Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/080206

On Global Attractors for a Class of Parabolic Problems

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Received: 10 Mar. 2013, Revised: 11 Jul. 2013, Accepted: 13 Jul. 2013
Published online: 1 Mar. 2014

Abstract: This paper is devoted to study the existence of global attractor in $H_0^1(\Omega)$ and uniform bounds of it in $L^\infty(\Omega)$ for a class of parabolic problems with homogeneous boundary conditions wich involves a uniform strongly elliptic operator of second order in the domain $\Omega \subset \mathbb{R}^n$. The main tools used to prove the existence of global attractor are the techniques used in Hale [8] and Cholewa [5], and for the uniform bound of the attractor we use the Alikakos-Moser iteration procedure [1].

Keywords: Parabolic equation, sectorial operator, global attractor, uniform boundness

1 Introduction

Today, the concept of global attractor is a very useful tool for studying the asymptotic behavior of differential equations, that is, an attractor is a nonempty subset $\mathscr{A}$ of the phase space which is compact, invariant under the flow and attracts every bounded set under the semigroup associated to the PDE (for more details see [8], [10] and the reference therein). For example, suppose that the parabolic problem

$$\begin{cases} u_t = Lu + f(u),\ t>0,\ x\in\Omega \\ u(t,x) = 0, \quad t>0,\ x\in\partial\Omega \\ u(0,\cdot) = u^0(\cdot)\in H_0^1(\Omega), \end{cases} \tag{1}$$

models a certain phenomenon. Here $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, L is second order elliptic operator given by

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + \sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x_j} + (c(x)+\lambda)u,$$

with coefficients $a_{ij}, b_j, c : \overline{\Omega} \to \mathbb{R}$ smooth, $a_{ij} = a_{ji}$, $i,j = 1,\cdots,n$, $\lambda \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear function of class $C^2(\mathbb{R})$ satisfying the condition

$$|f'(s)| \leqslant C(1+|s|^{\gamma-1}),\ \forall s\in\mathbb{R}, \tag{C}$$

where $1 \leqslant \gamma < \frac{n+2}{n-2}$ if $n \geqslant 3$, and $\gamma \geqslant 1$ if $n=2$, and

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} \leqslant 0. \tag{D}$$

We assume that L is uniformly strongly elliptic operator, that is, there is a constant $\vartheta > 0$, such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \vartheta\Big(\sum_{k=1}^{n}\xi_k^2\Big), \tag{2}$$

for all $x\in\Omega$ and $\xi = (\xi_1,\ldots,\xi_n)\in\mathbb{R}^n$.

The main goal of this work is to prove the existence and uniform boundedness of global attractor for the initial-boundary value problem (1) where the operator L presents lower order terms. The presence of such that terms in the operator L, in particular $b_j u_{x_j}$ makes it difficult to obatins directly the Lyapunov function as in the works of Hale [8].

Let $X = L^2(\Omega)$ be a Hilbert space and define the linear operator $A : D(A) \subset X \to X$ by

$$D(A) = H^2(\Omega)\cap H_0^1(\Omega),$$
$$Au = -Lu,\ \ \forall u\in D(A).$$

There are many studies on PDE's for (1) with the operator A being self-adjoint (see [5], [8], [9]). In our case the operator A is not self-adjoint.

We will show that the operator A is sectorial and assuming that λ is chosen such that $\mathrm{Re}\,\sigma(A) > 0$, we can define the fractional powers A^α and the corresponding fractional power spaces $X^\alpha := D(A^\alpha)$, $\alpha > 0$, endowed with the graph norm (see [5, Section 1.3.3]). X^α is a Hilbert space with the inner product $\langle\varphi,\psi\rangle_\alpha = \int_\Omega (A^\alpha\varphi)(A^\alpha\psi)$. Then, $X^1 = D(A)$, $X^0 = L^2(\Omega)$ and $X^{1/2} = H_0^1(\Omega)$.

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With this notation, the problem (1) can be written in the abstract form

$$\begin{cases} \dot{u}+Au=F(u) \\ u(0)=u^0\in X^{1/2}, \end{cases} \tag{3}$$

where $F: X^{1/2}\to X$ is the Nemytskii operator given by $F(u(t))x = f(u(x,t))$. From Henry's theory [9], the equation (3) defines a semigroup $T(t,\cdot)$ on $X^{1/2}$, for $t\geqslant 0$.

Next, we describe the contents of the paper. In section 2 we will show that the operator A is sectorial and that F and F' is is locally Lipschitz continuous. In section 3 we prove the existence of local and global solution of (1) and finally in section 4 we prove the existence of the global attractor and the uniform boundedness for it.

2 The sectoriality of operator A

Firstly, we will be prove the following.

Lemma 1. *Let $A_0 : D(A_0) = D(A)\subset X\to X$ be a linear operator given by*

$$A_0u=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial u}{\partial x_j}\big).$$

Then, A_0 is sectorial in X.

Furthermore, there is a constant $\rho>0$ such that

$$\|u\|_{H^2(\Omega)}\leqslant \rho\|A_0u\|_X,\ \ \forall u\in X. \tag{4}$$

Proof. Let $\lambda_0>0$. We will show that $A_0+\lambda_0 I$ é setorial, then by the Remark 1.3.1 in [5, p.32] it follows that A_0 is sectorial. Indeed, we have $D(A_0)=D(A_0+\lambda_0 I)$ and for $u,v\in D(A_0)$,

$$\begin{aligned}\langle (A_0+\lambda_0 I)u,v\rangle_X &= \sum_{i,j=1}^{n}\int_\Omega a_{ji}(x)\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_j}dx+\lambda_0\int_\Omega uvdx\\ &=-\sum_{i,j=1}^{n}\int_\Omega u\frac{\partial}{\partial x_j}\big(a_{ji}(x)\frac{\partial v}{\partial x_i}\big)dx+\lambda_0\int_\Omega uvdx\\ &=\langle u,(A_0+\lambda_0 I)v\rangle_X,\end{aligned}$$

thus, $A_0+\lambda_0 I$ is symmetric operator.

Using the Sobolev embeddings (see [5, p. 23]) and the density of $C_0^\infty(\Omega)$ in $L^p(\Omega)$, $1\leqslant p<\infty$, we conclude that $A_0+\lambda_0 I$ is densely defined in X. Therefore, $D(A_0)$ is densely defined in X.

Now, let $f\in X$ be such that $(A_0+\lambda_0 I)u=f$. Following the Example 3 in [4, p. 294] taking $a_0=\lambda_0>0$, we conclude that there is $u\in D(A_0)$ satisfying $(A_0+\lambda_0 I)u=f$. Thus, $R(A_0+\lambda_0 I)=X$. Using Theorem 13.11 item (d) in [11, p.334] we conclude that $A_0+\lambda_0 I$ is selfadjoint.

Also, for $u\in D(A_0)$, we obtain

$$\begin{aligned}\langle (A_0+\lambda_0 I)u,u\rangle_X &= \sum_{i,j=1}^{n}\int_\Omega a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}dx+\lambda_0\int_\Omega u^2dx\\ &\geqslant \vartheta\sum_{k=1}^{n}\int_\Omega\Big|\frac{\partial u}{\partial x_k}\Big|^2dx+\lambda_0\int_\Omega|u|^2dx\geqslant\lambda_0\|u\|_X^2.\end{aligned}$$

Here we have used (2). Now, by density it follows that $A_0+\lambda_0 I$ is bounded below and by Proposition 1.3.3 in [5, p. 39] $A_0+\lambda_0 I$ is sectorial in X.

Finally using the results of elliptic regularity (taking $m=1$, $k=0$, $p=2$ and $B_j=I, j=1$ in [12, p. 14]), we have

$$\|u\|_{H^2(\Omega)}\leqslant\rho\|A_0u\|_{L^2(\Omega)},$$

where $\rho>0$ and $\ker(A_0)=\{0\}$. This last result is extend to X by density of $D(A_0)$. □

Lemma 2. *The operator A is sectorial in X.*

Proof. Let $B: D(B)\subset X\to X$ be a linear operator given by

$$-Bu=\sum_{j=1}^{n}b_j(x)\frac{\partial u}{\partial x_j}+(c(x)+\lambda)u,$$

where $D(B)=\{u\in L^2(\Omega): Bu\in L^2(\Omega),\, u=0 \text{ in } \partial\Omega\}=H_0^1(\Omega)$.

By Gagliardo-Nirenberg's inequality (Theorem 10.1 in [7, p. 27]) with $m=r=p=2$, $j=1$ and $\theta=1/2$, we have

$$\|u\|_{H_0^1(\Omega)}\leqslant C\|u\|_{H^2(\Omega)}^{\frac12}\|u\|_{L^2(\Omega)}^{\frac12}, \tag{5}$$

where C is a constant.

By other side, from the Young's inequality and (5), we obtain

$$\|u\|_{H_0^1(\Omega)}\leqslant\frac{\varepsilon}{2}\|u\|_{H^2(\Omega)}+\frac{C^2}{2\varepsilon}\|u\|_{L^2(\Omega)},\ \forall\varepsilon>0. \tag{6}$$

Substituting (4) in (6) we get

$$\|u\|_{H_0^1(\Omega)}\leqslant\frac{\rho\varepsilon}{2}\|A_0u\|_{L^2(\Omega)}+\frac{C^2}{2\varepsilon}\|u\|_{L^2(\Omega)}. \tag{7}$$

Thus,

$$\|Bu\|_X\leqslant\frac{\tau\rho\varepsilon}{2}\|A_0u\|_{L^2(\Omega)}+\frac{\tau C^2}{2\varepsilon}\|u\|_{L^2(\Omega)}, \tag{8}$$

where in the last inequality we have used (7) and $\tau=\max\big\{\max_{x\in\overline{\Omega}}|b_j(x)|,\max_{x\in\overline{\Omega}}|c(x)+\lambda|\big\}$.

Observe that $A=A_0+B$ where A_0 is sectorial in X, $D(A_0)\subset D(B)$. Therefore, by Theorem 2.6.3 in [12, p. 69] and (8) follows that A is a sectorial in X. □

Now, we prove that F is locally Lipschitz continous.

Lemma 3. *If $f:\mathbb{R}\to\mathbb{R}$ is a function of class $C^1(\mathbb{R})$, the assumption (C) is satisfied and $n\geqslant 1$ then $F: X^{1/2}\to X$ is locally Lipschitz continous.*

Proof. The case $n=1$ was proved in [8, p.75]. Now, let $n \geqslant 2$. Using the mean value theorem and assumption (C), we have

$$\begin{aligned}|f(s)-f(r)| &= |f'(\theta s+(1-\theta)r)||s-r|,\ \ 0\leqslant\theta\leqslant 1\\ &\leqslant C(1+|\theta s+(1-\theta)r|^{\gamma-1})|s-r|\\ &\leqslant C_1(1+|s|^{\gamma-1}+|r|^{\gamma-1})|s-r|, \qquad (9)\end{aligned}$$

where $C_1=C_1(\gamma)$, $1\leqslant\gamma<\frac{n+2}{n-2}$ if $n\geqslant 3$ and $\gamma\geqslant 1$ if $n=2$.

Now, we will to show that for $p>1$ satisfying $p\gamma=\frac{2n}{n-2}$, the function $F: L^{\frac{2n}{n-2}}(\Omega)\to L^p(\Omega)$ is locally Lipschitz continuous. Indeed, using (9) we get

$$\begin{aligned}\|F(u)-F(v)\|^p_{L^p(\Omega)} &= \int_\Omega |f(u(x))-f(v(x))|^p dx\\ &\leqslant C_1\int_\Omega (1+|u(x)|^{\gamma-1}+|v(x)|^{\gamma-1})^p|u(x)-v(x)|^p dx\\ &\leqslant C_2\Big(\int_\Omega |u(x)-v(x)|^{p\gamma}dx\Big)^{\frac{1}{\gamma}}\\ &\quad\times\Big(\int_\Omega (1+|u(x)|^{p(\gamma-1)}+|v(x)|^{p(\gamma-1)})^r dx\Big)^{\frac{1}{r}}\\ &\leqslant C_3\Big(\int_\Omega |u(x)-v(x)|^{\frac{2n}{n-2}}dx\Big)^{\frac{p(n-2)}{2n}}\\ &\quad\times\Big(\int_\Omega (1+|u(x)|^{p(\gamma-1)r}+|v(x)|^{p(\gamma-1)r})dx\Big)^{\frac{1}{r}},\end{aligned}$$

where in the second inequality we have used the Hölder inequality for γ and r, with $\frac{1}{r}=1-\frac{p(n-2)}{2n}$ because $u,v\in L^{\frac{2n}{n-2}}(\Omega)=L^{p\gamma}(\Omega)$. Since γ and r are conjugate we have $p(\gamma-1)r=p\gamma$, thus $(u-v)^p\in L^\gamma(\Omega)$, and since $u^{p(\gamma-1)}$, $v^{p(\gamma-1)}\in L^r(\Omega)$ we have also $1+u^{p(\gamma-1)}+v^{p(\gamma-1)}\in L^r(\Omega)$. Then,

$$\begin{aligned}\|F(u)-F(v)\|^p_{L^p(\Omega)} &\leqslant C_4\|u-v\|^p_{L^{p\gamma}(\Omega)}\\ &\quad\times\Big(1+\|u\|^{p\gamma}_{L^{p\gamma}(\Omega)}+\|v\|^{p\gamma}_{L^{p\gamma}(\Omega)}\Big)^{\frac{1}{r}}\\ &\leqslant C_5\|u-v\|^p_{L^{\frac{2n}{n-2}}(\Omega)}\Big(1+\|u\|_{L^{\frac{2n}{n-2}}(\Omega)}+\|v\|_{L^{\frac{2n}{n-2}}(\Omega)}\Big)^{p(\gamma-1)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|F(u)-F(v)\|_{L^p(\Omega)} &\leqslant C_6\|u-v\|_{L^{\frac{2n}{n-2}}(\Omega)}\\ &\quad\times\Big(1+\|u\|_{L^{\frac{2n}{n-2}}(\Omega)}+\|v\|_{L^{\frac{2n}{n-2}}(\Omega)}\Big)^{\gamma-1}.\end{aligned}$$

In particular, for $p=2$, using the immersions $H^1_0(\Omega)\hookrightarrow L^\sigma(\Omega)$, $\sigma\in\big[1,\frac{2n}{n-2}\big]$ for $n\geqslant 3$ and $H^1_0(\Omega)\hookrightarrow L^\sigma(\Omega)$, $\sigma\in[1,\infty)$ for $n=2$, we have

$$\|F(u)-F(v)\|_X\leqslant C_7\|u-v\|_{X^{1/2}}\big(1+\|u\|_{X^{1/2}}+\|v\|_{X^{1/2}}\big)^{\gamma-1}.$$

Finally, given $\delta>0$ such that $u,v\in X^{1/2}$ with $\|u\|_{X^{1/2}},\|v\|_{X^{1/2}}\leqslant\delta$ we have

$$\|F(u)-F(v)\|_X\leqslant C_{\delta,\gamma}\|u-v\|_{X^{1/2}},$$

where $C_{\delta,\gamma}=C_7(1+2\delta)^{\gamma-1}$. Therefore F is locally Lipschitz continuous. □

Lemma 4. *Let $f:\mathbb{R}\to\mathbb{R}$ be a function of class $C^2(\mathbb{R})$ satisfying (C). Then, $F: X^{1/2}\to X$ is a function of class C^1 with $F'(u)\in\mathscr{L}(X^{1/2},X)$ given by $[F'(u)h](x)=f'(u(x))h(x)$, for all $u,h\in X^{1/2}$.*

Moreover, there is a constante $C>0$ such that

$$\begin{aligned}\|F'(u)-F'(v)\|_{\mathscr{L}(X^{1/2},X)} &\leqslant C\|u-v\|_{X^{1/2}}\\ &\quad\times\big(1+\|u\|_{X^{1/2}}+\|v\|_{X^{1/2}}\big)^{\gamma-2}, \qquad (10)\end{aligned}$$

for all $u,v\in X^{1/2}$.

Proof. First we obtain the Gateaux differential of the Nemytskii's operator. Thus, for $u,h\in X^{1/2}$ we have

$$\begin{aligned}\delta F(u,h)(x) &= \lim_{t\to 0}\frac{f(u(x)+th(x))-f(u(x))}{t}\\ &= \lim_{t\to 0}\int_0^1\frac{f'(u(x)+sth(x))th(x)}{t}ds\\ &= \int_0^1 f'(u(x))h(x)ds = f'(u(x))h(x),\end{aligned}$$

where we have used the Dominated Convergence Theorem of Lebesgue and the fact that $f\in C^2(\mathbb{R})$ with $s=s(x)\in[0,1]$.

From assumption (C), we obtain

$$|f''(s)|\leqslant C_\gamma(1+|s|^{\gamma-2}), \qquad \text{(CC)}$$

where $s\in\mathbb{R}$ with $2\leqslant\gamma\leqslant\frac{n+2}{n-2}$ if $n\geqslant 3$ and $\gamma\geqslant 2$ if $n=2$.

Similarly, as was done for (9) using (CC), we have

$$|f'(s)-f'(r)|\leqslant C_{1,\gamma}(1+|s|^{\gamma-2}+|r|^{\gamma-2})|s-r|, \qquad (11)$$

where $2\leqslant\gamma<\frac{n+2}{n-2}$ if $n\geqslant 3$ and $\gamma\geqslant 2$ if $n=2$.

Now, we can show that F is of class C^1 in the sense of Fréchet. We begin by defining

$$\Phi(x):=F(u+h)(x)-F(u)(x)-f'(u(x))h(x),$$

for $u,h\in L^{\frac{2n}{n-2}}$ and $x\in\Omega$. Thus, for $p>1$ such that $p\gamma=\frac{2n}{n-2}$ and (11) we obtain

$$\begin{aligned}\|\Phi\|^p_{L^p(\Omega)} &= \int_\Omega\Big|\int_0^1[f'(u(x)+sh(x))-f'(u(x))]dsh(x)\Big|^p dx\\ &\leqslant C^p_{1,\gamma}\int_\Omega\Big|\int_0^1(1+|u(x)+sh(x)|^{\gamma-2}+|u(x)|^{\gamma-2})|sh(x)|\\ &\quad\times|h(x)|ds\Big|^p dx\\ &\leqslant C^p_{2,\gamma}\int_\Omega |h(x)|^{2p}(1+|u(x)|^{\gamma-2}+|h(x)|^{\gamma-2})^p dx\\ &\leqslant C^p_{3,\gamma}\int_\Omega |h(x)|^{2p}(1+|u(x)|^{p(\gamma-2)}+|h(x)|^{p(\gamma-2)})dx\\ &\leqslant C^p_{4,\gamma}\|h\|^{2p}_{L^{p\gamma}(\Omega)}\Big(\int_\Omega(1+|u(x)|^{p\gamma}+|h(x)|^{p\gamma})dx\Big)^{\frac{\gamma-2}{\gamma}}\\ &\leqslant C^p_{5,\gamma}\|h\|^{2p}_{L^{p\gamma}(\Omega)}\big(1+\|u\|^{p\gamma}_{L^{p\gamma}(\Omega)}+\|h\|^{p\gamma}_{L^{p\gamma}(\Omega)}\big)^{\frac{\gamma-2}{\gamma}}\\ &\leqslant C^p_{6,\gamma}\|h\|^{2p}_{L^{p\gamma}(\Omega)}\big(1+\|u\|_{L^{p\gamma}(\Omega)}+\|h\|_{L^{p\gamma}(\Omega)}\big)^{p(\gamma-2)},\end{aligned}$$

where we have used Hölder's inequality for $\frac{\gamma}{2}$ and $\frac{\gamma}{\gamma-2}$. Then,

$$\|\Phi\|_{L^p(\Omega)} \leqslant C_{6,\gamma}\|h\|^2_{L^{p\gamma}(\Omega)}\left(1+\|u\|_{L^{p\gamma}(\Omega)}+\|h\|_{L^{p\gamma}(\Omega)}\right)^{\gamma-2}.$$

Now, by the Sobolev embeddings we obtain

$$\frac{\|\Phi\|_{L^p(\Omega)}}{\|h\|_{H^1_0(\Omega)}} \leqslant C_{7,\gamma}\|h\|_{H^1_0(\Omega)}\left(1+\|u\|_{H^1_0(\Omega)}+\|h\|_{H^1_0(\Omega)}\right)^{\gamma-2}.$$

Therefore, $\frac{\|\Phi\|_{L^p(\Omega)}}{\|h\|_{H^1_0(\Omega)}} \to 0$ when $\|h\|_{H^1_0(\Omega)} \to 0$.

In particular, for $p=2$ the function F is of class C^1 in the sense of Fréchet with $[F'(u)h](x) = f'(u(x))h(x)$, for all $u,h \in H^1_0(\Omega)$.

Now we will show the estimate (10). In fact, for $u,v,h \in L^{p\gamma}(\Omega)$ denoting

$$\Delta F(u,v) := F'(u) - F'(v)$$

and using (11) we obtain

$$\begin{aligned}
\|\Delta F(u,v)h\|^p_{L^p(\Omega)} &= \int_\Omega |f'(u(x)) - f'(v(x))|^p |h(x)|^p dx \\
&\leqslant C^p_{1,\gamma}\int_\Omega \left(1+|u(x)|^{\gamma-2}+|v(x)|^{\gamma-2}\right)^p |u(x)-v(x)|^p|h(x)|^p dx \\
&\leqslant C^p_{8,\gamma}\int_\Omega \left(1+|u(x)|^{p(\gamma-2)}+|v(x)|^{p(\gamma-2)}\right)|u(x)-v(x)|^p \\
&\quad \times |h(x)|^p dx \Big(\int_\Omega \left(1+|u(x)|^{p(\gamma-2)}+|v(x)|^{p(\gamma-2)}\right)^\theta dx\Big)^{\frac{1}{\theta}} \\
&\leqslant C^p_{9,\gamma}\|u-v\|^p_{L^{p\gamma}(\Omega)}\|h\|^p_{L^{p\gamma}(\Omega)} \\
&\quad \times \left(1+\|u\|^{p\theta(\gamma-2)}_{L^{p\theta(\gamma-2)}(\Omega)}+\|v\|^{p\theta(\gamma-2)}_{L^{p\theta(\gamma-2)}(\Omega)}\right)^{\frac{1}{\theta}} \\
&\leqslant C^p_{10,\gamma}\|u-v\|^p_{L^{p\gamma}(\Omega)}\|h\|^p_{L^{p\gamma}(\Omega)} \\
&\quad \times \left(1+\|u\|_{L^{p\theta(\gamma-2)}(\Omega)}+\|v\|_{L^{p\theta(\gamma-2)}(\Omega)}\right)^{p(\gamma-2)} \\
&\leqslant C^p_{10,\gamma}\|u-v\|^p_{L^{\frac{2n}{n-2}}(\Omega)}\|h\|^p_{L^{\frac{2n}{n-2}}(\Omega)} \\
&\quad \times \left(1+\|u\|_{L^{\frac{2n}{n-2}}(\Omega)}+\|v\|_{L^{\frac{2n}{n-2}}(\Omega)}\right)^{p(\gamma-2)},
\end{aligned}$$

where $\theta=\theta(\gamma)$ and we have used the Hölder's inequality for γ, γ and θ with $\theta = \frac{\gamma}{\gamma-2}$, then $p(\gamma-2)\theta = p\gamma$ and $u^{p(\gamma-2)}, v^{p(\gamma-2)} \in L^\theta(\Omega)$.

Therefore,

$$\begin{aligned}
\|F'(u)h-F'(u)h\|_{L^p(\Omega)} &\leqslant C_{10,\gamma}\|u-v\|_{L^{\frac{2n}{n-2}}(\Omega)}\|h\|_{L^{\frac{2n}{n-2}}(\Omega)} \\
&\quad \times \left(1+\|u\|_{L^{\frac{2n}{n-2}}(\Omega)}+\|v\|_{L^{\frac{2n}{n-2}}(\Omega)}\right)^{\gamma-2}.
\end{aligned}$$

Finally by the Sobolev embedding we obtain

$$\begin{aligned}
\|F'(u)h-F'(u)h\|_{L^p(\Omega)} &\leqslant C_{11,\gamma}\|u-v\|_{H^1_0(\Omega)} \\
&\quad \times \left(1+\|u\|_{H^1_0(\Omega)}+\|v\|_{H^1_0(\Omega)}\right)^{\gamma-2}\|h\|_{H^1_0(\Omega)}.
\end{aligned}$$

From this last inequality taking $p=2$ immediately follows (10). □

Now let's get uniform bounds for the solutions of (3).

Lemma 5. *Consider the problem (1) under all the hypothesis to get the existence of smooth solutions and satisfying*

$$uf(u) \leqslant \zeta u^2 + \tau, \tag{12}$$

where ζ and τ are positive constants. Then,

$$\sup_{t\geqslant 0}\|u(t,u^0)\|_{L^\infty(\Omega)} \leqslant \delta^{\frac{3}{2}}2^{\frac{3n}{4}+1}\max\left\{\|u^0\|_{L^2(\Omega)},1\right\}, \tag{13}$$

where $\delta>0$ is defined in (28) below.

Proof. We use Alikakos-Moser iteration (see [1] e [5]) which allows us to obtain estimates on $L^\infty(\Omega)$ for the solutions of parabolic equations of second order. Indeed, multiplying (1) by u^{2^k-1}, $k=1,2,\ldots$ and integrating over Ω we get

$$\begin{aligned}
\int_\Omega u_t u^{2^k-1}dx &= \underbrace{\int_\Omega \sum_{i,j=1}^n \frac{\partial}{\partial x_j}\Big(a_{ij}(x)\frac{\partial u}{\partial x_i}\Big)u^{2^k-1}dx}_{=:\mathrm{I}} \\
&+ \underbrace{\int_\Omega \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i}u^{2^k-1}dx}_{=:\mathrm{II}} + \int_\Omega (c(x)+\lambda)u^{2^k}dx \\
&+ \int_\Omega f(u)u^{2^k-1}dx.
\end{aligned} \tag{14}$$

For I, we have

$$\begin{aligned}
\mathrm{I} &= -\int_\Omega \sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial(u^{2^k-1})}{\partial x_j}dx \\
&= -\frac{(2^k-1)}{2^{2k-2}}\int_\Omega \sum_{i,j=1}^n a_{ij}(x)\frac{\partial(u^{2^{k-1}})}{\partial x_i}\frac{\partial(u^{2^{k-1}})}{\partial x_j}dx.
\end{aligned} \tag{15}$$

Similarly, for II, we have

$$\begin{aligned}
\mathrm{II} &= \int_\Omega \sum_{i=1}^n b_i(x)u^{2^{k-1}-1}\frac{\partial u}{\partial x_i}u^{2^{k-1}}dx \\
&= \frac{1}{2^{k-1}}\int_\Omega \sum_{i=1}^n b_i(x)u^{2^{k-1}}\frac{\partial(u^{2^{k-1}})}{\partial x_i}dx.
\end{aligned} \tag{16}$$

Replacing (15) and (16) in (14) we get

$$\begin{aligned}
\frac{1}{2^k}\frac{d}{dt}\int_\Omega u^{2^k}dx &= \frac{(1-2^k)}{2^{k-2}}\int_\Omega \sum_{i,j=1}^n a_{ij}(x)\frac{\partial(u^{2^{k-1}})}{\partial x_i}\frac{\partial(u^{2^{k-1}})}{\partial x_j}dx \\
&+ 2^k\int_\Omega uf(u)u^{2^k-2}dx + 2\int_\Omega \sum_{i=1}^n b_i(x)\frac{\partial(u^{2^{k-1}})}{\partial x_i}u^{2^{k-1}}dx \\
&+ 2^k\int_\Omega (c(x)+\lambda)u^{2^k}dx.
\end{aligned}$$

Now, using (2) and (12) we get

$$\frac{d}{dt}\int_\Omega u^{2^k}dx \leqslant -\frac{(2^k-1)}{2^{k-2}}\vartheta\int_\Omega\sum_{i=1}^n\Big[\frac{\partial(u^{2^{k-1}})}{\partial x_i}\Big]^2dx$$
$$+2^k\int_\Omega(\zeta u^2+\tau)u^{2^k-2}dx+2B_\infty^*\int_\Omega\sum_{i=1}^n\frac{\partial(u^{2^{k-1}})}{\partial x_i}u^{2^{k-1}}dx$$
$$+2^k(C_\infty^*(x)+\lambda)\int_\Omega u^{2^k}dx, \tag{17}$$

where $B_\infty^* := \max\limits_{1\leqslant i\leqslant n}\sup\limits_{x\in\Omega}|b_i(x)|$ and $C_\infty^* := \sup\limits_{x\in\Omega}|c(x)|$. Using Hölder's inequality in (17), we obtain

$$\frac{d}{dt}\int_\Omega u^{2^k}dx \leqslant -\frac{(2^k-1)}{2^{k-2}}\vartheta\int_\Omega\sum_{i=1}^n\Big[\frac{\partial(u^{2^{k-1}})}{\partial x_i}\Big]^2dx$$
$$+2^k\zeta\int_\Omega u^{2^k}dx+2^k\tau\int_\Omega u^{2^k-2}dx$$
$$+2B_\infty^*\Big(\int_\Omega\Big[\sum_{i=1}^n\frac{\partial(u^{2^{k-1}})}{\partial x_i}\Big]^2dx\Big)^{\frac12}\Big(\int_\Omega(u^{2^{k-1}})^2dx\Big)^{\frac12}$$
$$+2^k(C_\infty^*(x)+\lambda)\int_\Omega u^{2^k}dx. \tag{18}$$

Since $u^{2^k-2}\leqslant u^{2^k}+1$, $\forall k=1,2,\dots$ and $(d_1+\dots+d_n)^r\leqslant n^{r-1}(d_1^r+\dots+d_n^r)$, $d_i\geqslant 0$ and $r\geqslant 1$ in (18) we obtain

$$\frac{d}{dt}\int_\Omega u^{2^k}dx \leqslant -\frac{(2^k-1)}{2^{k-2}}\vartheta\int_\Omega\sum_{i=1}^n\Big[\frac{\partial(u^{2^{k-1}})}{\partial x_i}\Big]^2dx$$
$$+2^k\tau\int_\Omega(u^{2^k}+1)dx+2B_\infty^*\Big(\int_\Omega n\sum_{i=1}^n\Big[\frac{\partial(u^{2^{k-1}})}{\partial x_i}\Big]^2dx\Big)^{\frac12}$$
$$\times\Big(\int_\Omega u^{2^k}dx\Big)^{\frac12}+2^k(\zeta+C_\infty^*(x)+\lambda)\int_\Omega u^{2^k}dx$$
$$\leqslant -\frac{(2^k-1)}{2^{k-2}}\vartheta\int_\Omega\sum_{i=1}^n\Big[\frac{\partial(u^{2^{k-1}})}{\partial x_i}\Big]^2dx+2^k(\zeta+\tau+C_\infty^*(x)$$
$$+\lambda)\int_\Omega u^{2^k}dx+2B_\infty^*\Big(\int_\Omega\sum_{i=1}^n\Big[\frac{\partial(u^{2^{k-1}})}{\partial x_i}\Big]^2dx\Big)^{\frac12}\Big(n\int_\Omega u^{2^k}dx\Big)^{\frac12}$$
$$+2^k\tau|\Omega|. \tag{19}$$

Now, using the Cauchy's inequality in (19) with $\overline{\varepsilon}=\frac{\vartheta}{B_\infty^*}$ we get

$$\frac{d}{dt}\int_\Omega u^{2^k}dx \leqslant -\frac{(2^k-1)}{2^{k-2}}\vartheta\int_\Omega\sum_{i=1}^n\Big[\frac{\partial(u^{2^{k-1}})}{\partial x_i}\Big]^2dx$$
$$+2^k(\zeta+\tau+C_\infty^*(x)+\lambda)\int_\Omega u^{2^k}dx$$
$$+2B_\infty^*\Big(\frac{\overline{\varepsilon}}{2}\int_\Omega\sum_{i=1}^n\Big[\frac{\partial(u^{2^{k-1}})}{\partial x_i}\Big]^2dx+\frac{1}{2\overline{\varepsilon}}n\int_\Omega u^{2^k}dx\Big)+2^k\tau|\Omega|$$
$$=\Big(-\frac{(2^k-1)}{2^{k-2}}\vartheta+\vartheta\Big)\int_\Omega\sum_{i=1}^n\Big[\frac{\partial(u^{2^{k-1}})}{\partial x_i}\Big]^2dx+\Big[2^k(\zeta$$
$$+\tau+C_\infty^*(x)+\lambda)+\frac{n(B_\infty^*)^2}{\vartheta}\Big]\int_\Omega u^{2^k}+2^k\tau|\Omega|. \tag{20}$$

Since $2\leqslant\frac{2^k-1}{2^{k-2}}\leqslant 4$ for all $k=1,2,\dots$, we can write (20) as

$$\frac{d}{dt}\int_\Omega u^{2^k}dx \leqslant -\vartheta\int_\Omega\sum_{i=1}^n\Big[\frac{\partial(u^{2^{k-1}})}{\partial x_i}\Big]^2dx+\Big[2^k(\zeta+\tau$$
$$+C_\infty^*(x)+\lambda)+\frac{n(B_\infty^*)^2}{\vartheta}\Big]\int_\Omega u^{2^k}+2^k\tau|\Omega|. \tag{21}$$

Taking $j=0$, $p=2$, $m=1$, $r=2$, $q=1$, $\theta=\frac{n}{n+2}$ in Gagliardo-Nirenberg inequality we obtain

$$\|v\|_{L^2(\Omega)}\leqslant\widetilde{C}\|v\|_{H^1(\Omega)}^{\theta}\|v\|_{L^1(\Omega)}^{1-\theta},\ \forall v\in H^1(\Omega), \tag{22}$$

where $\widetilde{C}=\widetilde{C}(\Omega,n)$. Again, using the Young's inequality in (22) with $m=\frac{1}{\theta}>1$ and $\varepsilon\in(0,1)$ we obtain

$$\|v\|_{L^2(\Omega)}^2\leqslant\frac{1}{m}\varepsilon^m\|v\|_{H^1(\Omega)}^{2\theta m}+\frac{m-1}{m}\varepsilon^{\frac{-m}{m-1}}\widetilde{C}^{\frac{2m}{m-1}}\|v\|_{L^1(\Omega)}^{\frac{2m(1-\theta)}{m-1}}$$
$$\leqslant\varepsilon\|v\|_{H^1(\Omega)}^2+C_\varepsilon\|v\|_{L^1(\Omega)}^2, \tag{23}$$

where $C_\varepsilon := \xi\varepsilon^{-\frac{n}{2}-1}$ and $\xi := \frac{2}{n+2}\widetilde{C}^{n+2}$. From the definition of norm in $H^1(\Omega)$ we can see that (23) can be written as

$$\frac{1-\varepsilon}{\varepsilon}\|v\|_{L^2(\Omega)}^2-\frac{C_\varepsilon}{\varepsilon}\|v\|_{L^1(\Omega)}^2\leqslant\|v_x\|_{H^1(\Omega)}^2. \tag{24}$$

Now, using (24) in (21) with $v=u^{2^{k-1}}$ we get

$$\frac{d}{dt}\int_\Omega u^{2^k}dx\leqslant-\frac{\vartheta(1-\varepsilon)}{\varepsilon}\int_\Omega u^{2^k}dx+\frac{\vartheta C_\varepsilon}{\varepsilon}\Big(\int_\Omega u^{2^k}dx\Big)^2$$
$$+\Big[2^k(\zeta+\tau+C_\infty^*(x)+\lambda)+\frac{n(B_\infty^*)^2}{\vartheta}\Big]\int_\Omega u^{2^k}+2^k\tau|\Omega|$$
$$=\Big[-\frac{\vartheta(1-\varepsilon)}{\varepsilon}+2^k(\zeta+\tau+C_\infty^*(x)+\lambda)+\frac{n(B_\infty^*)^2}{\vartheta}\Big]$$
$$\times\int_\Omega u^{2^k}dx+\frac{\vartheta C_\varepsilon}{\varepsilon}\Big(\int_\Omega u^{2^k}dx\Big)^2+2^k\tau|\Omega|.$$

Then, fixing $\varepsilon=\varepsilon_k$, $k=1,2,\dots$ such that

$$-\frac{\vartheta(1-\varepsilon)}{\varepsilon}+2^k(\zeta+\tau+C_\infty^*(x)+\lambda)+\frac{n(B_\infty^*)^2}{\vartheta}\leqslant-2^k$$

and $\varepsilon_k=\rho 2^{-k}$ where ρ is a positive constant, we have

$$\frac{d}{dt}\int_\Omega u^{2^k}dx\leqslant-2^k\int_\Omega u^{2^k}dx+\mu(2^k)^{\frac{n}{2}+2}\Big(\int_\Omega u^{2^k}dx\Big)^2+2^k\tau|\Omega|, \tag{25}$$

where $\mu := \vartheta\xi\rho^{-\frac{n}{2}-2}$. Applying Lemma 1.2.5 in [5, p. 17] with $y(t)=\int_\Omega u^{2^k}dx$ in (25) we obtain

$$\int_\Omega u^{2^k}dx\leqslant\max\Big\{\int_\Omega(u^0)^{2^k}dx,\mu(2^k)^{\frac{n}{2}+1}\Big(\int_\Omega u^{2^k}dx\Big)^2+\tau|\Omega|\Big\}$$
$$\leqslant\max\Big\{\int_\Omega(u^0)^{2^k}dx,\mu(2^k)^{\frac{n}{2}+1}m_{k-1}^{2^k}+\tau|\Omega|\Big\}, \tag{26}$$

where $m_{k-1} := \sup_{t\geqslant 0}\left(\int_\Omega u^{2^{k-1}}dx\right)^{2^{-k+1}}$. Taking the 2^k–th root on both sides of (26) and then the supremum on the left hand we get

$$m_k \leqslant \max\left\{\|u^0\|_{L^{2^k}(\Omega)}, \left(\mu(2^k)^{\frac{n}{2}+1}m_{k-1}^{2^k}+\tau|\Omega|\right)^{\frac{1}{2^k}}\right\}. \quad (27)$$

Since, $\|u^0\|_{L^{2^k}(\Omega)} \leqslant \sup_{x\in\Omega}|u^0(x)|\left(\int_\Omega dx\right)^{\frac{1}{2^k}} \leqslant \|u^0\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{2^k}} =: \mathscr{K}$, we can see that the first term of (27) is uniform bounded by $\mathscr{K}$, for all $k\in\mathbb{N}$. Now, enlarging μ to the value

$$\delta := \max\{\mu, 1, \tau|\Omega|, \mathscr{K}^2\}. \quad (28)$$

Enlarging also m_1 (which is defined by $\sup_{x\in\Omega}\|u(t,\cdot)\|_{L^2(\Omega)}$) to the value $x_1 := \max\{m_1, 1\}$. Then, using (28) in (27) we have

$$m_k \leqslant \max\left\{\mathscr{K}, \left(\delta(2^k)^{\frac{n}{2}+1}m_{k-1}^{2^k}+\delta\right)^{\frac{1}{2^k}}\right\}, k=2,3,\dots$$

We can see that the numbers m_k, $k=1,2,\dots$ are bounded by the corresponding x_k satisfying the recurrence relation

$$x_k = \max\left\{\mathscr{K}, \left(\delta(2^k)^{\frac{n}{2}+1}x_{k-1}^{2^k}+\delta\right)^{\frac{1}{2^k}}\right\}, k=2,3,\dots \quad (29)$$

Since $\delta\geqslant 1$ and $x_1\geqslant 1$ we can see that sequence $\{x_k\}_{k\in\mathbb{N}}$ is increasing. Furthermore, for $k=2$ the second term of (29) is bounded because

$$\begin{aligned} x_2 &= \max\left\{\mathscr{K}, \left(\delta(2)^{n+2}x_1^4+\delta\right)^{\frac{1}{4}}\right\} \\ &\leqslant \max\left\{\mathscr{K}, \left(2\delta(2)^{n+2}x_1^4\right)^{\frac{1}{4}}\right\} \\ &\leqslant \max\left\{\mathscr{K}, \left(2\delta(2)^{n+2}\right)^{\frac{1}{4}}\max\{m_1,1\}\right\}. \end{aligned}$$

By other side, the sequence $\{x_k\}_{k\in\mathbb{N}}$ is dominated by the sequence $\{z_k\}_{k\in\mathbb{N}}$ given by

$$\begin{cases} z_1 = x_1 \geqslant 1 \\ z_k = \left(2\delta(2^k)^{\frac{n}{2}+1}\right)^{\frac{1}{2^k}}z_{k-1}, k=2,3,\dots \end{cases}$$

Taking the limit when $k\to\infty$ we have $x_\infty := \lim_{k\to\infty}x_k \leqslant z_\infty := \lim_{k\to\infty}z_k$. Thus,

$$\begin{aligned} &\sup_{t\geqslant 0}\|u(t,\cdot)\|_{L^\infty(\Omega)} \leqslant z_1\prod_{k=2}^{\infty}\left(2\delta(2^k)^{\frac{n}{2}+1}\right)^{\frac{1}{2^k}} \\ &\leqslant z_1\left(\delta(2)^{\frac{n}{2}+2}\right)^{-\frac{1}{2}}\prod_{k=1}^{\infty}(2\delta)^{\frac{1}{2^k}}\left[\prod_{k=1}^{\infty}2^{\frac{k}{2^k}}\right]^{\frac{n}{2}+1} \\ &\leqslant z_1\left(\delta(2)^{\frac{n}{2}+2}\right)^{-\frac{1}{2}}(2\delta)^{\lim_{k\to\infty}\left(1-\frac{1}{2^k}\right)}\left[2^{\lim_{k\to\infty}\left(2-\frac{k+2}{2^k}\right)}\right]^{\frac{n}{2}+1} \\ &\leqslant \delta^{\frac{1}{2}}2^{\frac{3n}{4}+1}\max\left\{\sup_{t\geqslant 0}\|u(t,\cdot)\|_{L^2(\Omega)}, 1\right\}. \end{aligned} \quad (30)$$

Finally we will show that $\sup_{t\geqslant 0}\|u(t,u^0)\|_{L^2(\Omega)} \leqslant \gamma$, where $\gamma>0$. Indeed, similarly as we did above, we multiply the equation of the problem (1) for u and integrate over Ω we obtain

$$\begin{aligned} \int_\Omega u_t u\,dx = &-\int_\Omega\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx + \int_\Omega\sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i}u\,dx \\ &+\int_\Omega(c(x)+\lambda)u^2dx + \int_\Omega f(u)u\,dx. \end{aligned}$$

From (2) and (12) we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx \leqslant &-\vartheta\int_\Omega\sum_{i=1}^n\left(\frac{\partial u}{\partial x_i}\right)^2dx + B_\infty^*\int_\Omega\sum_{i=1}^n\frac{\partial u}{\partial x_i}u\,dx \\ &+(C_\infty^*+\lambda)\int_\Omega u^2dx + \int_\Omega(\zeta u^2+\tau)dx, \end{aligned}$$

where B_∞^* e C_∞^* are defined as above. Using Poincare's and Hölder's inequality we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^2dx \leqslant &-2\bar{C}\vartheta\int_\Omega u^2dx + 2B_\infty^*\left(\int_\Omega\sum_{i=1}^n\left(\frac{\partial u}{\partial x_i}\right)^2dx\right)^{\frac{1}{2}} \\ &\times\left(n\int_\Omega u^2dx\right)^{\frac{1}{2}} + 2(C_\infty^*+\lambda)\int_\Omega u^2dx \\ &+2\zeta\int_\Omega u^2dx + 2\tau|\Omega|. \end{aligned} \quad (31)$$

By Cauchy's inequality with $\varepsilon_0>0$ in (31), we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^2dx \leqslant 2&\left(-\bar{C}\vartheta + \frac{\bar{C}B_\infty^*\varepsilon_0}{2} + C_\infty^* + \lambda + \zeta + \frac{nB_0^*}{\varepsilon_0}\right) \\ &\times\int_\Omega u^2dx + 2\tau|\Omega|, \end{aligned} \quad (32)$$

where in the last inequality we have used Poincaré's inequality.

Choosing ε_0 such that

$$-\bar{C}\vartheta + \frac{\bar{C}B_\infty^*\varepsilon_0}{2} + C_\infty^* + \lambda + \zeta + \frac{nB_\infty^*}{\varepsilon_0} \leqslant -1.$$

Thus, in (32) we have

$$\frac{d}{dt}\int_\Omega u^2dx \leqslant -2\int_\Omega u^2dx + 2\tau|\Omega|. \quad (33)$$

Using Lemma 1.2.4 in [5, p. 17] to (33), we get

$$\sup_{t\geqslant 0}\|u(t,u^0)\|_{L^2(\Omega)} \leqslant \max\left\{\|u^0\|_{L^2(\Omega)}, \tau|\Omega|\right\}. \quad (34)$$

Therefore, using (34) in (30) we obtain

$$\sup_{t\geqslant 0}\|u(t,u^0)\|_{L^\infty(\Omega)} \leqslant \delta^{\frac{3}{2}}2^{\frac{3n}{4}+1}\max\left\{\|u^0\|_{L^2(\Omega)}, 1\right\}.$$

Thus we get the result. □

Lemma 6. *Let $f:\mathbb{R}\to\mathbb{R}$ be a function of class $C^2(\mathbb{R})$ satisfying (D). Then given $\varepsilon>0$ there exists $m_\varepsilon=:m>0$ such that*

$$vf(v)\leqslant \varepsilon v^2+m,$$

for all $v\in\mathbb{R}$.

Proof. From the assumption (D), follows that is there is $M>0$ such that $vf(v)\leqslant\varepsilon v^2$, for all $|v|\geqslant M$. Since the set $\{x\in\overline{\Omega}:|v|\leqslant M\}$ is bounded in $\mathbb{R}^n$ where we take $v=v(x)$ and f be a continuous function. Thus, we have $|vf(v)|\leqslant m_\varepsilon$. Therefore, joining these two facts follow the result. □

3 Local and global solution

Lemma 7(Local solution). *Under the growth assumption (C) the problem (3) has local solution in $X^{1/2}$.*

Proof. By Lemma 3 and Lemma 2 we can see that the hypothesis of Theorem 2.1.1 in [5, p. 62] are satisfied, thus we have the result. □

Theorem 1(Bounded global solution). *Assume the growth conditions (C) and dissipation (D) holds, then the solution of the problem (3) with $u^0\in B_R=\{v\in X^{1/2}:\|v\|_{H_0^1(\Omega)}\leq R\}$ is define globally and there exist a constant $K_1>0$ such that*

$$\limsup_{t\to\infty}\|u(t,u^0)\|_{H_0^1(\Omega)}\leq K_1.$$

Proof. From (13) and Lemma 5, we obtain that $u\in L^\infty(\Omega)$, then $\|F(u)\|_X\leq C(\|u\|_{L^\infty(\Omega)})$. Thus, if $u^0\in X^\alpha$, then u is a local solution in X^α of the problem (3) satisfying the constants variation formula. Since, the operator A is sectorial positive, we have

$$\begin{aligned}\|u(t,u^0)\|_{X^\alpha}&\leq\|A^\alpha e^{-At}u^0\|_X\\&\quad+\int_0^t\|A^\alpha e^{-A(t-s)}\|_{\mathscr{L}(X)}\|F(u(s,u^0))\|_X ds\\&\leq c_0e^{-\beta t}\|u^0\|_{X^\alpha}+\int_0^t c_1(t-s)^{-\alpha}e^{-\beta(t-s)}\|F(u(s,u^0))\|_X ds\\&\leq c_0\|u^0\|_{X^\alpha}+c_1C(\|u\|_{L^\infty(\Omega)})\int_0^t(t-s)^{-\alpha}ds,\end{aligned}$$

where c_0, c_1 are positive constants with $\alpha\in(0,1)$ and $\operatorname{Re}\sigma(A)>\beta>0$. It follows that $u(t,u^0)$ in the norm X^α is limited to finite intervals of time with $\alpha<1$. Therefore, for $\alpha=1/2$ a solution is global.

As before, using the formula of the constants variation and since $\|u(t,u^0)\|_{L^\infty(\Omega)}\leq K_\infty$ for all $t\geq0$, we obtain

$$\begin{aligned}\|u(t,u^0)\|_{X^\alpha}&\leq c_\alpha t^{-\alpha}e^{-\beta t}\|u^0\|_X\\&\quad+\int_0^t c_1(t-s)^{-\alpha}e^{-\beta(t-s)}C(K_\infty)ds\\&\leq c_\alpha t^{-\alpha}e^{-\beta t}\|u^0\|_{L^\infty(\Omega)}+c_1C(K_\infty)\int_0^t(t-s)^{-\alpha}e^{-\beta(t-s)}ds.\end{aligned}$$

From this, it follows

$$\begin{aligned}\limsup_{t\to\infty}\|u(t,u^0)\|_{X^\alpha}&\leq\limsup_{t\to\infty}c_1C(K_\infty)\int_0^t r^{-\alpha}e^{-\beta r}ds\\&=c_1C(K_\infty)\Gamma(1-\alpha)\beta^{\alpha-1}=:K_1.\square\end{aligned}$$

Let $\{T(t,\cdot):t\geqslant0\}$ be a semigroup in $X^{1/2}$ given by

$$T(t,u^0)=u(t,u^0),\ \ \forall t\geqslant0,$$

where u is the unique global solution of (3). This semigroup $\{T(t,\cdot):t\geqslant0\}$ is a C_0-semigroup in $X^{1/2}$.

As a consequence of Theorem 1 results

Corollary 1. *Under the hypotheses of Theorem 1, it follows that semigroup $\{T(t,u^0):t\geq0\}$ is point dissipative.*

4 Existence of global attractor

Theorem 2. *The C_0-semigroup $\{T(t,\cdot):t\geqslant0\}$ associated to the problem (3) has global attractor $\mathscr{A}$ in $X^{1/2}$.*

Proof. We show first that C_0-semigroup $\{T(t,u^0):t\geq0\}$ is compact in $X^{1/2}$. In fact, we see that the resolvent of A is compact, since $X^1=D(A)$ is embedding compactly in $X=L^2(\Omega)$ and of the Proposition 4.25 in [6, p. 118], the result follows. Now, using the Theorem 3.3.1 in [5, p. 80], we have the C_0-semigroup $\{T(t,u^0):t\geq0\}$ is compact in $X^{1/2}$. Finally, by Corollary's 1 and 1.1.6 in [5, p.13] the result follows. □

As a consequence of Lemma 5 and Lemma 6, we have

Theorem 3. *Assuming the same hipotheses of Lemma 13, we have*

$$\sup_{u\in\mathscr{A}}\|u\|_{L^\infty(\Omega)}<K_0,$$

where $K_0=K_0(\Omega,n,\zeta,\tau,\|u^0\|_{L^\infty(\Omega)})$ is a positive constant.

Acknowledgement

The first author was partially supported by FAPESP(Brazil) through the research grant 2009/08088-9 and the second author was partially supported by FAPESP(Brazil) through the research grant 2009/08435-0.

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