau) p_c(\tau), h^*(t-\tau) \rangle |_{t=mT_2} \\ &= \int_{-\infty}^{+\infty} \beta(\tau) p_c(\tau) h(t-\tau) d\tau |_{t=mT_2}, m \in [1, M] \end{aligned} \quad (2)$$

where  $T_2$  is the sample interval of ADC2 and  $M$  is the number of measurements in a frame. Substituting for  $\beta(\tau)$  in (1), the relations below are obtained.

$$d[m] = \sum_{n=1}^N d(t)|_{nT_1} \cdot \int_{-\infty}^{+\infty} p_c(\tau) \cdot \psi_i(n) \cdot h(mT_2 - \tau) d\tau$$

and

$$\mathbf{d}_m = \Phi_{m,n} \mathbf{d}_n \quad (3)$$

in matrix format (the index  $i$  and  $\tau$  are correlated), where

$$\begin{aligned} \Phi_{m,n} &= \int_{-\infty}^{+\infty} p_c(\tau) \cdot \psi_i(n) \cdot h(mT_2 - \tau) d\tau, \\ & m \in [1, M], n \in [1, N] \end{aligned} \quad (4)$$

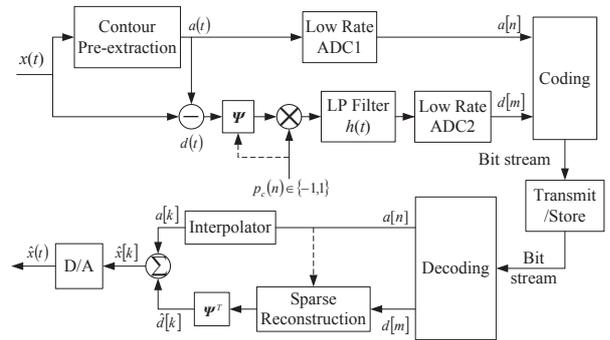


Fig. 1: The structure of MAIC based on contour pre-exaction

is the measurement matrix as an analog system.

The extended measurement matrix in (4) satisfies the RIP [10].

In the following section, we will improve the algorithm of AIC signal reconstruction by introducing signal effectivity judgment, contour protection and detail sparsifying.

## 3 Adaptive Piece-wise Basis Pursuit

### 3.1 The Principle of Adaptive Segmentation

An analog signal measured by a number of parallel BMIs is first segmented in time into  $M$  segments uniformly [8]. In this paper, different from former ones, signal segmentation depends on the coarse contour information and the signal effectivity which defines whether the signal contains useful information or only noise is judged at the same time. Take voice signal acquisition for example, the speaker's voice is an effective signal and the interval among his voice is not. In general, the input is usually a real-time streaming signal with unknown effectivity and the MAIC can be always continuing theoretically, so this kind of signal should be separated into several pieces to reconstruct live. The MAIC must work online rather than offline like CS processing.

In this paper, we detect the signal effectivity and segment the detail measurements into  $K$ -sparse ones through its contour information. The algorithm schematic diagram is shown as Fig.2.

Before segment, the contour differential between current value and previous one is calculated. The difference is zero (or very tiny) means it is a breathing space and the frame can be lengthened without increasing the sparsity, while the difference is nonzero implies the signal is effective and the detail measurements are supposed to be segmented once the number of these nonzero differentials reach a certain value. Associated with detail segment, the iteration which is a PH

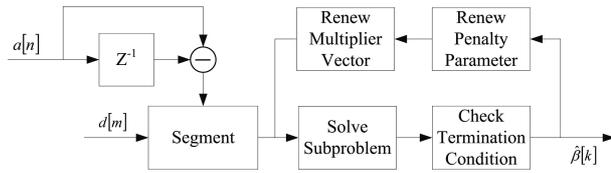


Fig. 2: The schematic diagram of APBP algorithm

process [11] (proposed by Powell and Hestenes) is applied to global optimization of sparse signal reconstruction under BP principle, and adaptive piece-wise basis pursuit algorithm is proposed.

### 3.2 Adaptive Piece-wise Basis Pursuit Algorithm

The principle of BP is to find a representation of the signal whose coefficients have minimal  $\ell_1$  norm [12]. Formally, one solves the problem

$$\min \|\beta\|_1 \quad \text{subject to} \quad \left\| d_m - \Phi_{m,n} \Psi_{k,n}^T \beta_n \right\|_2 \leq \delta \quad (5)$$

Let  $A^{CS} = \Phi_{m,n} \Psi_{k,n}^T$  be the measurement matrix of sparse coefficients  $\beta_n$ .

Input: Contour samples  $a[n]$ , detail measurements  $d[m]$  and measure matrix  $\Phi_{m,n}$ .

Output: The estimation of signal sparse coefficients  $\hat{\beta}_n$ .

**Step 0** (Initialization): Segment detail measurements, preset  $\beta_0 \in \mathbb{R}^n$ ,  $\lambda_1 \in \mathbb{R}$ ,  $\sigma_k > 0$ ,  $0 \leq \varepsilon \ll 1$ ,  $\vartheta \in (0, 1)$ ,  $\eta > 1$ . Let  $k := 1$ .

**Step 1** (Solve Sub-problem): Solve the minimum point  $\beta_k$  of the unconstrained minimization

$$\min H(\beta, \lambda_k, \sigma_k) = \min \left\{ \|\beta\|_1 - \lambda_k^T \cdot h(\beta) + \frac{\sigma_k}{2} \|h(\beta)\|_2^2 \right\} \quad (6)$$

with the initial value  $\beta_{k-1}$  and

$$h(\beta) = \left\| d_m - A^{CS} \beta_n \right\|_2^2 - \eta.$$

**Step 2** (Check Termination Condition): If  $h(\beta_k) \leq \varepsilon$ , the iteration stops and exports  $\beta_k$  as the approximately minimum point of the minimization, or else go to **Step 3**.

**Step 3** (Renew Penalty Parameter): If  $\left\| A^{CS} \beta_k - d_m \right\|_2^2 \geq \vartheta \left\| A^{CS} \beta_{k-1} - d_m \right\|_2^2$ , let  $\sigma_{k+1} := \eta \sigma_k$ , or else  $\sigma_{k+1} := \sigma_k$ .

**Step 4** (Renew Multiplier Vector): Let  $\lambda_{k+1} = \lambda_k - \sigma_k h(\beta)$  and  $k := k + 1$ . Go to **Step 1**.

PH process is a global optimization of sparse signal reconstruction by introducing Lagrange function and proper penalty function. By contrast, the augmented

objective function of previous penalty function algorithm under BP principle becomes more and more ill-posed when penalty parameter  $\sigma_k \rightarrow +\infty$ .

### 3.3 Theoretical analysis of APBP

There is a connection between BP and LP, and the BP problem (5) can be equivalently reformulated as a perturbed linear program defined in terms of a variable  $x \in \mathbb{R}^p$  in the standard form [12]

$$\min f(x) \quad \text{subject to} \quad h(x) = 0,$$

specifically

$$\min c^T x \quad \text{subject to} \quad \|Ax - b\|_2^2 = \eta, \quad x \geq 0 \quad (7)$$

by making the following translations:

$$p \Leftrightarrow 2N, \quad A \Leftrightarrow \begin{pmatrix} A^{CS} \\ -A^{CS} \end{pmatrix}, \quad b \Leftrightarrow d_m, \quad c \Leftrightarrow (1, 1)^T, \quad x \Leftrightarrow \begin{pmatrix} u \\ v \end{pmatrix}, \quad \beta = u - v, \quad \eta \text{ is the relaxation factor which change inequality constraints into equality ones.}$$

Hence the solution of (5) can be obtained by solving an equivalent perturbed linear program. Perturbed LP is really quadratic programming, but it retains a structure similar to LP. To prove the validity of APBP algorithm, the KT point  $(x^*, \lambda^*)$  of equality constraint optimization (7) should be discussed.

**Theorem 1.** *The Kuhn-Tucker point  $(x^*, \lambda^*)$  of equality constraint optimization (7) satisfies the 2<sup>nd</sup> order sufficiency condition, that is,  $\nabla L(x^*, \lambda^*) = 0$  and any  $0 \neq d \in \mathbb{R}^{2N}$  which satisfies  $\nabla h_i(x^*)^T d = 0$  makes  $d^T \nabla_{xx}^2 L(x^*, \lambda^*) d > 0$ .*

**Proof.** The objective function and constraint condition of optimization (7) are twice continuously differentiable ( $f''(x) \equiv 0$ ).

According to the definition, KT point  $(x^*, \lambda^*)$  satisfies  $\nabla_x L(x^*, \lambda^*) = 0$ , i.e.,  $\nabla f(x^*) - \lambda^* \nabla h(x^*) = 0$  and  $x^*$  meet the constraint condition that  $h(x^*) = 0$ .

$$\text{Obviously, } \nabla L(x^*, \lambda^*) = \begin{bmatrix} \nabla_x L(x^*, \lambda^*) \\ -h(x^*) \end{bmatrix} = 0.$$

$$\begin{aligned} \because h(x) &= \|Ax - b\|_2^2 - \eta = \|Ax\|_2^2 - 2b^T Ax + \|b\|_2^2 - \eta \\ \therefore \nabla^2 \|Ax\|_2^2 &= 2A^T A, \quad \nabla h(x) = \nabla \|Ax\|_2^2 - 2A^T b, \\ \nabla^2 h(x) &= \nabla^2 \|Ax\|_2^2 = 2A^T A \end{aligned}$$

$$\begin{aligned} \therefore \nabla_{xx}^2 L(x, \lambda) &= \nabla^2 f(x) - \lambda \nabla^2 h(x) = -2\lambda A^T A \\ \therefore d^T \nabla_{xx}^2 L(x^*, \lambda^*) d &= -2\lambda^* d^T A^T A d = -2\lambda^* \|Ad\|_2^2 \end{aligned}$$

Because of  $\|Ad\|_2^2 \geq 0$ ,  $d \neq 0$ , the decision rule  $d^T \nabla_{xx}^2 L(x^*, \lambda^*) d > 0$ , if  $\lambda^* < 0$ ,  $Ad \neq 0$ .

In summary, The Kuhn-Tucker point  $(x^*, \lambda^*)$ ,  $\lambda^* < 0$  of equality constraint optimization (7) satisfies the 2<sup>nd</sup> order sufficiency condition.

According to literature [11], we know that if Kuhn-Tucker point  $(x^*, \lambda^*)$  satisfies the 2<sup>nd</sup> order

sufficiency condition,  $\mathbf{x}^*$  is the rigid local minimum of augmented objective function  $H(\mathbf{x}, \lambda^*, \sigma)$  (defined in (6)). Because the solution of (5) can be obtained by solving optimization (7), the validity APBP algorithm is proved.

#### 4 Simulation Results and Discussion

To verify the effectiveness of the MAIC and APBP reconstruction algorithm, we use a group of signals (sparse and approximately sparse ones) to simulate in Matlab/Simulink environment. This section also draws a reconstruction comparison among several typical algorithms, such as CoSaMp[13], OMP[10,14], CVX[15] (internally using SeDuMi), Simplex and Interior-point.

The most signals in our daily lives are not time sparse, and they may be approximately sparse in some transformation domain. The recovery performance of approximately sparse signal sampled by MAIC is checked.

Fig.3 describes the situation in dimension  $N = 256$  and the sparsity level  $K = 10$ . It shows the Mean Square Error as a function of measurement rate that the ratio of measurements amount to total amount. Each curve represents a different reconstruction algorithm. As expected, when the ratio increases, the MSE between the reconstruction signal and the original one decreases, and under the same ratio, the MSE of MAIC is lower than that of AIC. The result may not very satisfactory, because the MSEs are larger than  $10^{-4}$  even MAIC is used.

Fig.4 shows the MSEs of MAIC reconstruction versus the ratio of the number of measurements to the number of whole frame points ( $N = 256$ ). The results are shown for three different sparsity levels ( $K$ ) of 15, 30 and 45. It can be seen from the figure that better recovery quality is achieved by using the APBP algorithm as compared to the CVX optimization toolbox, and the MSEs are all less than  $10^{-10}$ , i.e., the signal is reconstructed precisely. As expected, the recovery performance in the case of the low sparsity is slightly worse than that in the case of the high sparsity. The number of measurements had better be treble the sparsity level  $K$ , obviously, if  $M < K$ , the reconstruction will be failure.

#### 5 Conclusions

A modified analog-to-information conversion structure and a new approximately sparse signal reconstruction algorithm are proposed. The contour of the original signal is pre-extracted and the detail of signal multiplies by transformation matrix and then is measured by conventional AIC. To reconstruct the signal, adaptive piece-wise basis pursuit is used. As is shown in the simulation, APBP reconstruction in the MAIC framework can be applied to the signal whether it is a time-sparse

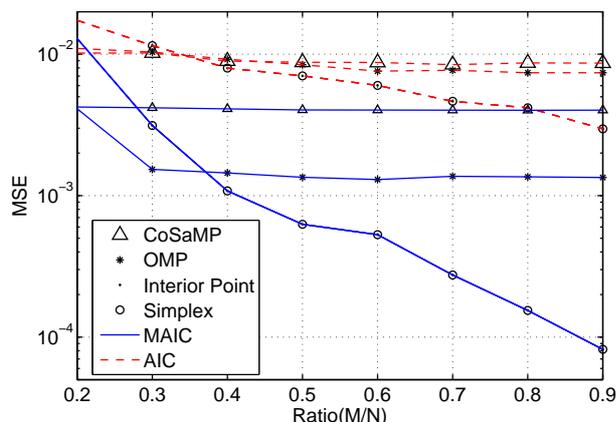


Fig. 3: Recovery comparison of the approximately sparse signal based on several typical algorithms: MSE versus  $M/N$

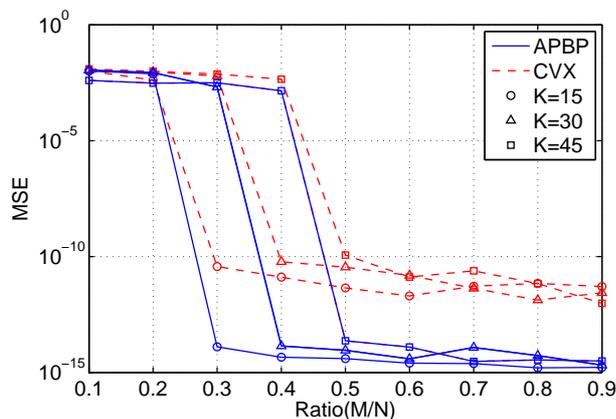


Fig. 4: Recovery comparison of the approximately sparse signal based on APBP and CVX: MSE versus  $M/N$

one or approximately sparse one in some transformation domain and the mean square error of the recovery is of the order of about  $10^{-14}$ . The validity, effectiveness, and excellent performance of the proposed MAIC framework and APBP algorithm are also justified based on our simulation results.

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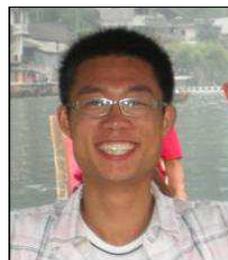
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