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The q-Fibonacci Hyperbolic Functions

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Abstract: In 2005 Stakhov and Rozin introduced a new class of hyperbolic functions which is called Fibonacci hyperbolic functions. In this paper, we study q-analogue of Fibonacci hyperbolic functions. These functions can be regarded as q extensions of classical hyperbolic functions. We introduce the q-analogue of classical Golden ratio as follow $\phi_q = \frac{1+\sqrt{1+4q^{n-2}}}{2}$, $n \geq 2$. Making use of this q-analogue of the Golden ratio, we defined $\sin F_q h(x)$ and $\cos F_q h(x)$ functions, and also investigated some properties and gave some relationships between these functions.

Keywords: Fibonacci numbers; Fibonacci hyperbolic functions; q-calculus; q-analogue

1 Introduction

One of the simplest and most celebrated integer sequences is the Fibonacci sequence. The Fibonacci sequence is $F_n = \{0,1,1,2,3,5,...\}$ where in each term is the sum of the two proceeding terms, beginning with the values $F_0 = 0$, and $F_1 = 1$. It is interesting to emphasize the fact that the ratio of two consecutive Fibonacci numbers converges to the Golden Mean, or Golden Section, $\phi = \frac{1+\sqrt{5}}{2}$, which appears in modern research in many fields from architecture to physics of the high energy particles or theoretical physics other than mathematical areas (for instance, [1] and [2,3] *et al.*). For example, Fibonacci numbers and Golden mean are popular in many scientific disciplines, from quasi-crystals through models of DNA sequences, phyllotaxis, to the research on brain activity (e.g. EEG signals). After fractals, chaos, power laws: Minutes from an infinite paradise were introduced [9], in 2008, Yamagishi et. al. proposed evidence of Fibonacci based organization and verified it at a statistical global level accross the whole human genome. Then, they said that Chargaff's second parity rule is valid in all human chromosomes. According to the rule, the division of the frequency of one nucleotide by the sum of the frequencies of the remaining nucleotides is in the proportion of three Fibonacci numbers [17]. Also, Pletzer et al demonstrated that the pattern of excitatory phase meetings provided by the golden mean as the "most irrational" number is least frequent and most irregular. Thus, in a physiological sense, the golden mean provides (i) the highest physiologically possible desynchronized state in the resting brain, (ii) the possibility for spontaneous and most irregular(!) coupling and uncoupling between rhythms and (iii) the oppurtunity for a transition from resting state to activity [7].

Furhermore, Perez showed fractal behavior in the genome at the logical DNA analysis level. This fractal behavior provides an exhaustive analysis of codon frequencies which are clustered around 2 fractal-like attractors, strongly linked to the golden ratio, on a whole human genome scale [6].

First formula in what we now call q-calculus were obtained by Euler in the eighteenth century. Many remarkable results (like Jakobi's triple product identity and the theory of q-hypergeometric functions) were obtained in the nineteenth century. A q-analogue, also called a q-extension or q-generalization, is a mathematical expression parameterized by a quantity q that generalized a known expression and reduces to the known expression in the limit $q \to 1^-$. There are q-analogues for the fractional, binomial coefficient, derivative, integral, Fibonacci numbers and so on.

On the other hand, recently, the Fibonacci hyperbolic functions have been defined and studied as $\sin Fh(x)$ and $\cos Fh(x)$ [13] and [11,12,14,15,16].

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A sequence of polynomials $F_n(q)$ was firstly introduced as follows [8]:

$$F_n(q) = \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ F_{n-1}(q) + q^{n-2}F_{n-2}(q), & \text{if } n \geq 2. \end{cases}$$

Obviously $F_n(q)$ is the q-analogue of the Fibonacci numbers. Arithmetic properties of q-Fibonacci numbers were given [5].

Throughout this paper, we will assume that q satisfies the condition $0 < |q| < 1$. The q-derivative $D_q f$ of an arbitrary function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x},$$

where $x \neq 0$. Clearly, if f is differentiable, then

$$\lim_{q \to 1^-} (D_q f)(x) = \frac{df(x)}{dx}.$$

Before we continue, let us introduce some notation that is used in the remainder of the paper. For any real number α,

$$[\alpha] := \frac{q^\alpha - 1}{q - 1}.$$

In particular, if $n \in \mathbb{Z}^+$, it is denoted as follows [4]:

$$[n] = \frac{q^n - 1}{q - 1} = q^{n-1} + \ldots + q + 1.$$

In this paper, q-analogues of $\sin Fh(x)$ and $\cos Fh(x)$ Fibonacci hyperbolic functions are defined. Some basic properties are studied of these functions. Also, q-analogues of some equalities like Pythagorean theorem, sum and difference, half argument etc. are obtained.

Now, we introduce basic definitions which will be used throughout the paper.

2 Fibonacci Hyperbolic Functions

There are a number of fundamental results in the Fibonacci numbers theory. One of them was found in 19th century by the famous French mathematician Binet. Studying the Golden Section, Fibonacci numbers, he discovered remarkable formulas, Binet's formulas, connecting Fibonacci numbers with the Golden Section $\phi = \frac{1+\sqrt{5}}{2}$. Where ϕ is the root of the characteristic equation $(\phi^2 = \phi + 1)$ associated to the Fibonacci sequence. Binet's formulas give connections between the "extended" Fibonacci Numbers and the Golden Section may be written in the following form [10]:

$$F_n = \begin{cases} \frac{\phi^{2k+1} + \phi^{-(2k+1)}}{\sqrt{5}} & , n = 2k+1 \\ \frac{\phi^{2k} - \phi^{-2k}}{\sqrt{5}} & , n = 2k \end{cases} \tag{1}$$

Where the discrete variable k takes its values from the set $0, \pm1, \pm2, \pm3, \ldots$. The formula (1) was replaced with the continuous variable x taken its values from the set of the real numbers and then the following continuous functions called the Fibonacci hyperbolic functions were introduced [12].

Definition 1.$([12])$ *Let ϕ be Golden ratio. The functions $\sin Fh(x)$ and $\cos Fh(x)$ are defined as:*

$$\sin Fh(x) = \frac{\phi^{2x} - \phi^{-2x}}{\sqrt{5}},$$

$$\cos Fh(x) = \frac{\phi^{2x+1} + \phi^{-(2x+1)}}{\sqrt{5}},$$

and are called Fibonacci hyperbolic functions.

Notice that these functions verify the property that, if x is an even number, $x = 2n$ then $\sin Fh(x) = F_{2n}$, while if x is an odd number, $x = 2n+1$ then $\cos Fh(x) = F_{2n+1}$. Note that $\sin Fh(x)$ is symmetric with respect to the origin, while the graphics of $\cos Fh(x)$ presents a symmetry with respect to the axis $x = -\frac{1}{2}$. By making a change of variable $2x + 1 = t$ in functions $\sin Fh(x)$ and $\cos Fh(x)$ and also using $\phi + \phi^{-1} = \sqrt{5}$, another representations of Fibonacci hyperbolic *sine* and *cosine* functions are, respectively, given as follows by [12].

Definition 2.$([12])$ *Let ϕ be Golden ratio. The functions $\sin Fh(x)$ and $\cos Fh(x)$ are defined as:*

$$\sin Fh(x) = \frac{\phi^x - \phi^{-x}}{\phi + \phi^{-1}},$$

$$\cos Fh(x) = \frac{\phi^x + \phi^{-x}}{\phi + \phi^{-1}},$$

and are called Fibonacci hyperbolic functions.

Now, equalities introduced [12] can naturally be related with the q-Fibonacci hyperbolic functions studied before.

The following correlations that are similar to the equation $[\cos h(x)]^2 - [\sin h(x)]^2 = 1$ are valid for the Fibonacci hyperbolic functions:

$$[\cos Fh(x)]^2 - [\sin Fh(x)]^2 = \frac{4}{5}. \tag{2}$$

Equality (2) is called Pythagorean Theorem.

Sum and difference formulas are as follows:

$$\cos Fh(x+y) = \frac{\sqrt{5}}{2}(\cos Fh(x)\cos Fh(y) + \sin Fh(x)\sin Fh(y)) \tag{3}$$

$$\cos Fh(x-y) = \frac{\sqrt{5}}{2}(\cos Fh(x)\cos Fh(y) - \sin Fh(x)\sin Fh(y)) \tag{4}$$

$$\sin Fh(x+y) = \frac{\sqrt{5}}{2}(\sin Fh(x)\cos Fh(y) + \sin Fh(y)\cos Fh(x)) \tag{5}$$

$$\sin Fh(x-y) = \frac{\sqrt{5}}{2}(\sin Fh(x)\cos Fh(y) - \sin Fh(y)\cos Fh(x)) \tag{6}$$

By doing $x = y$ in (3) and (5) formulas, we have the following equations.

$$\cos Fh(2x) = \frac{\sqrt{5}}{2}((\cos Fh(x))^2 + (\sin Fh(x))^2) \tag{7}$$

$$\sin Fh(2x) = \sqrt{5}(\sin Fh(x)\cos Fh(x)) \tag{8}$$

Equations mentioned above give the relationship between double argument and argument. Also, from equations (2) and (7) it is deduced, respectively, by summing up and subtracting:

$$[\cos Fh(x)]^2 = \frac{1}{\sqrt{5}}\left(\cos Fh(2x) + \frac{2}{\sqrt{5}}\right)$$

$$[\sin Fh(x)]^2 = \frac{1}{\sqrt{5}}\left(\cos Fh(2x) - \frac{2}{\sqrt{5}}\right).$$

For any integer r, Catalan's identity can be given as in the following equation:

$$\cos Fh(x-r)\cos Fh(x+r) - (\cos Fh(x))^2 = (\sin Fh(r))^2$$

The following results can be obtained in a similar way.

$$\cos Fh(x-r)\cos Fh(x+r) - (\sin Fh(x))^2 = (\cos Fh(r))^2$$

$$\sin Fh(x-r)\sin Fh(x+r) - (\sin Fh(x))^2 = -(\sin Fh(r))^2$$

$$\sin Fh(x-r)\sin Fh(x+r) - (\cos Fh(x))^2 = -(\cos Fh(r))^2.$$

For $r = 1$,

$$\cos Fh(x-1)\cos Fh(x+1) - (\sin Fh(x))^2 = 1$$

$$\sin Fh(x-1)\sin Fh(x+1) - (\cos Fh(x))^2 = -1$$

can be obtained. These identities are called Cassini's or Simson's identity. Furthermore, d'Ocagne's identity is given in the following:

$$\cos Fh(x)\cos Fh(y+r) - \sin Fh(x+r)\sin Fh(y) = \cos Fh(r)\cos Fh(x-y) \tag{9}$$

$$\cos Fh(x)\sin Fh(y+r) - \cos Fh(x+r)\sin Fh(y) = \sin Fh(r)\cos Fh(x-y) \tag{10}$$

Notice that by taking $r = 1$ in equation (9) the following identity:

$$\cos Fh(x)\cos Fh(y+1) - \sin Fh(x+1)\sin Fh(y) = \cos Fh(x-y)$$

is obtained.

3 q-Analogues of Fibonacci Hyperbolic Functions

We start with the definition of q-Golden ratio which will be used in the sequal of the paper. The q-Golden ratio is positive root of characteristic equation which can be obtained by using recurrence of q-analogues of the Fibonacci numbers.

Definition 3.*Let $F_n(q)$ be a sequence of polynomials and*

$$\frac{F_n(q)}{F_{n-1}(q)} = \phi_q,$$

that is,

$$\frac{F_n(q)}{F_{n-1}(q)} = \frac{F_{n-1}(q)}{F_{n-1}(q)} + q^{n-2}\frac{F_{n-2}(q)}{F_{n-1}(q)},\ n \geqslant 2\,.$$

From now on, $\phi_q^2 = \phi_q + q^{n-2}$ $(n \geqslant 2)$ characteristic equation is obtained, where $\phi_{q_1} = \frac{1-\sqrt{1+4q^{n-2}}}{2}$ and $\phi_{q_2} = \frac{1+\sqrt{1+4q^{n-2}}}{2}$. Then, q-Golden ratio which is the positive root of characteristic equation denoted by ϕ_q, is defined as

$$\phi_q = \frac{1+\sqrt{1+4q^{n-2}}}{2},\ n \geqslant 2\,.$$

Now, we give basic definitions called $\sin F_q h(x)$ and $\cos F_q h(x)$, respectively, where $\sin F_q h(x)$ is q-analogue of *sine* Fibonacci hyperbolic function and in a similar way, $\cos F_q h(x)$ is q-analogue of *cosine* Fibonacci hyperbolic function.

Definition 4.*Let ϕ_q be Golden ratio, q-analogue of Golden ratio. For $n \geq 2$, then we define*

$$\sin F_q h(x) = \frac{\phi_q^{2x} - \phi_q^{-2x}}{\sqrt{1+4q^{n-2}}},$$

$$\cos F_q h(x) = \frac{\phi_q^{2x+1} + \phi_q^{-(2x+1)}}{\sqrt{1+4q^{n-2}}}.$$

We will call $\sin F_q h(x)$ is q-analogue of sine Fibonacci hyperbolic function and $\cos F_q h(x)$ is q-analogue of cosine Fibonacci hyperbolic function. Notice that these functions verify the property that, if x is an even number, $x = 2n$ then $\sin F_q h(x) = F_{2n}(q)$, while if x is an odd number, $x = 2n+1$ then $\cos F_q h(x) = F_{2n+1}(q)$. Note that $\sin F_q h(x)$ is symmetric with respect to the origin, while the graphics of $\cos F_q h(x)$ presents a symmetry with respect to the axis $x = -\frac{1}{2}$. Another representations q-analogue of Fibonacci hyperbolic sine and cosine functions are, respectively, given as follows.

Definition 5. *Let ϕ_q be Golden ratio, q-analogue of Golden ratio. For $n \geq 2$, then we define*

$$\sin F_q h(x) = \frac{\phi_q^x - \phi_q^{-x}}{\phi_q + \phi_q^{-1}}, \tag{11}$$

$$\cos F_q h(x) = \frac{\phi_q^x + \phi_q^{-x}}{\phi_q + \phi_q^{-1}}, \tag{12}$$

since $\phi_q + \phi_q^{-1} = \sqrt{1+4q^{n-2}}$.

Now, the graphics of these the new q-Fibonacci hyperbolic functions against the Fibonacci hyperbolic functions and for $k = 2$ the k-Fibonacci hyperbolic functions [3] are given for two values of q where the thin, medium and thick of graphics are, respectively, shown the Fibonacci hyperbolic functions, for $k = 2$ the k-Fibonacci hyperbolic functions and the new q-Fibonacci hyperbolic functions.

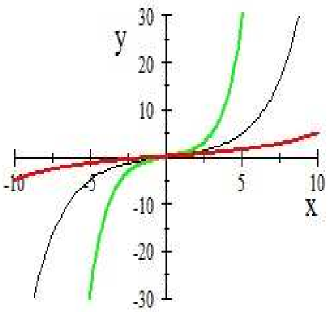

Fig. 1: (a) The thin of graphics is $\sin Fh(x)$, (b) The medium of graphics is $\sin F_2 h(x)$, (c) The thick of graphics is $\sin F_{(1/3)} h(x)$, for $n = 3$

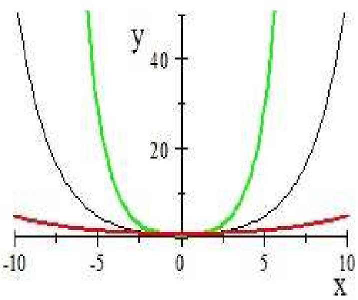

Fig. 2: (a) The thin of graphics is $\cos Fh(x)$, (b) The medium of graphics is $\cos F_2 h(x)$, (c) The thick of graphics is $\cos F_{(1/3)} h(x)$, for $n = 3$

For $q = \frac{1}{3}$ and $n = 3$, the graphics of $\sin F_{\frac{1}{3}} h(x)$ and $\cos F_{\frac{1}{3}} h(x)$ are shown in Fig. 1 and Fig. 2, respectively.

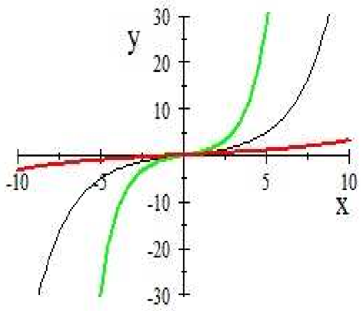

Fig. 3: (a) The thin of graphics is $\sin Fh(x)$, (b) The medium of graphics is $\sin F_2 h(x)$, (c) The thick of graphics is $\sin F_{(1/2)} h(x)$,for $n = 4$,

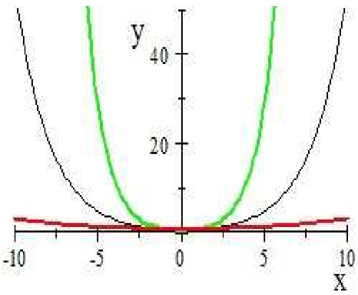

Fig. 4: (a) The thin of graphics is $\cos Fh(x)$, (b) The medium of graphics is $\cos F_2 h(x)$, (c) The thick of graphics is $\cos F_{(1/2)} h(x)$,for $n = 4$

Similarly, For $q = \frac{1}{2}$ and $n = 4$, the graphics of $\sin F_{\frac{1}{2}} h(x)$ and $\cos F_{\frac{1}{2}} h(x)$ are shown in Fig. 3 and Fig. 4, respectively.

In the sequel we present the main properties of these functions in a similar way in which the similar properties of the Fibonacci hyperbolic functions are usually presented. For $n \geq 2$, we have (13) which may be regarded as a version of the Pythagorean theorem.

Theorem 1. *(q-Fibonacci Pythagorean theorem). The main property of these functions which may be called as a version of the Pythagorean theorem is*

$$[\cos F_q h(x)]^2 - [\sin F_q h(x)]^2 = \frac{4}{1+4q^{n-2}} \tag{13}$$

for $n \geq 2$.

*Proof.*Using the definitions of q-analogue of the Fibonacci hyperbolic sine and cosine functions of (11) and (12), we get

$$\begin{aligned}
&[\cos F_q h(x)]^2 - [\sin F_q h(x)]^2 \\
&= \left(\frac{\phi_q^x + \phi_q^{-x}}{\phi_q + \phi_q^{-1}}\right)^2 - \left(\frac{\phi_q^x - \phi_q^{-x}}{\phi_q + \phi_q^{-1}}\right)^2 \\
&= \frac{\phi_q^{2x} + 2\phi_q^x\phi_q^{-x} + \phi_q^{-2x} - \phi_q^{2x} + 2\phi_q^x\phi_q^{-x} - \phi_q^{-2x}}{(\phi_q + \phi_q^{-1})^2} \\
&= \frac{4}{(\phi_q + \phi_q^{-1})^2} = \frac{4}{1+4q^{n-2}}.
\end{aligned}$$

Theorem 2.*(q-Fibonacci Sum and Difference) Let* $\sin F_q h(x)$ *and* $\cos F_q h(x)$ *be two functions of q-analogue of Fibonacci hyperbolic function. For* $n \geq 2$, $x, y \in \mathbb{R}$

$$\begin{aligned}
\cos F_q h(x+y) = \frac{\sqrt{1+4q^{n-2}}}{2}&(\cos F_q h(x)\cos F_q h(y) \\
&+ \sin F_q h(x)\sin F_q h(y))
\end{aligned} \tag{14}$$

$$\begin{aligned}
\cos F_q h(x-y) = \frac{\sqrt{1+4q^{n-2}}}{2}&(\cos F_q h(x)\cos F_q h(y) \\
&- \sin F_q h(x)\sin F_q h(y))
\end{aligned} \tag{15}$$

$$\begin{aligned}
\sin F_q h(x+y) = \frac{\sqrt{1+4q^{n-2}}}{2}&(\sin F_q h(x)\cos F_q h(y) \\
&+ \sin F_q h(y)\cos F_q h(x))
\end{aligned} \tag{16}$$

$$\begin{aligned}
\sin F_q h(x-y) = \frac{\sqrt{1+4q^{n-2}}}{2}&(\sin F_q h(x)\cos F_q h(y) \\
&- \sin F_q h(y)\cos F_q h(x))
\end{aligned} \tag{17}$$

Proof.(14) the correlation similar to the equation

$$\begin{aligned}
\cos Fh(x+y) = \frac{\sqrt{5}}{2}&(\cos Fh(x).\cos Fh(y) \\
&+ \sin Fh(x).\sin Fh(y))
\end{aligned}$$

is valid for the q-analogue of the Fibonacci hyperbolic functions. We only give the proof of the (14), because the proof of (15)-(17) is similar.

$$\begin{aligned}
&\cos F_q h(x).\cos F_q h(y) + \sin F_q h(x).\sin F_q h(y) \\
&= \left(\frac{\phi_q^x + \phi_q^{-x}}{\phi_q + \phi_q^{-1}}\right)\left(\frac{\phi_q^y + \phi_q^{-y}}{\phi_q + \phi_q^{-1}}\right) + \left(\frac{\phi_q^x - \phi_q^{-x}}{\phi_q + \phi_q^{-1}}\right)\left(\frac{\phi_q^y - \phi_q^{-y}}{\phi_q + \phi_q^{-1}}\right) \\
&= \frac{\phi_q^{x+y} + \phi_q^{x-y} + \phi_q^{-x+y} + \phi_q^{-x-y}}{(\phi_q + \phi_q^{-1})^2} \\
&\quad + \frac{\phi_q^{x+y} - \phi_q^{x-y} - \phi_q^{-x+y} + \phi_q^{-x-y}}{(\phi_q + \phi_q^{-1})^2} \\
&= \frac{2(\phi_q^{x+y} + \phi_q^{-x-y})}{(\phi_q + \phi_q^{-1})^2} = \frac{2}{\sqrt{1+4q^{n-2}}}\cos F_q h(x+y).
\end{aligned}$$

By setting $x = y$ in the (14) and (16), we have the following corollary.

Corollary 1.*(q-Fibonacci Double Argument)*

$$\begin{aligned}
\cos F_q h(2x) = \frac{\sqrt{1+4q^{n-2}}}{2}&\Big((\cos F_q h(x))^2 \\
&+ (\sin F_q h(x))^2\Big),
\end{aligned} \tag{18}$$

$$\sin F_q h(2x) = \sqrt{1+4q^{n-2}}(\sin F_q h(x)\cos F_q h(x)). \tag{19}$$

From (18) and (13), we obtain the following corollary.

Corollary 2.*(q-Fibonacci Half Argument) For* $n \geq 2$

$$\begin{aligned}
[\cos F_q h(x)]^2 = \frac{1}{\sqrt{1+4q^{n-2}}}&\Big(\cos F_q h(2x) \\
&+ \frac{2}{\sqrt{1+4q^{n-2}}}\Big),
\end{aligned} \tag{20}$$

$$\begin{aligned}
[\sin F_q h(x)]^2 = \frac{1}{\sqrt{1+4q^{n-2}}}&\Big(\cos F_q h(2x) \\
&- \frac{2}{\sqrt{1+4q^{n-2}}}\Big).
\end{aligned} \tag{21}$$

*Proof.*Similarly, we only give proof the identity (20), because the proof of the identity (21) is similar. We start with the identity (18)

$$\begin{aligned}
\cos F_q h(2x) &= \frac{\sqrt{1+4q^{n-2}}}{2}\Big((\cos F_q h(x))^2 + (\sin F_q h(x))^2\Big) \\
\cos F_q h(2x) &= \frac{\sqrt{1+4q^{n-2}}}{2}(\cos F_q h(x))^2 \\
&\quad + \frac{\sqrt{1+4q^{n-2}}}{2}(\sin F_q h(x))^2
\end{aligned}$$

$$\begin{aligned}
\frac{\sqrt{1+4q^{n-2}}}{2}(\cos F_q h(x))^2 &= \cos F_q h(2x) \\
&\quad - \frac{\sqrt{1+4q^{n-2}}}{2}(\sin F_q h(x))^2 \\
(\cos F_q h(x))^2 &= \frac{2}{\sqrt{1+4q^{n-2}}}\cos F_q h(2x) \\
&\quad - (\sin F_q h(x))^2
\end{aligned}$$

Using the formula (13)

$$\begin{aligned}
(\cos F_q h(x))^2 &= \frac{2}{\sqrt{1+4q^{n-2}}}\cos F_q h(2x) \\
&\quad - (\cos F_q h(x))^2 + \frac{4}{1+4q^{n-2}} \\
(\cos F_q h(x))^2 &= \frac{1}{\sqrt{1+4q^{n-2}}}\cos F_q h(2x) + \frac{2}{1+4q^{n-2}}.
\end{aligned}$$

It is necessary that we give q-Catalan's recurrence to obtain q-Catalan's identity of q-Fibonacci hyperbolic functions. However, for $n \in \mathbb{N}$ and $r \in \mathbb{Z}$ we first recall Catalan's identity for Fibonacci numbers stated in [2]

$$F_{1,n-r}F_{1,n+r} - F_{1,n}^2 = (-1)^{n+1-r}F_{1,r}^2.$$

q-Catalan's identity for the q-Fibonacci numbers is given by

$$F_{1,n-r}(q)F_{1,n+r}(q) - F_{1,n}^2(q) = (-1)^{n+1-r}F_{1,r}^2(q)$$

For the q-Fibonacci hyperbolic functions we have the following result:

Theorem 3.*(q-Fibonacci Catalan's Identity)*

$$\cos F_q h(x-r)\cos F_q h(x+r) - (\cos F_q h(x))^2 = (\sin F_q h(r))^2. \tag{22}$$

*Proof.*By definitions (3.1) and (3.2) of q-Fibonacci hyperbolic functions

$$\begin{aligned}
&\cos F_q h(x-r)\cos F_q h(x+r) - \left(\cos F_q h(x)\right)^2 \\
&= \left(\frac{\phi_q^{x-r}+\phi_q^{-(x-r)}}{\phi_q+\phi_q^{-1}}\right)\left(\frac{\phi_q^{x+r}+\phi_q^{-(x+r)}}{\phi_q+\phi_q^{-1}}\right) - \left(\frac{\phi_q^{x}+\phi_q^{-x}}{\phi_q+\phi_q^{-1}}\right)^2 \\
&= \frac{\phi_q^{2x}+\phi_q^{-2r}+\phi_q^{2r}+\phi_q^{-2x}-\phi_q^{2x}-2-\phi_q^{-2x}}{\left(\phi_q+\phi_q^{-1}\right)^2} \\
&= \left(\frac{\phi_q^{r}-\phi_q^{-r}}{\phi_q+\phi_q^{-1}}\right)^2 \\
&= \frac{(\phi_q^{r}-\phi_q^{-r})^2}{1+4q^{n-2}}, n \geqslant 2 \\
&= \left(\sin F_q h(r)\right)^2
\end{aligned}$$

The following results can be obtained in a similar way.

Corollary 3.

$$\cos F_q h(x-r)\cos F_q h(x+r) - (\sin F_q h(x))^2 = (\cos F_q h(r))^2 \tag{23}$$

$$\sin F_q h(x-r)\sin F_q h(x+r) - (\sin F_q h(x))^2 = -(\sin F_q h(r))^2 \tag{24}$$

$$\sin F_q h(x-r)\sin F_q h(x+r) - (\cos F_q h(x))^2 = -(\cos F_q h(r))^2 \tag{25}$$

From Lemma 1, By setting $r = 1$ into q-Catalan's identity, it is straightforwardly obtained q-Cassini's or Simson's identity for the q-Fibonacci numbers:

$$F_{1,n-1}(q)F_{1,n+1}(q) - F_{1,n}^2(q) = (-1)^n.$$

The corresponding identity for the q-Fibonacci hyperbolic functions is as follows:

Proposition 1.*(q-Fibonacci Cassini's or Simson's Identity)*

$$\cos F_q h(x-1)\cos F_q h(x+1) - (\sin F_q h(x))^2 = 1,$$

$$\sin F_q h(x-1)\sin F_q h(x+1) - (\cos F_q h(x))^2 = -1.$$

*Proof.*Setting $r = 1$ in (23) and (25), we have

$$\cos F_q h(x-1)\cos F_q h(x+1) - (\sin F_q h(x))^2 = 1$$

and

$$\sin F_q h(x-1)\sin F_q h(x+1) - (\cos F_q h(x))^2 = -1.$$

It is necessary that we give q-d'Ocagne's recurrence to obtain q-d'Ocagne's identity of q-Fibonacci hyperbolic functions. However, for $m, n \in \mathbb{N}$ and $m > n$ we first provide d'Ocagne's identity for Fibonacci numbers stated in [2]

$$F_{1,m}F_{1,n+1} - F_{1,m+1}F_{1,n} = (-1)^n F_{1,m-n}.$$

q-d'Ocagne's identity for the q-Fibonacci number is

$$F_{1,m}(q)F_{1,n+1}(q) - F_{1,m+1}(q)F_{1,n}(q) = (-1)^n F_{1,m-n}(q)$$

Theorem 4.*(q-Fibonacci d'Ocagne's Identity)*

$$\cos F_q h(x)\cos F_q h(y+r) - \sin F_q h(x+r)\sin F_q h(y) = \cos F_q h(r)\cos F_q h(x-y) \tag{26}$$

$$\cos F_q h(x)\sin F_q h(y+r) - \cos F_q h(x+r)\sin F_q h(y) = \sin F_q h(r)\cos F_q h(x-y) \tag{27}$$

*Proof.*We only give proof the identity (26), because the proof of the identity (27) is similar.

$$\begin{aligned}
&\cos F_q h(x)\cos F_q h(y+r) - \sin F_q h(x+r)\sin F_q h(y) \\
&= \left(\frac{\phi_q^{x}+\phi_q^{-x}}{\phi_q+\phi_q^{-1}}\right)\left(\frac{\phi_q^{y+r}+\phi_q^{-(y+r)}}{\phi_q+\phi_q^{-1}}\right) \\
&\quad - \left(\frac{\phi_q^{x+r}-\phi_q^{-(x+r)}}{\phi_q+\phi_q^{-1}}\right)\left(\frac{\phi_q^{y}-\phi_q^{-y}}{\phi_q+\phi_q^{-1}}\right) \\
&= \frac{\phi_q^{x-y-r}+\phi_q^{-x+y+r}+\phi_q^{x-y+r}+\phi_q^{-x+y-r}}{\left(\phi_q+\phi_q^{-1}\right)^2} \\
&= \frac{\phi_q^{-r}(\phi_q^{x-y}+\phi_q^{-(x-y)})+\phi_q^{r}(\phi_q^{x-y}+\phi_q^{-(x-y)})}{\left(\phi_q+\phi_q^{-1}\right)^2} \\
&= \left(\frac{\phi_q^{r}+\phi_q^{-r}}{\phi_q+\phi_q^{-1}}\right)\left(\frac{\phi_q^{x-y}+\phi_q^{-(x-y)}}{\phi_q+\phi_q^{-1}}\right) \\
&= \frac{\left(\phi_q^{r}+\phi_q^{-r}\right)\left(\phi_q^{x-y}+\phi_q^{-(x-y)}\right)}{\left(\phi_q+\phi_q^{-1}\right)^2} \\
&= \frac{\left(\phi_q^{r}+\phi_q^{-r}\right)\left(\phi_q^{x-y}+\phi_q^{-(x-y)}\right)}{1+4q^{n-2}}, n \geqslant 2 \\
&= \cos F_q h(r)\cos F_q h(x-y)
\end{aligned}$$

Notice that by taking $r = 1$ in (26), we obtain the following identity.

Corollary 4.

$$\cos F_q h(x) \cos F_q h(y+1) - \sin F_q h(x+1) \sin F_q h(y) = \cos F_q h(x-y)$$

4 Conclusion

The formulas of classical hyperbolic and Fibonacci hyperbolic identities that are already present:

Classical Hyperbolic Functions
$[\cosh(x)]^2 - [\sinh(x)]^2 = 1$
$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$
$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y)$
$\sinh(x+y) = \sinh(x)\cosh(y) + \sinh(y)\cosh(x)$
$\sinh(x-y) = \sinh(x)\cosh(y) - \sinh(y)\cosh(x)$
$\cosh(2x) = [\cosh(x)]^2 + [\sinh(x)]^2$
$\sinh(2x) = 2.\sinh(x)\cosh(x)$

Fibonacci Hyperbolic Functions
$[\cos Fh(x)]^2 - [\sin Fh(x)]^2 = \frac{4}{5}$
$\cos Fh(x+y) = \frac{\sqrt{5}}{2}(\cos Fh(x)\cos Fh(y) + \sin Fh(x)\sin Fh(y))$
$\cos Fh(x-y) = \frac{\sqrt{5}}{2}(\cos Fh(x)\cos Fh(y) - \sin Fh(x)\sin Fh(y))$
$\sin Fh(x+y) = \frac{\sqrt{5}}{2}(\sin Fh(x)\cos Fh(y) + \sin Fh(y)\cos Fh(x))$
$\sin Fh(x-y) = \frac{\sqrt{5}}{2}(\sin Fh(x)\cos Fh(y) - \sin Fh(y)\cos Fh(x))$
$\cos Fh(2x) = \frac{\sqrt{5}}{2}([\cos Fh(x)]^2 + [\sin Fh(x)]^2)$
$\sin Fh(2x) = \sqrt{5}(\sin Fh(x).\cos Fh(x))$

Some identities given throughout the paper by benefitting from the formulas above have been listed as follows:

q-Fibonacci Hyperbolic Functions(for $n \geq 2$)
$[\cos F_q h(x)]^2 - [\sin F_q h(x)]^2 = \frac{4}{1+4q^{n-2}}$
$\cos F_q h(x+y) = \frac{\sqrt{1+4q^{n-2}}}{2}(\cos F_q h(x)\cos F_q h(y) + \sin F_q h(x)\sin F_q h(y))$
$\cos F_q h(x-y) = \frac{\sqrt{1+4q^{n-2}}}{2}(\cos F_q h(x)\cos F_q h(y) - \sin F_q h(x)\sin F_q h(y))$
$\sin F_q h(x+y) = \frac{\sqrt{1+4q^{n-2}}}{2}(\sin F_q h(x)\cos F_q h(y) + \sin F_q h(y)\cos F_q h(x))$
$\sin F_q h(x-y) = \frac{\sqrt{1+4q^{n-2}}}{2}(\sin F_q h(x)\cos F_q h(y) - \sin F_q h(y)\cos F_q h(x))$
$\cos F_q h(2x) = \frac{\sqrt{1+4q^{n-2}}}{2}\left((\cos F_q h(x))^2 + (\sin F_q h(x))^2\right)$
$\sin F_q h(2x) = \sqrt{1+4q^{n-2}}(\sin F_q h(x)\cos F_q h(x))$

The main result of the present paper is a strong mathematical proof of the deep connection between the Golden Section, Fibonacci and hyperbolic functions. A new class of the hyperbolic functions based on the Golden Section could have far going consequences for future progress of mathematics, physics, biology and cosmology. In the first place, the hyperbolic Fibonacci which is the extension of Binet's formulas for the q-Fibonacci numbers in continuous domain transform the Fibonacci numbers theory into "continuous" theory because every identity for the Fibonacci hyperbolic functions has its discrete analogy in the framework of the Fibonacci number theory. In other words, the theory of Fibonacci numbers are being the "discrete" case of the theory of the Fibonacci hyperbolic functions. Considering the fundamental role of the classical hyperbolic functions in the mathematical tools of the modern science, it is possible to suppose that the new theory of the hyperbolic functions will bring the new results and interpretations in various spheres of natural science.

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