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Parametric Prediction Limits for Generalized Exponential Distribution Using Record Observations

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In this paper, based on a set of upper record values from the generalized exponential (GE), Bayesian, non-Bayesian and empirical Bayes estimate is derived for the parameters of the Generalized Exponential (GE) model based on record statistics. The estimate is obtained using the squared error loss and Varian's linear-exponential (LINEX) loss functions, and compared with the corresponding maximum likelihood and Bayes estimates. Empirical Bayes prediction bounds for future record values are also obtained. Finally, practical examples using real record values are given to illustrate the application of the results.

Keywords: Record statistics, maximum likelihood, Bayes estimation, empirical Bayes, squared error loss, LINEX, exponential model, quadratic loss, Monte Carlo simulation.

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1 Introduction

Recently a new distribution, named generalized exponential distribution has been introduced and studied quite extensively by authors. Generalized exponential distribution can be used as an alternative to gamma or Weibull distribution in many situations. In a companion paper, the author considered the maximum likelihood estimation of the different parameters of the generalized exponential distribution and discussed some of the testing of hypothesis problems. In this paper, the empirical Bayes estimate is derived for the parameters of the generalized exponential model based on record statistics. The two parameter generalized exponential distribution has been introduced by Gupta and Kundu (1999a,b). The GE distribution has the distribution function

$$F(x;\alpha,\beta) = (1-e^{-\beta x})^{\alpha}; \qquad \alpha,\ \beta,\ x>0. \tag{1.1}$$

Therefore, GE distribution has a density function

$$f(x;\alpha,\beta) = \alpha\beta(1 - e^{-\beta x})^{\alpha-1}e^{-\beta x}, \tag{1.2}$$

a reliability function

$$R(x;\alpha,\beta) = 1 - (1 - e^{-\beta x})^{\alpha} \tag{1.3}$$

and a hazard function

$$H(x;\alpha,\beta) = \frac{\alpha\beta(1 - e^{-\beta x})^{\alpha-1}e^{-\beta x}}{1 - (1 - e^{-\beta x})^{\alpha}}. \tag{1.4}$$

Here α is the shape parameter and β is the scale parameter. GE distribution with the shape parameter α and the scale parameter β will be denoted by $GE(\alpha,\beta)$. $GE(1,\beta)$ represents the exponential distribution with the scale parameter β. It is observed in Gupta and Kundu (1999a) that the two-parameter $GE(\alpha,\beta)$ can be used quite effectively in analyzing many lifetime data, especially in place of two-parameter gamma and two-parameter Weibull distributions. The two-parameter $GE(\alpha,\beta)$ can have increasing and decreasing failure rates depending on the shape parameter. Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed random variables with cdf $F(x)$ and pdf $f(x)$. Set $Y_{\ m} = \max(X_1, X_2, X_3, \ldots, X_n)$, $n \geq 1$, we say that X_j is an upper record and it is denoted by $X_{U(j)}$ if $Y_j > Y_{j-1}$, $j > 1$. Let $X_{U(1)}, X_{U(2)}, X_{U(3)}, \ldots, X_{U(n)}$ be the first n upper record values arising from a sequence $\{X_i\}$ of i.i.d GE variables with pdf (1.1) and cdf (1.2). For more details on record values, see Arnold, Balakrishnan and Nagaraja (1998), Chandler (1952) and Soliman, Abd Ellah and Sultan (2006). As most statisticians are interested mainly in controlling the amount of variability, it has become standard practice to consider squared error loss function (*s.e.l*) (symmetric). The symmetric nature of this function gives equal weight to overestimation and underestimation, while in the estimation of parameters of life time model overestimation may be more serious than underestimation or vice-versa. For example, in the estimation of reliability and failure rate functions, an overestimate is usually much more serious than underestimate, and the use of symmetric loss function may be inappropriate as has been recognized by Basu and Ebrahimi (1992). This leads us to think that an asymmetrical loss function may be more appropriate. A number of asymmetric loss functions have been proposed for use. One of the most popular asymmetric loss function is (linear-exponential) loss function (LINEX), which was introduced by Varian (1975). Various authors including Basu and Ebrahimi (1992) have used this loss function in different estimation problems. This function rises approximately exponentially on one side of zero and approximately linearly on the other side. Under the assumption that the minimal loss occurs at $\phi^* = \phi$, the LINEX loss function for $\phi = \phi(\alpha,\beta)$ can be expressed as

$$L(\Delta) \propto \exp(c\Delta) - c\Delta - 1 \qquad c \neq 0. \tag{1.5}$$

where $\Delta = (\phi^* - \phi)$, ϕ^* is an estimate of ϕ. The sign and magnitude of the shape parameter c represents the direction and degree of symmetry respectively. (if $c > 0$, the overestimation is more serious than underestimation, and vice-versa). For c closed to zero, the LINEX loss is approximately squared error loss and therefore almost symmetric.

The posterior expectation of the LINEX loss function (1.5) is

$$E_\phi[L(\phi^* - \phi)] \propto \exp(c\phi^*)E_\phi[\exp(-c\phi)] - c(\phi^* - E_\phi(\phi)) - 1, \tag{1.6}$$

where $E_\phi(\cdot)$ denotes posterior expectation with respect to the posterior density of ϕ. The Bayes estimator of ϕ, denoted by ϕ^*_{BL} under the LINEX loss function is the value ϕ^* that minimizes (1.6). It is found to be

$$\phi^*_{BL} = -\frac{1}{c}\ln\{E_\phi[\exp(-c\phi)\}, \tag{1.7}$$

provided that the expectation $E_\phi[\exp(-c\phi)]$ exists and is finite [Calabria and Pulcini (1996)].

The objective of this paper is to obtain and compare several types of estimation based on record statistics for the two unknown parameters of the generalized exponential distribution, and the survival time parameters, namely the hazard and Reliability functions. A discussion of the maximum likelihood estimators is also included in section 2. In section 3, the Bayes estimators of the parameters of the model as well as the reliability and hazard functions are derived based on upper record values, using the conjugate prior on the shape parameter and discretizing the scale parameter to a finite number of values. The estimates are obtained using both the symmetric loss function (*s.e.l.*) and the asymmetric loss function (varian's linear - exponential (LINEX)). The maximum likelihood and Bayes estimates are compared via Monte Carlo simulation study. The section 6 provides Bayes prediction for future record with numerical example.

2 Maximum Likelihood Estimation

The joint density function of the first n upper record values $x \equiv (X_{u(1)}, X_{u(2)}, \ldots, X_{u(n)})$ is given by

$$f_{1,2,\ldots,n}(x_{U(1)}, x_{U(2)}, x_{U(3)}, \ldots, x_{U(n)}) = f(x_{U(n)}) \prod_{i=1}^{n-1} \frac{f(x_{U(i)})}{1 - F(x_{U(i)})},$$
$$-\infty < X_{U(1)} < X_{U(2)} < \cdots < X_{U(n)} < \infty, \tag{2.1}$$

where $f(\cdot)$ and $F(\cdot)$ are given, respectively, by (1.1) and (1.2) after replacing x by $x_{U(i)}$. The likelihood function (2.1) reduces to

$$\ell(\alpha, \beta|\underline{x}) = \alpha^n\beta^n e^{-\beta\sum_{i=1}^n x_{U(i)}} \prod_{i=1}^{n}(1 - e^{-\beta x_{U(i)}})^{\alpha-1}. \tag{2.2}$$

The log-likelihood function is

$$L(\alpha,\beta|\underline{x}) \equiv \ln \ell = n\ln(\alpha) + nln\beta + (\alpha-1)\sum_{i=1}^{n} ln(1-e^{-\beta x_{U(i)}}) - \beta\sum_{i=1}^{n} x_{U(i)}. \quad (2.3)$$

The normal equations become

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} ln(1-e^{-\beta x_{U(i)}}) = 0, \quad (2.4)$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} + (\alpha-1)\sum_{i=1}^{n} \frac{x_{U(i)}e^{-\beta x_{U(i)}}}{(1-e^{-\beta x_{U(i)}})} - \sum_{i=1}^{n} x_{U(i)} = 0. \quad (2.5)$$

From equation (2.4), we obtain the maximum likelihood estimate (MLE) of α as function of β, say $\hat{\alpha}(\beta)$, where

$$\hat{\alpha}(\beta)_{ML} = \frac{-n}{\sum_{i=1}^{n} ln(1-e^{-\beta x_{U(i)}})}. \quad (2.6)$$

The corresponding MLE's $\hat{R}_{ML}\ (t)$ and $\hat{H}_{ML}\ (t)$ of $R(t)$ and $H(t)$ are given respectively by equations (1.3) and (1.4) after replacing α by $\hat{\alpha}_{ML}$. See Cohen and Whitten (1998), Gertsbakh (1989), Meeker and Escobar (1998) and Hastings (2001).

3 Bayes Estimation

For Bayesian estimation, we assume a gamma (conjugate prior) density for α with parameters a, b, and pdf

$$g(\alpha) = \frac{b^a}{\Gamma(a)}\alpha^{a-1}e^{-b\alpha}, \qquad \alpha > 0,\ a > 0,\ b > 0. \quad (3.1)$$

It follows, from (2.2) and (3.1), that the posterior density of α, for a given $\underline{x}$, is given by

$$P(\alpha|\underline{x}) = \frac{(b+x_{U(n)})^{n+a}}{\Gamma(n+a)}\alpha^{n+a-1}e^{-(b+x_{U(n)})},\ \alpha > 0. \quad (3.2)$$

Under the squared error loss function, the Bayes estimator of α, denoted by $\hat{\alpha}_{BS}$, is the mean of the posterior distribution which can be shown to be

$$\hat{\alpha}_{BS} = \frac{n+a}{b+x_{U(n)}}. \quad (3.3)$$

Under the LINEX loss function (1.5), when $\Delta = \hat{\alpha} - \alpha$, the Bayes estimate $\hat{\alpha}$ of α is obtained by using (1.7) as

$$\hat{\alpha}_{BL} = \frac{n+a}{c}\ln(1+\frac{c}{b+x_{U(n)}}), \qquad c \neq 0. \quad (3.4)$$

4 Empirical Bayes Estimation

When the prior parameters a and b are unknown, we may use the empirical Bayes approach to get their estimates. Since the prior density (3.1) belongs to a parametric family with unknown parameters, such parameters are to be estimated using past samples. Applying these estimates in (2.4) and (2.5), we obtain the empirical Bayes estimates of the parameter α based on squared error and LINEX loss functions, respectively. For more details on the empirical Bayes approach, see Maritz and Lwin (1989).

When the current (informative) sample is observed, suppose that there are available m past similar samples $X_{j,U(1)}, X_{j,U(2)}, X_{j,U(3)}, \ldots, X_{j,U(n)},\ j = 1, 2, \ldots, m$ with past realizations $\alpha_1, \alpha_2, \cdots \alpha_m$ of the random variable α. Each sample is assumed to be an upper record sample of size n obtained from the exponential distribution with pdf given by (1.1). The LF of the j^{th} sample is given by (2.1) with $x_{U(n)}$ being replaced by $x_{j,U(n)}$. For a sample j, $j = 1, 2, \ldots, m$, the maximum likelihood estimate of the parameter α_j is obtained from (2.2) and written as

$$\hat{\beta}_j = n/x_{j,U(n)}. \tag{4.1}$$

The pdf of $x_{j,U(n)}, j = 1, 2, \ldots, m$, is given, by

$$\begin{aligned} f_{j,U(n)} &= f(x)\frac{(-\ln(1-F(x)))^{n-1}}{(n-1)!} \\ &= \frac{\alpha_j^n}{\Gamma(n)} x_{j,U(n)} e^{-\alpha_j x_{j,U(n)}}, x_{j,U(n)} > 0, \end{aligned} \tag{4.2}$$

which is gamma with parameters (n, α_j). Therefore, the conditional pdf of Z_j for a given α_j is obtained from (4.2) and is given by

$$f(z_j|\alpha_j) = \frac{(n\alpha_j)^n}{\Gamma(n) z_j^{n+1}} e^{-n\alpha_j/z_j}, \qquad z_j > 0, \tag{4.3}$$

which is the inverted gamma with parameters (n, α_j). Following Schafer and Feduccia (1972) and using (3.1) and (4.3), the marginal pdf of $Z_j, j = 1, 2, \ldots, m$, can be shown to be

$$\begin{aligned} f(z_j) &= \int_0^\infty f(z_j|\alpha_j) g(\alpha_j)\, d\alpha_j \\ &= \frac{b^a n^n z_j^{a-1}}{\beta(n,a)(n + bz_j)^{n+a}}, \qquad z_j > 0. \end{aligned} \tag{4.4}$$

Therefore, the moments estimates of the parameters a and b may be obtained by using (3.1) and are of the forms

$$\hat{a} = \frac{(n-1)S_1^2}{(n-2)S_2 - (n-1)S_1^2}, \tag{4.5}$$

$$\hat{b} = \frac{nS_1}{(n-2)S_2 - (n-1)S_1^2}, \tag{4.6}$$

where $S_1 = \sum_{j=1}^{m} z_j/m$ and $S_1 = \sum_{j=1}^{m} z_j^2/m$. Therefore, the empirical Bayes estimates of the parameter α under the squared error and LINEX loss functions are given, respectively, by

$$\overset{\wedge}{\alpha}_{EBS} = \frac{n + \hat{a}}{\hat{b} + x_{U(n)}}, \tag{4.7}$$

$$\overset{\wedge}{\alpha}_{EBL} = \frac{n + \hat{a}}{c} \ln\left(1 + \frac{c}{\hat{b} + x_{U(n)}}\right), \quad c \neq 0, \tag{4.8}$$

where $\hat{a}$ and $\hat{b}$ are given by (4.5) and (4.6).

5 Prediction of Future Record Values

In the context of prediction of the future record observations, the prediction intervals provide bounds to contain the results of a future record based upon the results of the previous record observed from the same sample, see Abd Ellah (2003), Abd Ellah and Sultan (2005), Escobar and Meeker(1999). This section is devoted to drive the Bayes predictive density function, which is necessary to obtain bounds for predictive interval of future record. Suppose that we observe only the first n upper recorded observations $x \equiv (\ X_{u(1)},\ X_{u(2)}, \ldots,\ X_{u(n)})$, and the goal is to obtain the Bayes predictive interval for the s^{th} future upper record , where $1 \leq n < s$. Let $Y \equiv X_{u(s)}$ be the s^{th} upper record value, the conditional density function of Y for given $x_n = X_{u(n)}$ is given by

$$f(y|x_n;\alpha) = \frac{[w(y) - w(x_n)]^{s-n-1}}{\Gamma(s-n)} \frac{f(y)}{1 - F(x_n)}, \tag{5.1}$$

where $w(\cdot) = -\ln[1 - F(\cdot)]$.

Using the exponential distribution, with pdf given by (1.1), the conditional density function (5.1) is

$$f(y|x_n) = \frac{(y - x_n)^{s-n-1}}{\Gamma(s-n)} e^{-\alpha(y - x_n)}, \quad y > x_n. \tag{5.2}$$

The Bayes predictive density function of y given the observed record $\underline{x}$ is given by

$$f(y|\underline{x}) = \int\limits_{\alpha} f(y|\alpha) P(\alpha|\underline{x}) d\alpha, \tag{5.3}$$

where $f(y|x_n)$ is the conditional pdf of Y for the given parameter α and $P(\alpha|\underline{x})$ is the posterior density function of α for the given informative data. The Bayes predictive density function of the future record is obtained by substituting from (3.2) and (5.2) into (5.3).

$$f(y|\underline{x}) = \frac{(b+x_n)^{n+a}}{\beta(n+a,s-n)}(y-x_n)^{s-n-1}(b+y)^{-(s+a)}, \quad y > x_n, \tag{5.4}$$

where $\beta(\cdot,\cdot)$ is the beta function. Bayesian prediction bounds of $Y = X_{U(n)}$, given the previous data, are obtained by evaluating $Pr(Y \geq t|\underline{x})$, for some positive t. It follows from (5.4) that

$$\begin{aligned} Pr(Y \geq t|\underline{x}) &= \int_t^\infty f(y|x_n)dy \\ &= \frac{IB(n+a,s-n,\gamma(t))}{\beta(n+a,s-n)}, \end{aligned} \tag{5.5}$$

where $IB(\cdot,\cdot)$ is the incomplete beta function and $\gamma(t) = b + x_n/(b+t)$. It can be easily shown that $f(y|x_n)$ is a density function on the positive half of the real line by proving that $Pr(Y \geq t|\underline{x}) = 1$ and $IB(z_1, z_2, \xi)$ is the incomplete beta function defined by

$$InBet(z_1, z_2, \xi) = \int_\xi^\infty t^{z_1-1}/(1+t)^{z_1+z_2}dt.$$

To obtain the lower and upper $100\tau\%$ prediction bounds for $Y = X_{U(s)}$ by finding t from equation (5.5), we use

$$Pr[LL(\underline{x}) < Y < UL(\underline{x})] = \tau, \tag{5.6}$$

where $LL(\underline{x})$ and $UL(\underline{x})$ are the lower and upper limits, respectively, satisfying

$$Pr[Y > LL(\underline{x})|\underline{x}] = (1+\tau)/2, \qquad Pr[Y > UL(\underline{x})|\underline{x}] = (1-\tau)/2. \tag{5.7}$$

For a special case, it is often important to predict the first unobserved record value $X_{U(n+1)}$. The predictive survival function for $Y_{n+1} = X_{U(n+1)}$ is obtained from (5.5) by setting $s = n+1$, in the form

$$Pr(Y_{n+1} \geq t_1|x) = (\frac{b+x_n}{b+t_1})^{n+a}. \tag{5.8}$$

Iterative numerical methods are also needed to obtain prediction bounds for Y_{n+1}, the lower and upper limits satisfying

$$LL(\underline{x}) = (b+x_n)(\frac{1+\tau}{2})^{-1/(n+a)} - b, \tag{5.9}$$

$$UL(\underline{x}) = (b+x_n)(\frac{1-\tau}{2})^{-1/(n+a)} - b. \tag{5.10}$$

6 Numerical Examples

In this section, the maximum likelihood and Bayes (squared error and LINEX) estimates are compared based on the Monte Carlo simulation study and examples are given to illustrate the result of prediction, see Sinha and Gutman (1976), Sugita (2002), Lawless (1982), Lindsey (1996).

Example 6.1. Let us consider the first seven upper record values simulated from an exponential distribution (1.1) with scale parameter $\alpha = 1.297, \beta = 1$:

$$0.39165, 1.36774, 1.46127, 1.6482, 1.81426, 1.89462, 1.9464.$$

Using this record values the different estimates of α, $R(t)$, and $H(t)$ are computed according to the following steps:

1. We approximate the prior for a, b over the intervals $(2.5, 3.4)$ and $(.235, 1.413)$. There is no further prior information about a nonparametric procedure can be use to estimate any two different values of the reliability function $R(t_1)$ and $R(t_2)$, see Maritz and Waller (1982, p.105).

2. Based on the generated value α, an upper record sample of size $n = 7$ is then generated from the density of the exponential $\alpha = 1.297, \beta = 1$ distribution defined by (1.1), which is considered to be the informative sample.

3. Using these data, 97.5% Bayes prediction interval for the future upper record values $x_{U(8)}$ is computed using (5.8) and given by $(1.9564, 3.342)$.

4. For given values of a, b and n, we generate a random sample (past sample) $z_{j,n}, j = 1, 2, \ldots, m$ of size $m = 10$ from the marginal density of $z_{j,n}$ given by (1.16). We substitute the values of $(R(t_1), t_1), (R(t_2), t_2)$ obtained in step 1 into equation (4.4), where a_j and b_j are solved numerically for each given $b_j, j = 1, 2, \ldots, 10$, and use the Newton-Raphson method. The resulting values of the hyper parameters (a_j, b_j) in the gamma prior as well as the posterior probabilities for each b_j (see equation (3.9)), are given in Table 6.2.

5. The ML estimates $(\cdot)_{ML}$, and the Bayes estimates $((\cdot)_{BS}, (\cdot)_{BL})$ of α, β, $R(t)$, and $H(t)$, are computed using results in section 4. The results are presented in Table 6.3.

Example 6.2. The above prediction procedure is demonstrated by using a simulated sets of record from the exponential model (1.1). Samples of upper records values of size $n = 3, 5, 7$ are simulated from the exponential distribution with $\alpha = 1.023, 2.543, 3.754, \beta = 1$, which include the exponential. Using our results in equation (5.8), the lower and the upper 95% prediction bounds for the next record values $X_{U(n+1)}$, for the three cases $(n = 3, 5, 7)$ are obtained and displayed in Table 6.2.

Example 6.3. Based on the seven record values from example 6.2, with the corresponding hyper parameter values obtained in the same example and using the results in (3.3) and

n	previous record values	LL	UL	Width
3	$0.0038, 1.48863, 3.44035$	3.4611	7.8041	4.3429
5	$0.429169, 0.560242, 1.51925, 2.95113, 3.81204$	3.8268	6.5684	2.7416
7	$0.3248, 2.2954, 2.5172, 2.5972, 2.6925, 2.9841, 5.3252$	5.3399	7.9039	2.5640
3	$2.27949, 2.36535, 2.56455$	2.5739	4.4117	1.8378
5	$0.72853, 0.796644, 1.35383, 1.42606, 2.27987$	2.2853	3.2637	0.9784
7	$0.4050, 1.4127, 1.6041, 1.6950, 2.0951, 2.1519, 2.6618$	2.6664	3.4409	0.7745
3	$1.3778, 1.5266, 1.67946$	1.6838	2.4290	0.7452
5	$1.08744, 1.23323, 1.58884, 1.83328, 1.86588$	1.8690	2.3615	0.4926
7	$1.3375, 1.5768, 1.6170, 1.6826, 1.7228, 1.7830, 1.8848$	1.8870	2.2395	0.3525

Table 6.1: The lower (LL) the upper (UL) and the width of the 95% prediction intervals for the future upper record $X_{U(n+1)}$, $(n = 3, 5, 7)$.

(3.4), the lower and the upper 95% prediction bounds for the next record values $X_{U(8)}$ are 73.125 and 116.725, respectively.

7 Conclusion

Based on the set of the upper record values the present paper proposed classical and Bayesian approaches to estimate the two unknown parameters as well as the reliability and hazard functions for exponential model. We also considered the problem of predicting future record in a Bayesian setting. Bayes estimators are obtained using both symmetric and asymmetric loss functions. It appears to be clear from this study that the Bayes method of estimation based on record statistics is superior to the ML method. Comparisons are made between different estimators based on simulation study and practical example using a set of real record values. The effect of symmetric and asymmetric loss functions was examined and the following were observed:

1. Table 6.1 (the case of known a and b) shows that the Bayes estimates relative to LINEX loss function has the smallest (EV) as compared with quadratic Bayes estimates or the MLE's. This is valid for all number of record values n, and the estimated variances decreases as n increases.

2. For the case of unknown shape and scale parameters, the use of a discrete distribution for the shape parameter resulted in closed form expression for the posterior pdf. The equal probabilities chosen in the discrete distributions caused an element of uncertainly, which can be desirable in some cases. Tables 6.2, 6.3, 6.4 and 6.5 shows that the Bayes estimates based on symmetric and asymmetric loss functions perform better than the MLE's, and the asymmetric Bayes estimates are sensitive to the values

		ER($\hat{\alpha}_{ML}$)			ER($\hat{\alpha}_{EBS}$)			ER($\hat{\alpha}_{EBL}$)	
n	s	97.5%	99%	c	97.5%	99%	c	97.5%	99%
5	6	0.9005	0.9501	1	0.9910	0.9974	1	0.9847	0.9951
				2	0.9923	0.9979	2	0.9064	0.9660
				3	0.9905	0.9983	3	0.7944	0.8784
	7	0.9017	0.9538	4	0.9910	0.9963	4	0.9873	0.9952
				5	0.9908	0.9963	5	0.9470	0.9816
				6	0.9927	0.9966	6	0.8375	0.9179
	8	0.9014	0.9496	7	0.9854	0.9944	7	0.9803	0.9925
				8	0.9846	0.9944	8	0.9474	0.9802
				9	0.9848	0.9951	9	0.8530	0.9265
	9	0.8987	0.9499	11	0.9617	0.9843	11	0.9541	0.9821
				12	0.9615	0.9845	12	0.9212	0.9645
				13	0.9594	0.9842	13	0.8272	0.9149
	10	0.8988	0.9506	20	0.8726	0.9381	20	0.8648	0.9330
				30	0.8708	0.9334	30	0.8193	0.8980
				40	0.8673	0.9319	40	0.7124	0.8441
7	8	0.8982	0.9517	41	0.9729	0.9925	41	0.9870	0.9972
				42	0.9864	0.9960	42	0.9451	0.9861
				43	0.9538	0.9884	43	0.8167	0.9010
	9	0.9010	0.9511	1	0.9605	0.9817	1	0.9742	0.9903
				2	0.9715	0.9881	2	0.9379	0.9765
				3	0.9356	0.9734	3	0.8296	0.9132
	10	0.8986	0.9498	1	0.8562	0.9277	1	0.8881	0.9416
				2	0.8825	0.9406	2	0.8398	0.9123
				3	0.8193	0.9074	3	0.7069	0.8305
9	10	0.8984	0.9511	1	0.6914	0.8272	1	0.7628	0.8689
				2	0.8695	0.9314	2	0.7120	0.8433
				3	0.6314	0.7909	3	0.5144	0.6772

Table 6.2: Estimated risk (ER) of the estimates of α for different values of n , s , m and c and 10000 repetitions.

		ER($\hat{\alpha}_{ML}$)		ER($\hat{\alpha}_{EBS}$)			ER($\hat{\alpha}_{EBL}$)		
n	s	90%	95%	c	90%	95%	c	90%	95%
5	6	0.8998	0.9500	1	0.8992	0.9459	1	0.8751	0.9359
				2	0.8860	0.9408	2	0.8134	0.8863
				3	0.8716	0.9356	3	0.7014	0.7894
	7	0.9026	0.2527	.25	0.9392	0.9665	0.25	0.9270	0.9641
				0.50	0.9274	0.9690	0.50	0.8848	0.9392
				0.75	0.9206	0.9629	0.75	0.7686	0.8505
	8	0.9007	0.9516	1	0.9395	.9719	1	0.9338	0.9683
				2	0.9372	0.9717	2	0.9077	0.9516
				3	0.9284	0.9703	3	0.8052	0.8851
	9	0.8992	0.9507	0.25	0.9300	0.9694	0.25	0.9245	0.9661
				0.50	0.9269	0.9655	0.50	0.9013	0.9533
				0.75	0.9168	0.9609	0.75	0.8014	0.8902
	10	0.8989	0.9493	0.01	0.8651	0.9283	0.01	0.8614	0.9236
				0.02	0.8603	0.9261	0.02	0.8376	0.9127
				0.03	0.8471	0.9221	0.03	0.7433	0.8482
7	8	0.8995	0.2521	.001	0.8886	0.9418	0.001	0.8773	0.9343
				.002	0.8764	0.9370	0.002	0.8270	0.8948
				.003	0.8632	0.9295	0.003	0.7106	0.7978
	9	0.9023	0.9511	0.25	0.9022	0.9554	0.25	0.8936	0.9476
				0.50	0.8986	0.9508	0.50	0.8575	0.9320
				0.75	0.8915	0.9446	0.75	0.7564	0.8451
	10	0.8956	0.9521	10	0.8537	0.9137	10	0.8452	0.9088
				20	0.8343	0.9083	20	0.8094	0.8903
				30.75	0.8356	0.9013	30.75	0.7084	0.8131
9	10	0.8979	0.9518	2.8	0.7878	0.8745	2.8	0.7804	0.8690
				4.9	0.7839	0.8657	4.9	0.7557	0.8435
				7.5	0.7711	0.8585	7.5	0.6511	0.7564

Table 6.3: Estimated risk (ER) of the estimates of α for different values of n , s , m and c and 10000 repetitions.

		ER($\hat{\alpha}_{ML}$)		ER($\hat{\alpha}_{EBS}$)			ER($\hat{\alpha}_{EBL}$)		
n	s	97.5%	99%	c	97.5%	99%	c	97.5%	99%
5	6	0.5828	0.8170	1.1	2.7377	3.8718	1.1	2.3604	3.4057
				2.6	2.5511	3.6452	2.6	1.3380	2.0357
				3.2	2.3420	3.3834	3.2	0.5158	0.7462
	7	1.1692	1.5622	1	3.7619	5.0900	0.25	3.4422	4.7067
				2	3.5784	4.8710	0.50	2.4092	3.4050
				3	3.3643	4.6115	0.75	1.0466	1.4600
	8	1.9284	2.5281	2	4.4212	5.8767	2	4.1794	5.5803
				3.50	4.2524	5.6732	3.50	3.2972	4.4971
				4.75	4.0508	5.4299	4.75	1.6979	2.3422
	9	3.0726	3.9942	10	4.9140	6.4611	10	4.7442	6.2576
				20	4.7595	6.2798	20	4.0357	5.3920
				30	4.5722	6.0571	30	2.4821	3.3838
	10	5.4438	7.0957	25	5.3064	6.9309	25	5.2022	6.8025
				50	5.1681	6.7673	50	4.6535	6.1365
				75	4.9965	6.5599	75	3.3403	4.4824
7	8	0.9037	1.2393	1	4.5901	3.5746	1	2.4228	3.3764
				2	2.4525	3.4141	2	1.7994	2.6102
				3	2.3071	3.4214	3	0.7657	1.1273
	9	1.9934	2.6005	1.1	3.4978	4.6129	1.1	3.3757	4.2722
				2.50	3.3575	4.4505	2.50	2.8209	3.8151
				3.75	3.2104	4.2820	3.75	1.5621	2.1871
	10	4.1852	5.3952	.25	4.0714	5.2655	0.25	3.9999	5.1861
				0.50	3.9379	5.1114	0.50	3.5624	4.6756
				0.75	3.7991	4.9527	0.75	2.4205	3.2712
9	10	2.6072	3.5311	1.25	2.5135	3.4249	1.25	2.4510	3.3533
				1.5	2.4057	3.3006	1.50	2.1176	2.9614
				2.75	2.3072	3.1894	2.75	1.2815	1.8860

Table 6.4: Estimated risk (ER) of the estimates of α for different values of n , s , m and c and 10000 repetitions.

		ER($\hat{\alpha}_{ML}$)		ER($\hat{\alpha}_{EBS}$)			ER($\hat{\alpha}_{EBL}$)		
n	s	90%	95%	c	90%	95%	c	90%	95%
5	6	0.5867	0.8242	1.25	2.7591	3.9113	1.25	2.2901	3.2705
				2.50	2.4874	3.5410	2.50	1.2345	1.7789
				3.75	2.2474	3.2132	3.75	0.5045	0.7114
	7	1.1785	1.5770	13	4.9471	6.7056	13	4.2609	5.7990
				14	4.5140	6.1361	14	2.5129	3.4517
				15	4.1343	5.6366	15	1.0049	1.3555
	8	1.9443	2.5526	100	7.1201	9.4708	100	6.2753	8.3688
				200	6.5439	8.7236	200	3.9562	5.3154
				300	6.0430	8.0745	300	1.6233	2.1601
	9	3.0986	4.0343	10	9.3354	12.2843	10	8.3609	11.0195
				20	8.6235	11.3675	20	5.5364	7.3379
				30	8.0100	10.5785	30	2.4161	3.1977
	10	5.4903	7.1635	1	11.6039	15.1607	1	10.5173	13.7527
				2	10.7598	14.0779	2	7.2204	9.4795
				3	10.0382	13.1552	3	3.4033	4.4831
7	8	0.9161	1.2604	.25	2.6210	3.6360	0.25	2.4146	3.3659
				2.50	2.4618	3.4287	2.50	1.6791	2.3740
				3	2.2404	3.1339	3	0.7295	1.0208
	9	2.0253	2.6510	1.01	4.5105	5.9662	1.01	4.2491	5.6344
				1.50	4.2751	5.6708	1.50	3.1706	4.2441
				3	3.9307	5.2297	3	1.5000	2.0091
	10	4.2552	5.5036	1	6.3174	8.1877	1	6.0354	7.8318
				2.01	6.0199	7.8189	2.01	4.7088	6.1486
				3.01	5.5690	7.2503	3.01	2.4524	3.2236
9	10	2.6963	3.6835	5	2.5820	3.5396	5	2.4781	3.4046
				10	2.4406	3.3584	10	2.0765	2.8832
				15	2.2593	3.1211	15	1.2042	1.6923

Table 6.5: Estimated risk (ER) of the estimates of α for different values of n , s , m and c and 10000 repetitions.

of the shape parameter c of the LINEX loss function. The problem of choosing the value of the parameter c is discussed in Calabria and Pulcini (1996).

3. The analytical ease with which results can be obtained using asymmetric loss functions makes them attractive for use in applied problems and in assessing the effects of departures from assumed symmetric loss functions.

4. It is clear that the variance of the exponential α distribution tends to zero as α tends to infinity. This implies that as α gets larger, the observations concentrate on a shorter domain. It then follows that the width of the predictive interval decrease as α increase, see Table 6.5. For more details on numerical solutions see Rubinstein (1981), IMSL (1984), Nelson (1982) and Gelman, Carlin, Stern and Rubin (1995).

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