A Coalitional Value for Multichoice Games with a Coalition Structure

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Abstract: In this paper a new model of multichoice games with a coalition structure is proposed, which can be seen as an extension of the Owen coalition structure. A coalitional value on the given model is defined, which can be seen as an extension of the Owen value. Three axiomatic systems are studied. The first one is enlightened by Owen's characterization for the Owen value and Faigle and Kern's characterization for the Shapley value on games under precedence constraints. The second one is inspired by Bilbao's characterization for the Shapley value on games on convex geometries. The last one is an extension of Young's characterization for the Shapley value on traditional games. Furthermore, the relationship between the given coalitional value and the core of multichoice games with a coalition structure is discussed.

Keywords: multichoice game, coalition structure, Shapley value, core

1 Introduction

A multichoice game is a generalization of a traditional TU game in which each player has several activity levels. The reward that a group of players can obtain depends on the efforts of the cooperating players. Hsiao and Raghavan [1] introduced multichoice games in which players have the same number of activity levels, and with each level of activity a weight is associated in order to discriminate between those levels. van den Nouweland et al. [2] considered a more general case with different numbers of activity levels, and extended the notions of core, dominant core and Weber set. They also proposed an alternative extension for the Shapley value based on an extension of the probabilistic formula by orders, but they did not give additional support for this extension. Calvo and Santos [3] studied another value for multichoice games, which is focused on total payoff instead of payoff per level. Klijn et al. [4] have studied a new solution to multichoice games, which is based on the work of Derks and Peters [5]. Calvo and Santos [6] researched a value for multichoice games by restricting the Aumann-Shapley value for continuum games in framework of multichoice games. Peters and Zank [7] proposed the egalitarian solution for multichoice games with each player having

the same number of activity levels, which is an extension of the egalitarian solution proposed by Dutta and Ray [8] for traditional games. Recently, Hwang and Liao [9] investigated the weighted associated consistent value, and characterized it by means of the weighted balanced contributions and the associated consistency. More researches can be seen in [4,6,9,10].

As we know, in some cooperative games, such as Economic Community and Military Alliance, the players are joined in coalitions that form a partition or coalitional structure of the set of players. Aumann and Drze [11] first researched in this area and proposed a model of games with a coalition structure, where the players in a union are independent to other players. Different to Aumann and Drze [11], Owen [12] gave another model for games with a coalition structure, where the probability of cooperation among coalitions is considered, and provided the Owen value, which is an extension of the Shapley value. Later, many experts and scholars have studied deeply [13,14,15, 16,17,18,19].

All above researches are about multichoice games and traditional games with a coalition structure. Both contexts, TU games with a priori system of unions and multichoice games are unified by Albizuri [20] who introduced the concept of a coalition structure for

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multichoice games and defined a multichoice coalition value in this framework. This value is an extension of the Shapley value [21] for traditional games. Based on the coalition structure introduced by Albizuri [20], Jones and Wilson [22] defined another coalitional value in a similar way as the Owen value. Based on the building axiomatic system, the authors showed its existence and uniqueness. From the analysis [22], it shows that the calculation of the introduced coalitional value is complex in the setting of the given model [20] when each player has several participation levels. Further, the coalitional value gives the same weight with respect to the different participation levels. In this paper we introduce another coalition structure for multichoice games, which is an extension of the coalition structure proposed by Owen [12]. According to the works of van den Nouweland et al. [2] and Owen [12], a coalitional value for multichoice games with a coalition structure is proposed, and three axiomatic systems for the given coalitional value are discussed, by which its existence and uniqueness can be proved. When multichoice games are convex, the relationship between the introduced coalitional value and the core is examined, which coincides with the classical case.

This paper is organized as follows. In Section 2, some basic concepts and notions for multichoice games are briefly reviewed. In Section 3, the model of multichoice games with a coalition structure is presented, and a coalitional value is defined. In Section 4, three axiomatic systems for the given coalitional value are discussed. In Section 5, the relationship between the given coalitional value and the core of multichoice games with a coalition structure is examined. The conclusion is made in the last section.

2 Preliminaries

Let $N = \{1, 2, ..., n\}$ be a set of players, and suppose each player $i \in N$ has $m_i + 1$ activity levels at which he can play. We set $M_i = \{0, 1, ..., m_i\}$ as the action space of player $i \in N$, where the action 0 means no participating, and $M_i^+ = \{1, ..., m_i\}$ for any $i \in N$. A function $v : \Pi_{i \in N} M_i \to \Re^+$ with $v(\emptyset) = 0$ gives each coalition $s = (s_1, s_2, ..., s_n) \in M = \Pi_{i \in N} M_i$ the worth that the players can obtain when each player i plays at level $s_i \in M_i$. Let $m = (m_1, m_2, ..., m_n)$, and a multichoice game be denoted by a triple (N, m, v), if there is not confusion we will simply denote a multichoice game (N, m, v) by v. The set of all multichoice games with the player set N is denoted by MC^N. For all coalitions $s, t \in M$, let $s \wedge t = (s_i \wedge t_i)_{i \in N}$ and $s \vee t = (s_i \vee t_i)_{i \in N}$. Furthermore, we denote $s \leq t$ for all $s, t \in M$ if and only if $s_i \leq t_i$ for each $i \in N$. For any $s = (s_1, ..., s_n) \in M$, let $S = \{i \in N | s_i > 0\}$. e^S denotes the vector in N satisfying $e_i^S = 1$ for $i \in S$, otherwise, $e_i^S = 0$, where $S \subseteq N$. Especially, $e^\emptyset$ denotes the vector in N satisfying $e_i^S = 0$ for all $i \in N$.

van den Nouweland et al. [2] gave a value for multichoice games as follows:

$$\Phi(N, m, v) = \frac{\Pi_{i \in N}(m_i!)}{(\sum_{i \in N} m_i)!} \sum_\sigma v^\sigma, \qquad (1)$$

where σ is an admissible order for v, v^σ means the value of v with respect to the admissible order σ.

It is not difficult to know that Eq.(1) is equivalent to the following equation.

$$\Phi_{ij}(N, m, v) = \sum_{s \in M, s_i = j} h_{ij}(s)(v(s) - v(s - e^i))$$

$$\forall i \in N, \ j \in M_i^+, \qquad (2)$$

where

$$h_{ij}(s) = \left(\frac{(\sum_{k:(s|s_i - 1)_k \neq 0} s_k)!}{\Pi_{k:(s|s_i - 1)_k \neq 0}(s_k!)} \frac{(\sum_{k \in N}(m_k - s_k))!}{\Pi_{k \in N}((m_k - s_k)!)} \right) \Big/ \frac{(\sum_{k \in N} m_k)!}{\Pi_{k \in N}(m_k!)},$$

and $s | s_i - 1 = (s_1, ..., s_{i-1}, s_i - 1, s_{i+1}, ..., s_n)$.

$\frac{(\sum_{k:(s|s_i - 1)_k \neq 0} s_k)!}{\Pi_{k:(s|s_i - 1)_k \neq 0}(s_k!)}$ is the number of admissible orders from coalition $e^\emptyset$ to coalition s, where s_i is the last step, and $\frac{(\sum_{k \in N}(m_k - s_k))!}{\Pi_{k \in N}((m_k - s_k)!)}$ is the number of admissible orders from coalition s to coalition m.

3 A coalitional value for multichoice games with a coalition structure

A coalition structure on (N, m, v), given by Albizuri [20], is a family $\mathscr{B} = \{B_1, ..., B_w\}$ of coalitions on M such that $B_k = (x_1^k, x_2^k, ..., x_n^k)$ with $\sum_{k=1}^w B_k = (m_1, m_2, ..., m_n)$, where x_i^k denotes the activity level of the player i in coalition B_k. The author further supposed that the same players do not form two coalitions with different activity levels. However, the author does not give the explanation for this restriction.

Based on the coalition structure introduced by Albizuri [20], Jones and Wilson [22] introduced the following coalitional value, for any $i \in N$, expressed by

$$\Omega_i(N, v, \mathscr{B}) = \sum_{B_k \in A_i} \sum_{R \subseteq B_k \setminus i} \sum_{S \subseteq W \setminus k} \frac{|S|!(|W| - |S| - 1)!}{|W|!}$$

$$\times \frac{|R|!(|B_k| - |R| - 1)!}{|B_k|!}$$

$$\times \left[v\left(\mathbf{z} + \left(\mathbf{x}_{-i}, x_i^j\right)\right) - v(\mathbf{z} + (\mathbf{x}_{-i}, 0))\right], \quad (3)$$

where $|S|$, $|W|$, $|R|$ and $|B_k|$ respectively denote the cardinalities of S, W, R and B_k, $W = \{1, 2, ..., w\}$, $B_k = \{i \in N | x_i^k \in M_i^+\}$, $A_i = \{B_k | i \in B_k\}$, $\mathbf{z} = \sum_{l \in S} B_l$, and $\mathbf{x}_{-i} = (\varepsilon_1 x_1^j, ..., \varepsilon_{i-1} x_{i-1}^j, \varepsilon_{i+1} x_{i+1}^j, ..., \varepsilon_n x_n^j)$ with $\varepsilon_p = \begin{cases} 1 & p \in R \\ 0 & \text{otherwise} \end{cases}$.

It is easy to see that the coalitional value degenerates to be the Owen value when each player has only two participation levels 0 and 1. From Eq.(3), we know it is

focused on total payoff instead of payoff per level. Since the weights given by Eq.(3) only consider the number of active players, it gives the same weight with respect to the different participation levels.

From the coalition structure [20], we know the author considers the player participation levels satisfy addtitivity, e.g., let $M_i^+ = \{1,2\}$, then the player i's participation level 2 is equal to the sum of his two participation levels 1. This point seems to be unreasonable in some situations. In above example, if these two participation levels of the player i respectively denote a number of money, and the participation levels 1 and 2 respectively denote one million dollars and five million dollars. It is obvious that the participation level 2 is not equal to the sum of two participation levels 1. In the setting of Albizuri's model, the total money of the player i used for cooperation is two million dollars rather than five million dollars.

Different to the coalition structure given by Albizuri [20], we introduce another coalition structure on multichoice games. Owen [12] gave a coalition structure for traditional games as follows:

Definition 3.1. [12] $\Gamma = \{B_1, B_2, ..., B_h\}$ is said to be a coalition structure on player set N, if it satisfies $\cup_{1 \le k \le h} B_k = N$ and $B_k \cap B_l = \emptyset$ for all $k \ne l$, where $k, l \in P = \{1, 2, ..., h\}$.

Here, we apply the coalition structure introduced by Owen [12] in the framework of multichoice games. For any given $B_k \in \Gamma$, let $b_k \in M$ denote the "maximum coalition" on B_k, namely, $(b_k)_i = m_i$ for all $i \in B_k$, and $(b_k)_i = 0$, otherwise. Then $\{b_1, b_2, ..., b_h\}$ is a partition of m. Let (N, m, Γ) denote a coalition structure on (N, m).

Definition 3.2. $s \in M$ is said to be a feasible coalition for (N, m, Γ), if there exist $B_k \in \Gamma$ and $\cup_{l \in R \subseteq P \setminus k} B_l$ such that $s = t_1 \vee t_2$, where $t_1 \le b_k$ and $t_2 = \vee_{l \in R \subseteq P \setminus k} b_l$.

Let $Fe(N, m, \Gamma)$ be the set of all feasible coalitions in (N, m, Γ). From Definition 3.2, it is easy to see when the coalition structure (N, m, Γ) is restricted in the setting of the model given in [12], then $Fe(N, m, \Gamma)$ degenerates to be the set of feasible coalitions for Γ on N.

Example 3.1. Let the player set $N = \{1, 2, 3, 4\}$, $m = (2, 2, 2, 1)$ and $\Gamma = \{B_1, B_2\}$, where $B_1 = \{1, 2\}$ and $B_2 = \{3, 4\}$. For B_1 and $s = (1, 2, 0, 0) \le b_1$, when the players 1 and 2 cooperate with the players 3 and 4, then $s' = (1, 2, 2, 1)$ is the only feasible coalition with respect to s.

Example 3.2. Let the player set $N = \{1, 2, 3\}$, $m = (2, 2, 1)$ and $\Gamma = \{B_1, B_2\}$, where $B_1 = \{1\}$ and $B_2 = \{2, 3\}$, then $Fe(N, m, \Gamma) = \{(1,0,0), (0,1,0), (0,0,1), (2,0,0)(0,2,0), (0,1,1), (0,2,1), (2,1,0), (2,2,0), (2,0,1), (1,2,1), (2,1,1), (2,2,1)\}$.

Definition 3.3. Let $v \in MC^N$, $t \in M$ is said to be a carrier for v in M if $v(s) = v(s \wedge t)$ for all $s \in M$.

Definition 3.4. Let $v \in MC^N$ and $\Gamma = \{B_1, B_2, ..., B_h\}$. v^B is said to be a quotient game on Γ, if it satisfies $v^B(R) = v(\vee_{l \in R} b_l)$ for all $R \subseteq P = \{1, 2, ..., h\}$.

Definition 3.3 is an extension of the carrier proposed by Shapley [21], and Definition 3.4 is an extension of the quotient game introduced by Owen [12].

Let (N, m, v, Γ) be a multichoice game with the coalition structure Γ, and MC^{CN} be the set of all multichoice games with a coalition structure. Following the works of van den Nouweland et al. [2] and Owen [12], we introduce a coalitional value on MC^{CN} as follows:

$$\psi_{ij}(N, m, v, \Gamma) = \sum_{R \subseteq H \setminus k} \sum_{s \le b_k, s_i = j} \frac{r!(h-r-1)!}{h!} h_{ij}^{B_k}(s)$$

$$(v(s \vee t) - v(s \vee t - e^i)) \ \forall i \in N, \ j \in M_i^+, \quad (4)$$

where p and r respectively denote the cardinalities of P and R, and $t = \vee_{w \in R} b_w$. $h_{ij}^{B_k}(s)$ is the coefficient for the coalition s restricted on B_k. Namely, $h_{ij}^{B_k}(s) = \left(\frac{(\Sigma_{g:(s|s_i-1)_g \ne 0} s_g)!}{\Pi_{g:(s|s_i-1)_g \ne 0}(s_g!)} \frac{(\Sigma_{g \in B_k}(m_g-s_g))!}{\Pi_{g \in B_k}((m_g-s_g)!)} \right) / \frac{(\Sigma_{g \in B_k} m_g)!}{\Pi_{g \in B_k}(m_g!)}$ with $s|s_i - 1 = (s_{k_1}, ..., s_i - 1, ..., s_{k_t})$ and $\{k_1, ..., i, ..., k_t\} \subseteq B_k$.

Obviously, Eq.(4) degenerates to be the Owen value [12] when v is restricted in the setting of games with a coalition structure, namely, every player has only two activity levels 0 and 1. Furthermore, Eq.(4) degenerates to be the Shapley value for multichoice games [2] when h is equal to 1, namely, there is only one coalition in Γ, that is N. In other words, Eq.(4) can be seen as an extension both of them.

As we know, when a coalition structure for traditional games has n coalitions, the Owen value degenerates to be the Shapley value for traditional games. But this point does not hold any more when it turns to multichoice games. Namely, when a coalition structure Γ for a multichoice game has n coalitions, it can not guarantee the given coalitional value is equal to the Shapley value introduced by van den Nouweland et al. [2].

Example 3.3. Let $N = \{1, 2\}$, $m = (2, 2)$ and $\Gamma = \{B_1, B_2\}$, where $B_1 = \{1\}$ and $B_2 = \{2\}$. If the values for coalitions are given as follows: $v(1, 0) = v(0, 1) = 1$, $v(2, 0) = v(0, 2) = 3$, $v(1, 1) = 5$, $v(1, 2) = 8$, $v(2, 1) = 10$, $v(2, 2) = 15$. From Eq.(4), it has

$$\psi_{11}(N, m, v, \Gamma) = 3,$$

$$\psi_{12}(N, m, v, \Gamma) = 4.5,$$

$$\psi_{21}(N, m, v, \Gamma) = 4,$$

$$\psi_{22}(N, m, v, \Gamma) = 3.5.$$

From Eq.(2), it gets

$$\phi_{11}(N, m, v) = 2.67,$$

$$\phi_{12}(N, m, v) = 5.5,$$

$$\phi_{21}(N, m, v) = 3,$$

$$\phi_{22}(N, m, v) = 3.83.$$

From above, we know $\psi \ne \phi$.

4 Characterizations of the given coalitional value

In this section, we mainly discuss the axiomatic systems of the given coalitional value. Let φ be a coalitional value on MC^{CN}.

The analogue of unanimity games for multichoice games are minimal effort games $(N, m, u_s) \in MC^N$, for any $s \in M \backslash \{e^{\emptyset}\}$, defined by

$$u_s(t) = \begin{cases} 1 & s \leq t \\ 0 & \text{otherwise} \end{cases}$$

for all $t \in M \backslash \{e^{\emptyset}\}$.

The identity games for multichoice games are games $(N, m, \delta_s) \in MC^N$, for any $s \in M \backslash \{e^{\emptyset}\}$, defined by

$$u_s(t) = \begin{cases} 1 & s = t \\ 0 & \text{otherwise} \end{cases}$$

for all $t \in M \backslash \{e^{\emptyset}\}$.

According to the literature [12,23], we give the following properties for the coalitional value φ.

additivity (**ADD**). Let $v, w \in MC^{CN}$, then

$$\varphi(N, m, v + w, \Gamma) = \varphi(N, m, v, \Gamma) + \varphi(N, m, w, \Gamma).$$

efficiency (**EFF**). Let $v \in MC^{CN}$, then

$$\sum_{i \in N} \sum_{j=1}^{m_i} \varphi_{ij}(N, m, v, \Gamma) = v(m).$$

null property (**NP**). Let $v \in MC^{CN}$ and $i \in N$, if $j \in M_i \backslash \{e^{\emptyset}\}$ is a "null level", then $\varphi_{ij}(N, m, v, \Gamma) = 0$. Here, $j \in M_i \backslash \{e^{\emptyset}\}$ is said to be a "null level" if $v(s \vee (j-1)e^i) = v(s \vee je^i)$ for all $s \in \Pi_{k \in N \backslash \{i\}} M_k$.

hierarchical strength within coalition (**HST**). Let $v \in MC^{CN}$. For $B_k \in \Gamma$ and all $i, j \in B_k$, it has

$$\sum_{\substack{s \wedge b_k \leq x \leq b_k, \\ x_j = s_j}} h_{js_j}^{B_k}(x) \varphi_{is_i}(N, m, u_s, \Gamma)$$

$$= \sum_{\substack{s \wedge b_k \leq x \leq b_k, \\ x_i = s_i}} h_{is_i}^{B_k}(x) \varphi_{js_j}(N, m, u_s, \Gamma)$$

for any $s \in Fe(N, m, \Gamma)$ with $s_i \in M_i^+$ and $s_j \in M_j^+$.

symmetry in quotient game (**SQ**). Let $v \in MC^{CN}$. For all $B_k, B_s \in \Gamma$, if $v^B(R \cup k) = v^B(R \cup s)$ for any $R \subseteq P \backslash \{k, s\}$, then

$$\sum_{i \in B_k} \sum_{l=1}^{m_i} \varphi_{il}(N, m, v, \Gamma) = \sum_{j \in B_s} \sum_{h=1}^{m_j} \varphi_{jh}(N, m, v, \Gamma).$$

Remark 4.1. The properties **ADD**, **EFF**, **NP** and **SQ** degenerate to be the properties introduced by Owen [12] when we restrict the domain of CM^{CN} in the setting of traditional games with a coalition structure. Furthermore, the property **HST** degenerates to be the property proposed

by Faigle and Kern [23] when the domain of CM^{CN} is limited to traditional games.

Theorem 4.1. A coalitional value satisfies **ADD**, **EFF**, **NP**, **HST** and **SQ** if, and only if, it is equal to Eq.(4), *i.e.*, $\varphi_{ij}(N, m, v, \Gamma) = \psi_{ij}(N, m, v, \Gamma)$ for all $i \in N$ and all $j \in M_i^+$, where $v \in MC^{CN}$.

Proof. First we show that ψ satisfies these properties. ψ satisfies **ADD** and **NP** as is easily seen with Eq.(4).

In the following, we shall ψ show satisfies the rest characterizations.

From Eq.(4), it has

$$\sum_{i=1}^{N} \sum_{j=1}^{m_i} \psi_{ij}(N, m, v, \Gamma)$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{m_i} \sum_{R \subseteq P \backslash k} \sum_{s \leq b_k, s_i = j} \frac{r!(p-r-1)!}{p!}$$

$$\times h_{ij}^{B_k}(s)(v(s \vee t) - v(s \vee t - e^i)).$$

For any $s \leq b_k$, let $v^t(s) = v(s) - v(t)$. It gets

$$\sum_{i=1}^{N} \sum_{j=1}^{m_i} \psi_{ij}(N, m, v, \Gamma)$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{m_i} \sum_{R \subseteq P \backslash k} \sum_{s \leq b_k, s_i = j} \frac{r!(p-r-1)!}{p!}$$

$$\times h_{ij}^{B_k}(s)(v(s \vee t) - v(s \vee t - e^i))$$

$$= \sum_{k \in P} \sum_{R \subseteq P \backslash k} \frac{r!(p-r-1)!}{p!} v^t(b_k)$$

$$= \sum_{k \in P} \sum_{R \subseteq P \backslash k} \frac{r!(p-r-1)!}{p!} (v(b_k \vee t) - v(t))$$

$$= \sum_{k \in P} \sum_{R \subseteq P \backslash k} \frac{r!(p-r-1)!}{p!} (v^B(R \cup k) - v^B(R))$$

$$= v^B(P)$$

$$= v(\vee_{k \in P} b_k)$$

$$= v(m).$$

From above, it shows **EFF** holds.

From Eq.(4) and the conditions in **HST**, it gets

$$\psi_{is_i}(N, m, u_s, \Gamma)$$

$$= \sum_{R \subseteq P \backslash k} \sum_{x \leq b_k, x_i = s_i} \frac{r!(p-r-1)!}{p!} h_{is_i}^{B_k}(x)$$

$$\times (u_s(x \vee t) - u_s(x \vee t - e^i))$$

$$= \sum_{P' \backslash k \subseteq R \subseteq P \backslash k \wedge s \leq x \leq b_k, x_i = s_i} \frac{r!(p-r-1)!}{p!} h_{is_i}^{B_k}(x)$$

$$= \sum_{P'\backslash k \subseteq R \subseteq P\backslash k} \frac{r!(p-r-1)!}{p!} \sum_{s\wedge b_k \le x \le b_k, x_i = s_i} h_{is_i}^{B_k}(x)$$

$$= \frac{1}{p'} \sum_{s\wedge b_k \le x \le b_k, x_i = s_i} h_{is_i}^{B_k}(x),$$

where $\frac{1}{p'} = \sum_{P'\backslash k \subseteq R \subseteq P\backslash k} \frac{r!(p-r-1)!}{p!}$ and p' is the cardinality of $P' = \{l \in P | b_l \wedge s \ne e^{\emptyset}\}$.
For the same reason, it has

$$\psi_{js_j}(N,m,u_s,\Gamma) = \frac{1}{p'} \sum_{s\wedge b_k \le x \le b_k, x_j = s_j} h_{js_j}^{B_k}(x).$$

Now one can easily verify that ψ indeed satisfies the property of **HST**.
From the proof of **EFF**, it has

$$\sum_{i\in B_k} \sum_{l=1}^{m_i} \psi_{il}(N,m,v,\Gamma)$$

$$= \sum_{R\subseteq P\backslash k} \frac{r!(p-r-1)!}{p!} (v^B(R\cup k) - v^B(R))$$

and

$$\sum_{j\in B_s} \sum_{h=1}^{m_i} \psi_{jh}(N,m,v,\Gamma)$$

$$= \sum_{R\subseteq P\backslash s} \frac{r!(p-r-1)!}{p!} (v^B(R\cup s) - v^B(R)).$$

Since, $v^B(R\cup k) = v^B(R\cup s)$ for all $R \subseteq P\backslash\{k,s\}$, it gets **SQ**.
Uniqueness. Let φ be a coalitional value on MC^{CN}, and satisfy the above mentioned properties. In the following, we shall show $\varphi_{ij}(N,m,v,\Gamma) = \{\psi_{ij}\}(N,m,v,\Gamma)$ for all $i \in N$ and all $j \in M_i^+$.
From Lemma 3.1 in Derks and Peters [5] and **ADD,** it is sufficient to show that φ and ψ coincides on the class of the analogue of unanimity games. For any $s \in Fe(N,m,\Gamma)$, let $P' = \{l \in P | b_l \wedge s \ne e^{\emptyset}\}$ and $B'_l = B_l \cap S$. Define quotient game (P, u_s^P) as follows:

$$u_s^B(R) = \begin{cases} 1 & P' \subseteq R \\ 0 & \text{otherwise} \end{cases}$$

for all $R \subseteq P$.
Since, φ satisfies **NP**, we get $\varphi_{ij}(N,m,u_s,\Gamma) = 0$ for all $i \in N\backslash S$ with $j \in M_i^+$, and all $i \in S$ with $j \in M_i^+$ and $j \ne s_i$.
From **EFF**, **NP** and **SQ**, it gets

$$\sum_{i\in \dot{B}_k} \varphi_{is_i}(N,m,u_s,\Gamma) = \begin{cases} \frac{1}{p'} & k \in P' \\ 0 & k \notin P' \end{cases}$$

for all $i \in S$ with $j = s_i \in M_i^+$.
From **EFF** and **HST**, it has

$$\varphi_{ij}(N,m,u_s,\Gamma) = \begin{cases} \frac{1}{p'} \sum_{s\wedge b_k \le x \le b_k, x_i = s_i} h_{is_i}^{B_k}(x) & j = s_i \in M_i^+ \\ 0 & \text{otherwise} \end{cases}.$$

On the other hand, from Eq.(4) it gets

$$\psi_{ij}(N,m,u_s,\Gamma) = \begin{cases} \frac{1}{p'} \sum_{s\wedge b_k \le x \le b_k, x_i = s_i} h_{is_i}^{B_k}(x) & j = s_i \in M_i^+ \\ 0 & \text{otherwise} \end{cases}.$$

Hence, $\varphi(N,m,u_s,\Gamma) = \psi(N,m,u_s,\Gamma)$. From the randomicity of $s \in Fe(N,m,\Gamma)$, the desired result is obtained. $\square$
linearity (**L**). Let $v, w \in MC^{CN}$ and $\alpha, \beta \in \Re$, then

$$\varphi(N,m,\alpha v + \beta w, \Gamma) = \alpha\varphi(N,m,v,\Gamma) + \beta\varphi(N,m,w,\Gamma).$$

carrier (**C**). If $t \in M$ is a carrier for the multichoice game $v \in MC^{CN}$, then

$$\sum_{i\in T} \sum_{j=1}^{t_i} \varphi_{ij}(N,m,v,\Gamma) = v(t),$$

where $T = \{k | t_k \ne 0, k \in N\}$.
When we restrict the domain of MC^{CN} in the setting of traditional games, the property **C** degenerates to be the property proposed by Shapley [21].
Remark 4.2. From Theorem 4.1, we know Eq.(4) is the unique coalitional value on MC^{CN} that satisfies **ADD**, **C**, **HST** and **SQ**, as well as the unique coalitional value on MC^{CN} that satisfies **L**, **C**, **HST** and **SQ**.
Bilbao [24] gave a characterization of the Shapley value for games on convex geometries by using chain axiom. Here, we define it in the setting of multichoice games with a coalition structure and give the following property.
chain axiom (**CA**). Let $v \in MC^{CN}$. For all $B_k \in \Gamma$ and all $i, j \in B_k$ with $s_i \in M_i^+$ and $s_j \in M_j^+$, then

$$c^{B_k}[0, s | s_i - 1] \varphi_{js_j}(N,m,\delta_s,\Gamma)$$

$$= c^{B_k}[0, s | s_j - 1] \varphi_{is_i}(N,m,\delta_s,\Gamma)$$

and

$$((p'+l-1)!(p-p'-l)!)\, \varphi_{is_i}(N,m,\delta_s,\Gamma)$$

$$= ((p'-1)!(p-p')!)\, \varphi_{is_i}(N,m,\delta_{s'},\Gamma),$$

where $s, s' \in Fe(N,m,\Gamma)$ such that $s \vee_{r=0}^{l} b_r = s'$ and $s \wedge (\vee_{r=0}^{l} b_r) = e^{\emptyset}$. $s | s_i - 1$ and $s | s_j - 1$ as shown in Eq.(4). $c^{B_k}[0, s | s_i - 1] = \frac{(\sum_{g:(s|s_i-1)_g \ne 0} s_g)!}{\Pi_{g:(s|s_i-1)_g \ne 0} (s_g!)}$ and $c^{B_k}[0, s | s_j - 1] = \frac{(\sum_{g:(s|s_j-1)_g \ne 0} s_g)!}{\Pi_{g:(s|s_j-1)_g \ne 0} (s_g!)}$. p' is the cardinality of $P' = \{l \in P | b_l \wedge s \ne e^{\emptyset}\}$.
It is obviously that the chain axiom on MC^{CN} degenerates to be the property introduced by Bilbao [24] when we restrict the domain of MC^{CN} in the setting of traditional games on convex geometries.
Theorem 4.2. There is a unique solution φ defined on MC^{CN} that satisfies **L**, **EFF**, **NP**, **CA** and **SQ**, i.e., $\varphi_{ij}(N,m,v,\Gamma) = \psi_{ij}(N,m,v,\Gamma)$ for all $i \in N$ and all $j \in M_i^+$.

Proof. From Theorem 4.1 and Eq.(4), we know ψ satisfies **L**, **EFF**, **NP** and **SQ**. In the following, we will show ψ satisfies **CA**. From the definition of δ_s, it gets

$$\psi_{is_i}(N,m,\delta_s,\Gamma) = \frac{(p'-1)!(p-p')!}{p!} h_{is_i}^{B_k}(s \wedge b_k)$$
$$= \gamma \frac{(p'-1)!(p-p')!}{p!} c^{B_k}[0,s|s_i-1],$$

where $\gamma = \frac{(\sum_{g \in B_k}(m_g - s_g))!}{\Pi_{g \in B_k}((m_g - s_g)!)} \Big/ \frac{(\sum_{g \in B_k} m_g)!}{\Pi_{g \in B_k}(m_g!)}$.

Similarly, it has

$$\psi_{js_j}(N,m,\delta_s,\Gamma) = \gamma \frac{(p'-1)!(p-p')!}{p!} c^{B_k}[0,s|s_j-1].$$

Thus,

$$c^{B_k}[0,s|s_i-1]\varphi_{js_j}(N,m,\delta_s,\Gamma)$$
$$= c^{B_k}[0,s|s_j-1]\varphi_{is_i}(N,m,\delta_s,\Gamma).$$

From above, it shows

$$\psi_{is_i}(N,m,\delta_s,\Gamma) = \frac{(p'-1)!(p-p')!}{p!} h_{is_i}^{B_k}(s \wedge b_k)$$

and

$$\psi_{is_i}(N,m,\delta_{s'},\Gamma) = \frac{(p'+l-1)!(p-p'-l)!}{p!} h_{is_i}^{B_k}(s' \wedge b_k).$$

Since $h_{is_i}^{B_k}(s' \wedge b_k) = h_{is_i}^{B_k}(s \wedge b_k)$, it gets

$$\big((p'+l-1)!(p-p'-l)!\big)\,\varphi_{is_i}(N,m,\delta_s,\Gamma)$$
$$= \big((p'-1)!(p-p')!\big)\,\varphi_{is_i}(N,m,\delta_{s'},\Gamma).$$

Uniqueness. Let φ be a coalitional value on MC^{CN}, and satisfy the above mentioned properties. For any $v \in MC^{CN}$, there exists an unique set of coefficients $\{v(s) : s \in Fe(N,m,\Gamma)\}$ such that $v = \sum_{s \in Fe(N,m,\Gamma)} v(s)\delta_s$. From **L**, we only need to prove φ and ψ coincides on (N,m,δ_t,Γ) for any $t \in Fe(N,m,\Gamma)$. Similar to Theorem 4.1, for any $s \in Fe(N,m,\Gamma)$, let $P' = \{l \in P | b_l \wedge s \neq e^\emptyset\}$ and $B'_l = B_l \cap S$. Define quotient game (P, u_s^B) as follows:

$$u_s^B(R) = \begin{cases} 1 & P' \subseteq R \\ 0 & \text{otherwise} \end{cases}$$

for all $R \subseteq P$.

From Theorem 4.1, it gets

$$\sum_{i \in \acute{B}_k} \varphi_{is_i}(N,m,u_s,\Gamma) = \begin{cases} \frac{1}{p'} & k \in P' \\ 0 & k \notin P' \end{cases} \quad (5)$$

Since $u_s = \sum_{s \leq t} \delta_t$ for all $s \in Fe(N,m,\Gamma)$, it has

$$\varphi(N,m,u_s,\Gamma) = \varphi(N,m,\sum_{s \leq t}\delta_t,\Gamma)$$
$$= \sum_{s \leq t}\varphi(N,m,\delta_t,\Gamma)$$

and

$$\varphi_{is_i}(N,m,u_s,\Gamma) = \sum_{s \leq t, t_i = s_i} \varphi_{is_i}(N,m,\delta_t,\Gamma).$$

From **CA**, it gets

$$\varphi_{is_i}(N,m,u_s,\Gamma)$$

$$= \sum_{\substack{s \wedge b_k \leq t \wedge b_k, \\ t_i = s_i, t - b_k = s - b_k}} \left(\sum_{0 \leq l \leq p-p'} \frac{(p'+l-1)!(p-p'-l)!}{(p'-1)!(p-p')!} \right)$$
$$\times \varphi_{is_i}(N,m,\delta_t,\Gamma)$$

$$= \left(\sum_{0 \leq l \leq p-p'} \frac{(p'+l-1)!(p-p'-l)!}{(p'-1)!(p-p')!} \right) \sum_{\substack{s \wedge b_k \leq t \wedge b_k, \\ t_i = s_i, t - b_k = s - b_k}}$$
$$\times \varphi_{is_i}(N,m,\delta_t,\Gamma)$$

$$= \frac{1}{p'} \frac{p!}{(p'-1)!(p-p')!} \sum_{\substack{s \wedge b_k \leq t \wedge b_k, \\ t_i = s_i, t - b_k = s - b_k}} \varphi_{is_i}(N,m,\delta_t,\Gamma).$$

From above, we know $P' = \{l \in P | b_l \wedge s \neq e^\emptyset\} = \{l \in P | b_l \wedge t \neq e^\emptyset\}$.

From **CA**, it has

$$h_{it_i}^{B_k}(t)\varphi_{jt_j}(N,m,\delta_t,\Gamma) = h_{jt_j}^{B_k}(t)\varphi_{it_i}(N,m,\delta_t,\Gamma),$$

which is equal to

$$h_{it_i}^{B_k}(t \wedge b_k)\varphi_{jt_j}(N,m,\delta_t,\Gamma) = h_{jt_j}^{B_k}(t \wedge b_k)\varphi_{it_i}(N,m,\delta_t,\Gamma).$$

Namely,

$$\varphi_{it_i}(N,m,\delta_t,\Gamma) = \frac{h_{it_i}^{B_k}(t \wedge b_k)}{h_{jt_j}^{B_k}(t \wedge b_k)}\varphi_{jt_j}(N,m,\delta_t,\Gamma). \quad (6)$$

By Eq.(6), it has

$$\frac{1}{p'} = \frac{1}{p'}\frac{p!}{(p'-1)!(p-p')!}$$
$$\times \sum_{i \in \acute{B}_k} \sum_{\substack{s \wedge b_k \leq t \wedge b_k, \\ t_i = s_i, t - b_k = s - b_k}} \varphi_{is_i}(N,m,\delta_t,\Gamma) \quad (7)$$

and

$$\frac{(p'-1)!(p-p')!}{p!}$$
$$= \sum_{i \in \acute{B}_k} \sum_{\substack{s \wedge b_k \leq t \wedge b_k, \\ t_i = s_i, t - b_k = s - b_k}} \varphi_{is_i}(N,m,\delta_t,\Gamma). \quad (8)$$

When $\sum_{i \in S \cap B_k} s_i = \sum_{i \in B_k}(b_k)_i$, by Eqs.(6) and (7), it gets

$$\frac{(p'-1)!(p-p')!}{p!}$$

$$= \sum_{\substack{i \in B_k \ s \wedge b_k = t \wedge b_k = \\ b_k, t_j = s_j = m_j}} \varphi_{is_i}(N, m, \delta_t, \Gamma)$$

$$= \varphi_{jm_j}(N, m, \delta_t, \Gamma) + \sum_{i \in B_k \setminus j} \frac{h_{im_i}^{B_k}(b_k)}{h_{jm_j}^{B_k}(b_k)} \varphi_{jm_j}(N, m, \delta_t, \Gamma)$$

$$= \frac{\sum_{i \in B_k} h_{im_i}^{B_k}(b_k)}{h_{jm_j}^{B_k}(b_k)} \varphi_{jm_j}(N, m, \delta_t, \Gamma).$$

Since $\sum_{i \in B_k} h_{im_i}^{B_k}(b_k) = 1$, it has

$$\varphi_{jm_j}(N, m, \delta_t, \Gamma) = \frac{(p'-1)!(p-p')!}{p!} h_{jm_j}^{B_k}(b_k)$$

for all $j \in B_k$.

When $\sum_{i \in S \cap B_k} s_i = \sum_{i \in B_k}(b_k)_i - 1$, without loss of generality, let $s_j = m_j - 1$.

By Eqs.(6) and (8), it gets

$$\frac{(p'-1)!(p-p')!}{p!}$$

$$= \sum_{\substack{i \in \acute{B}_k \ s \wedge b_k = b_k - e^j \le t \wedge b_k, \\ t_i = s_i, t - b_k = s - b_k}} \varphi_{is_i}(N, m, \delta_t, \Gamma)$$

$$= \sum_{i \in B'_k \setminus j} \left(\varphi_{im_i}(N, m, \delta_{t=s}, \Gamma) + \varphi_{im_i}(N, m, \delta_{t=s \vee m_j e^j}, \Gamma) \right)$$

$$\quad + \varphi_{j(m_j-1)}(N, m, \delta_{t=s}, \Gamma)$$

$$= \sum_{i \in B_k} \varphi_{is_i}(N, m, \delta_{t=s}, \Gamma) + \sum_{i \in B'_k \setminus j} \varphi_{im_i}(N, m, \delta_{t=s \vee m_j e^j}, \Gamma)$$

$$= \frac{\sum_{i \in B_k} h_{is_i}^{B_k}(t \wedge b_k)}{h_{j(m_j-1)}^{B_k}(t \wedge b_k)} \varphi_{j(m_j-1)}(N, m, \delta_{t=s}, \Gamma)$$

$$\quad + \frac{(p'-1)!(p-p')!}{p!} \sum_{i \in B'_k \setminus j} h_{im_i}^{B_k}(b_k)$$

$$= \frac{\frac{(p'-1)!(p-p')!}{p!} \sum_{i \in B_k} h_{is_i}^{B_k}(t \wedge b_k)}{\frac{(p'-1)!(p-p')!}{p!} h_{j(m_j-1)}^{B_k}(t \wedge b_k)} \varphi_{j(m_j-1)}(N, m, \delta_{t=s}, \Gamma)$$

$$\quad + \frac{(p'-1)!(p-p')!}{p!} \sum_{i \in B'_k \setminus j} h_{im_i}^{B_k}(b_k).$$

Since

$$\sum_{i \in \acute{B}_k} h_{is_i}^{B_k}(t \wedge b_k) + \sum_{i \in \acute{B}_k} \sum_{\substack{s \wedge b_k \le t \wedge b_k, s \wedge b_k \ne t \wedge b_k \\ t_i = s_i, t - b_k = s - b_k}} h_{is_i}^{B_k}(t \wedge b_k)$$

$$= \sum_{i \in \acute{B}_k} \sum_{\substack{s \wedge b_k \le t \wedge b_k, \\ t_i = s_i, t - b_k = s - b_k}} h_{is_i}^{B_k}(t \wedge b_k)$$

$$= 1,$$

it has

$$\varphi_{js_j}(N, m, \delta_t, \Gamma) = \frac{(p'-1)!(p-p')!}{p!} h_{js_j}^{B_k}(t \wedge b_k).$$

From induction and $t_i = s_i$, it gets

$$\varphi_{jt_j}(N, m, \delta_t, \Gamma) = \begin{cases} \frac{(p'-1)!(p-p')!}{p!} h_{jt_j}^{B_k}(t \wedge b_k) & t_j \in M_j^+ \\ 0 & \text{otherwise} \end{cases}.$$

On the other hand, by Eq.(4) it has

$$\psi_{jt_j}(N, m, \delta_t, \Gamma) = \begin{cases} \frac{(p'-1)!(p-p')!}{p!} h_{jt_j}^{B_k}(t \wedge b_k) & t_j \in M_j^+ \\ 0 & \text{otherwise} \end{cases}.$$

Namely, φ and ψ coincides on (N, m, δ_T, Γ). The proof is finished. $\square$

Remark 4.3. From Theorem 4.2, we know Eq.(4) is the unique coalitional value on MC^{CN} that satisfies **ADD**, **C**, **CA** and **SQ**, as well as the unique coalitional value on MC^{CN} that satisfies **L**, **C**, **CA** and **SQ**.

Young [25] proposed a characterization of the Shapley value by using strongly monotonicity. According to the literature [25], we propose strongly monotonicity on MC^{CN} as follows:

strongly monotonicity (**SMON**). Let $v, w \in MC^{CN}$ and $i \in N$. If

$$v(s \vee je^i) - v(s \vee (j-1)e^i) \ge w(s \vee je^i) - w(s \vee (j-1)e^i)$$

for all $s \in Fe(N, m, \Gamma)$ with $s_i = 0$ and $j \in M_i^+$, then

$$\varphi_{ij}(N, m, v, \Gamma) \ge \varphi_{ij}(N, m, w, \Gamma).$$

Similar to the property of symmetry given by Shapley [21] and the property of symmetric in the unions introduced by Owen [12], we introduce the veto property on MC^{CN} as follows:

veto property (**VP**). Let $v \in MC^{CN}$. For $B_k \in \Gamma$ and all $i_1, i_2 \in T \cap B_k$, if $j_1 = t_{i_1} \in M_{i_1}^+$ and $j_2 = t_{i_2} \in M_{i_2}^+$, then

$$\sum_{\substack{x_{i_2} = j_2, \\ t \wedge b_k \le x \le b_k,}} h_{i_2 j_2}^{B_k}(x) \varphi_{i_1 j_1}(N, m, v, \Gamma)$$

$$= \sum_{\substack{x_{i_1} = j_1, \\ t \wedge b_k \le x \le b_k,}} h_{i_1 j_1}^{B_k}(x) \varphi_{i_2 j_2}(N, m, v, \Gamma),$$

where $t \in Fe(N, m, \Gamma)$ is a veto coalition, and $T = \{i \in N | t_i \ne 0\}$. Here, $t \in Fe(N, m, \Gamma)$ is said to be a veto coalition if it satisfies

$$v(s) = \begin{cases} v(t) & s \in Fe(N, m, \Gamma) \ s.t. \ t \le s \\ 0 & \text{otherwise} \end{cases}.$$

Theorem 4.3. A coalitional value φ satisfies **EFF**, **SMON**, **VP** and **SQ** if, and only if, it is equal to Eq.(4), i.e., $\varphi_{ij}(N, m, v, \Gamma) = \psi_{ij}(N, m, v, \Gamma)$ for all $i \in N$ and all $j \in M_i^+$, where $v \in MC^{CN}$.

Proof. From Theorem 4.1 and Eq.(4), It is known that ψ satisfies **EFF**, **SMON** and **SQ**.

In the following, we shall show Eq.(4) satisfies **VP**. For any $t \in Fe(N, m, \Gamma)$, without loss of generality, let

$t = \vee_{l \in P_t \setminus k} b_l \vee s$ such that $s \leq b_k$, where $P_t = \{l \in P | t \wedge b_l \neq 0\}$. Then,

$$\psi_{i_1 j_1}(N, m, v, \Gamma)$$

$$= \sum_{R \subseteq P \setminus k} \sum_{x \leq b_k, x_{i_1} = j_1} \frac{r!(p-r-1)!}{p!} h_{i_1 j_1}^{B_k}(x)$$

$$\times v((x \vee y) - v(x \vee y - e^{i_1}))$$

$$= \sum_{\substack{P_t \setminus k \subseteq R \\ \subseteq P \setminus k}} \sum_{\substack{s \leq x \leq b_k, \\ x_{i_1} = j_1}} \frac{r!(p-r-1)!}{p!} h_{i_1 j_1}^{B_k}(x) v(s \vee (\vee_{l \in R \subseteq P \setminus k} b_l))$$

$$= \sum_{\substack{P_t \setminus k \subseteq R \\ \subseteq P \setminus k}} \frac{r!(p-r-1)!}{p!} \sum_{\substack{s \leq x \leq b_k, \\ x_{i_1} = j_1}} h_{i_1 j_1}^{B_k}(x) v(s \vee (\vee_{l \in R \subseteq P \setminus k} b_l))$$

$$= \frac{1}{p_t} \sum_{\substack{s \leq x \leq b_k, \\ x_{i_1} = j_1}} h_{i_1 j_1}^{B_k}(x) v(t).$$

where $y = \vee_{w \in R} b_w$, and p_t is the cardinality of P_t.
For the same reason, it has

$$\psi_{i_2 j_2}(N, m, v, \Gamma) = \frac{1}{p_t} \sum_{s \leq x \leq b_k, x_{i_2} = j_2} h_{i_2 j_2}^{B_k}(x) v(t).$$

Now one easily verifies that ψ indeed satisfies the property of **VP**.
We now prove the uniqueness part. Form **SMON**, it gets if $v(s \vee je^i) - v(s \vee (j-1)e^i) = w(s \vee je^i) - w(s \vee (j-1)e^i)$ for all $s \in Fe(N, m, \Gamma)$, and all $i \in N$, where $s_i = 0$ and $j \in M_i^+$, then

$$\varphi_{ij}(N, m, v, \Gamma) = \varphi_{ij}(N, m, w, \Gamma). \tag{9}$$

Next consider the symmetric game w on (N, m, Γ) which is identically zero on $Fe(N, m, \Gamma)$. Thus $w(s \vee je^i) - w(s \vee (j-1)e^i) = 0$ for all $s \in Fe(N, m, \Gamma)$, all $i \in N$ and all $j \in M_i^+$, where $s_i = 0$. Since, any $t \in Fe(N, m, \Gamma)$ is a veto coalition, from **EFF**, **VP** and **SQ**, it has $\varphi_{ij}(N, m, w, \Gamma) = 0$ for all $i \in N$ and all $j \in M_i^+$. From Eq.(9), it has $\varphi_{ij}(N, m, w, \Gamma) = 0$ for all $i \in N$ and all $j \in M_i^+$. Namely, φ satisfies **NP**.
According to the proof of Lemma 3.1 in Derks and Peters [5], we know for all $v \in MC^{CN}$, v can be expressed by

$$v = \sum_{t \in Fe(N, m, \Gamma)} \alpha_t u_t, \tag{10}$$

where t is an analogue of unanimity game as given above. By Eq.(4), it gets

$$\psi_{ij}(N, m, u_s, \Gamma) = \begin{cases} \frac{1}{p_t} \sum_{\substack{t \wedge b_k \leq x \leq b_k, \\ x_i = t_i = j}} h_{ij}^{B_k}(x) & j = t_i \in M_i^+ \\ 0 & \text{otherwise} \end{cases}$$

for any $B_k \in \Gamma$, where p_t is the cardinality of $P_t = \{l \in P | t \wedge b_l \neq 0\}$.
Define the index I of v to be the minimum number of non-zero terms in some expression for v of form (10). As in

Young [25], the theorem is proved by induction on I.
Case (i). If $I = 0$, every $j \in M_i^+$ is a "null level" for all $i \in N$, from above, it has

$$\varphi_{ij}(N, m, w, \Gamma) = 0.$$

Case (ii). If $I = 1$, without loss of generality, let $v = \alpha_t u_t$ for some $t \in Fe(N, m, \Gamma)$ where $\alpha_t \neq 0$. From above, it has $\varphi_{ij}(N, m, w, \Gamma) = 0$ for all $i \in N$ and all $j \in M_i^+$ with $j \neq t_i$. When $i \in T = \{i \in N | t_i \neq 0\}$, since t is a veto coalition, from **EFF**, **VP** and **SQ**, it has

$$\varphi_{ij}(N, m, v, \Gamma) = \frac{\alpha_t}{p_t} \sum_{\substack{t \wedge b_k \leq x \leq b_k, \\ x_i = t_i = j}} h_{ij}^{B_k}(x),$$

where $i \in B_k \in \Gamma$.
Therefore $\psi = \varphi$, whenever the index of v is 0 or 1.
Assume now that $\psi = \varphi$, whenever the index of v is at most I, and let v have index $I + 1$ with expression

$$v = \sum_{r=1}^{I+1} \alpha_{t_r} u_{t_r},$$

where $\alpha_{t_r} \neq 0$, $t_r \in Fe(N, m, \Gamma)$ and p_{t_r} is the cardinality of $P_{t_r} = \{l \in P | t_r \wedge b_l \neq 0\}$ for all $r = 1, 2, ..., I + 1$.
Let $t = \wedge_{r=1}^{I+1} t_r$, for any $i \in N \setminus T$ or any $i \in T$ with $(t_r)_i = t_i \neq j \in M_i^+$ for all $r = 1, 2, ..., I + 1$.
Construct the game

$$w = \sum_{r:(t_r)_i \neq 0} \alpha_{t_r} u_{t_r},$$

where $T = \{i \in N | t_i \neq 0\}$.
The index of w is at most I, since $v(s \vee je^i) - v(s \vee (j-1)e^i) = w(s \vee je^i) - w(s \vee (j-1)e^i)$ for all $s \in Fe(N, m, \Gamma)$, where $s_i = 0$. By induction and **SMON**, it has

$$\varphi_{ij}(N, m, w, \Gamma) = \varphi_{ij}(N, m, v, \Gamma)$$

$$= \sum_{r:(t_r)_i \neq 0} \frac{\alpha_{t_r}}{p_{t_r}} \sum_{t_r \wedge b_k \leq x \leq b_k, x_i = (t_r)_i = j} h_{ij}^{B_k}(x)$$

$$= \psi_{ij}(N, m, v, \Gamma).$$

For any $i \in T$ with $(t_r)_i = t_i = j \in M_i^+$ and all $r = 1, 2, ..., I + 1$, t_r is a veto coalition for $\alpha_{t_r} u_{t_r}$, where $r = 1, 2, ..., I + 1$.
From **EFF**, **VP** and **SQ**, it gets

$$\varphi_{ij}(N, m, v, \Gamma) = \sum_{r=1}^{I+1} \frac{\alpha_{t_r}}{p_{t_r}} \sum_{t_r \wedge b_k \leq x \leq b_k, x_i = (t_r)_i = j} h_{ij}^{B_k}(x)$$

$$= \psi_{ij}(N, m, v, \Gamma).$$

$\square$

Remark 4.4. As Young [25] pointed, strongly monotonicity can be replaced by independence condition, namely, a player's value depends only on the vector of his marginal contributions. Khmelnitskaya and Yanovskaya [18] proposed a characterization of the Owen coalition

value by extending Owen's original axioms and replacing *additivity* with *marginilist* (***M***) (i.e. independence condition). From Theorem 4.3, we know Eq.(4) is the unique coalitional value on MC^{CN} that satisfies ***EFF***, ***M***, ***VP*** and ***SQ***.

Property 4.1. Let $v \in MC^{CN}$. If $v(s) - v(s - (j-1)e^i) = v(je^i)$ for all $i \in N$ and all $s \in Fe(N,m,\Gamma)$ with $j = s_i \in M_i^+$, then $\psi_{ij}(N,m,v,\Gamma) = v(je^i)$.

Proof. By Eq.(4), it has

$$\psi_{ij}(N,m,v,\Gamma)$$

$$= \sum_{R \subseteq P \setminus k} \sum_{s \leq b_k, s_i = j} \frac{r!(p-r-1)!}{p!} h_{ij}^{B_k}(s)$$

$$\times (v(s \vee t) - v(s \vee t - e^i))$$

$$= \sum_{R \subseteq P \setminus k} \sum_{s \leq b_k, s_i = j} \frac{r!(p-r-1)!}{p!} h_{ij}^{B_k}(s) v(je^j)$$

$$= \sum_{R \subseteq P \setminus k} \frac{r!(p-r-1)!}{p!} \sum_{s \leq b_k, s_i = j} h_{ij}^{B_k}(s) v(je^j)$$

$$= \sum_{R \subseteq P \setminus k} \frac{r!(p-r-1)!}{p!} v(je^j)$$

$$= v(je^j).$$

$\square$

Player $i \in N$ is called a *dummy player*, if for all $j \in M_i^+$ and all $s \in Fe(N,m,\Gamma)$ with $s_i = 0$, it has $v(s \vee je^i) - v(s - (j^1)e^i) = v(je^i)$. If $i \in N$ is a *dummy player*, then $\psi_{ij}(N,m,v,\Gamma) = v(je^i)$ for all $j \in M_i^+$.

Property 4.2. Let $v \in MC^{CN}$ and any $s \in Fe(N,m,\Gamma)$ with $s_i = j \in M_i^+$. If $v(s) - v(s - e^i) = v(je^i) - v((j-1)e^i)$, then $\psi_{ij}(N,m,v) = v(je^i) - v((j-1)e^i)$.

Proof. The proof of Property 4.2 is similar to that of Property 4.1. $\square$

From Property 4.2, we know if $v(s) - v(s - e^i) = v(je^i) - v((j-1)e^i)$ holds for all $s \in Fe(N,m,\Gamma)$ and all $j \in M_i^+$ with $s_i = j$, then $\sum_{j=1}^{m_i} \psi_{ij}(N,m,v,\Gamma) = v(m_i e^i)$.

5 The relationship between the given coalitional value and the core

Definition 5.1. The multichoice game (N,m,v) is called convex if it satisfies

$$v(s \vee t) + v(s \wedge t) \geq v(s) + v(t)$$

for all $s,t \in M$.

When we restrict the multichoice game (N,m,v) in the setting of traditional games, Definition 5.1 reduces to be the concept of convexity for traditional case. Similar to the definition of the core of games with a coalition structure, we define the core of multichoice games with a

coalition structure as follows:

Definition 5.2. The core $C(N,m,v,\Gamma)$ of $v \in MC^{CN}$ is denoted by

$$C(N,m,v,\Gamma) = \left\{ x | \sum_{i \in S} \sum_{j=1}^{s_i} x_{ij} \geq v(s), \forall s \in Fe(N,m,\Gamma), \right.$$

$$\left. \sum_{i \in N} \sum_{j=1}^{m_i} x_{ij} = v(m) \right\},$$

where $S = \{i | s_i \neq 0, i \in N\}$.

Theorem 5.1. Let $v \in MC^{CN}$ be convex, then $C(N,m,v,\Gamma) \neq \emptyset$.

Proof. Let $\Gamma = \{B_1, B_2, ..., B_h\}$, we rearrange the order of the players in (N,m,Γ), and get $\pi\Gamma = \{\pi B_1, \pi B_2, ..., \pi B_h\}$ such that $\pi B_1 = \{i_1, i_2, ..., i_{d_1}\}$, $\pi B_2 = \{i_{d_1+1}, i_{d_1+2}, ..., i_{d_1+d_2}\}, ..., \pi B_m = \{i_{\sum_{l=1}^{h-1} d_l + 1}, i_{\sum_{l=1}^{h-1} d_l + 2}, ..., i_n\}$, where d_i denotes the cardinality of B_i for all $i \in \{1, 2, ..., h\}$. Let

$$x_{i_1 1} = v(e^{i_1}), ..., x_{i_1 m_{i_1}} = v(m_{i_1} e^{i_1}) - v((m_{i_1} - 1)e^{i_1}), ...,$$

$$x_{i_{d_1} 1} = v(\vee_{l=i_1}^{d_1 - 1} m_l e^l \vee e^{d_1}) - v(\vee_{l=i_1}^{d_1 - 1} m_l e^l), ...,$$

$$x_{i_{d_1} m_{d_1}} = v(\vee_{l=i_1}^{d_1} m_l e^l) - v(\vee_{l=i_1}^{d_1 - 1} m_l e^l \vee (m_{d_1} - 1)e^{d_1}), ...,$$

$$x_{i_{d_1}+1} = v(\vee_{l=i_1}^{d_1} m_l e^l \vee e^{d_1+1}) - v(\vee_{l=i_1}^{d_1} m_l e^l), ...,$$

$$x_{i_n 1} = v(m - (m_{i_n} - 1)e^{i_n}) - v(m - m_{i_n} e^{i_n}), ...,$$

$$x_{i_n m_{i_n}} = v(m) - v(m - e^n).$$

It is obvious that $\sum_{i \in N} \sum_{j=1}^{m_i} x_{ij} = v(m)$ holds.

For any $s \in Fe(N,m,\Gamma)$, without loss of generality, let $s = \vee_{l \in R \subseteq P \setminus k} b_l \vee t$ with $t \leq b_k$. If $t \leq b_k$ and $t \neq b_k$, then it has

$$m - t = \vee_{l \in P \setminus (R \cup k)} b_l \vee b_k - t \wedge b_k.$$

Let $\{j_1, j_2, ..., j_d\} = B_k$ such that $j_1 \leq j_2 \leq ... \leq j_d$. From $t \leq b_k$ and $t \neq b_k$, without loss of generality, let $\{j_t, j_{t+1}, ..., j_{t+a}\} \subseteq B_k$, where $t_{j_c} \neq m_{j_c}$ for all $c \in \{t, t+1, ..., t+a\}$. According to above, we rearrange the order of the players in (N,m,Γ) such that $s = \vee_{l \in \{1,2,...k-1\} \subseteq P \setminus k} \pi b_l \vee t$, where $\pi b_l = \{m_{i_{\sum_{x=1}^{l} d_{x-1}+1}}, m_{i_{\sum_{x=1}^{l} d_{x-1}+2}}, ... m_{i_{\sum_{x=1}^{l+1} d_{x-1}}}\}$ for all $l \in \{1, 2, ..., k-1\}$. Let

$$r = \vee_{l \in \{1,2,...k-1\} \subseteq P \setminus k} \pi b_l \vee_{a=1}^{t-1} m_{j_a} e^{j_a} \vee (s_{j_t} + 1)e^{j_t},$$

then it gets

$$r \vee s = s \vee (s_{j_t} + 1)e^{j_t}, \quad r \wedge s = r - e^{j_t}.$$

From the convexity of v, it gets

$$v(s \vee (s_{j_t} + 1)e^{j_t}) + v(r - e^{j_t}) \geq v(s) + v(r)$$

and

$$x_{j_t(s_{j_t}+1)} = v(r) - v(r - e^{j_t})$$

$$\leq v(s \vee (s_{j_t} + 1)e^{j_t}) - v(s).$$

Namely,

$$(\sum_{i \in S} \sum_{j=1}^{s_i} x_{ij} + x_{j_t(s_{j_t}+1)}) - \sum_{i \in S} \sum_{j=1}^{s_i} x_{ij}$$

$$\leq v(s \vee (s_{j_t}+1)e^{j_t}) - v(s),$$

where $S = \{k | s_k \neq 0, k \in N\}$. Thus,

$$(\sum_{i \in S} \sum_{j=1}^{s_i} x_{ij} + x_{j_t(s_{j_t}+1)}) - v(s \vee (s_{j_t}+1)e^{j_t})$$

$$\leq \sum_{i \in S} \sum_{j=1}^{s_i} x_{ij} - v(s).$$

Repeat the above process $\sum_{i \in N} m_i - \sum_{i \in N} s_i$ times, it has

$$\sum_{i \in S} \sum_{j=1}^{s_i} x_{ij} - v(s) \geq \sum_{i \in N} \sum_{j=1}^{m_i} x_{ij} - v(m) = 0.$$

From Definition 5.1, it shows $x \in C(N, m, v, \Gamma)$.
If $t = b_k$, then let $r = \vee_{l \in \{1,2,...k-1,k\} \subseteq P} \pi b_l \vee e^j$, where $j \in B_{k+1}$ and $k+1 \in P$. Repeat the above process, one can also get $x \in C(N, m, v, \Gamma)$. $\square$
Theorem 5.2. Let $v \in MC^{CN}$ be convex, then $\psi(N, m, v, \Gamma) \in C(N, m, v, \Gamma)$.
Proof. From Eq.(4) and Theorem 5.1, we know the function $\psi(N, m, v, \Gamma)$ is a convex combination of $h_{\sum_{k \in P}(\sum_{i \in B_k} m_i)!} \big/ \prod_{i \in B_k} m_i!$ elements in $C(N, m, v, \Gamma)$. Thus, $\psi(N, m, v, \Gamma) \in C(N, m, v, \Gamma)$ for $C(N, m, v, \Gamma)$ being a convex set.$\square$

Remark 5.1. The relationship between the given coalitional value and the core coincides with that of the Shapley value and the core of traditional case. The proof of Theorem 5.1 is an extension of the classical case [26].
Example 5.1. Let $N = \{1, 2, 3, 4\}$, $m = (2, 3, 2, 1)$ and $\Gamma = \{B_1, B_2\}$, where $B_1 = \{1, 2\}$ and $B_2 = \{3, 4\}$. The coalition values for $v \in MC^{CN}$ are given by Table 1.

Table 1: Values for coalitions

S_0	$v_0(S_0)$	S_0	$v_0(S_0)$	S_0	$v_0(S_0)$
(1,0,0,0)	1	(2,1,0,0)	8	(2,3,1,0)	18
(0,1,0,0)	1	(2,1,0,0)	8	(2,3,2,0)	20
(0,0,1,0)	1	(2,2,0,0)	10	(1,1,2,1)	15
(0,0,0,1)	1	(1,3,0,0)	10	(1,2,2,1)	20
(2,0,0,0)	2	(2,3,0,0)	15	(1,3,2,1)	25
(0,2,0,0)	2	(1,0,2,1)	10	(2,1,2,1)	20
(0,0,2,0)	2	(2,0,2,1)	12	(2,3,1,1)	20
(0,3,0,0)	3	(0,1,2,1)	12	(2,2,2,1)	30
(1,1,0,0)	5	(0,2,2,1)	15	(2,3,2,1)	40
(0,0,1,1)	5	(0,3,2,1)	18		
(1,2,0,0)	8	(2,3,0,1)	15		

By Eq.(4), it has

$$\psi_{11}(N, m, v, \Gamma) = 3.45,$$

$$\psi_{12}(N, m, v, \Gamma) = 6.75,$$

$$\psi_{21}(N, m, v, \Gamma) = 3.55,$$

$$\psi_{22}(N, m, v, \Gamma) = 4,$$

$$\psi_{23}(N, m, v, \Gamma) = 5.75,$$

$$\psi_{31}(N, m, v, \Gamma) = 2.83,$$

$$\psi_{32}(N, m, v, \Gamma) = 8.17,$$

$$\psi_{41}(N, m, v, \Gamma) = 5.5$$

and

$$\sum_{i=1}^{4} \sum_{j=1}^{m_i} \psi_{ij}(N, m, v, \Gamma) = 40 = v(m).$$

From Definition 5.1, we know the game $v \in MC^{CN}$, given in example 5.1, is convex. Thus, $\psi(N, m, v, \Gamma)$ is an element in the core.

6 Conclusion

We have introduced and researched a coalitional value for multichoice games with a coalition structure, which can be regarded as an extension of the Owen value [12] and the Shapley value [2]. Furthermore, we discuss three characterizations of the given coalitional value, which will help us better understand it. Like other solutions, we can have other axiomatic systems to define the given coalitional value. The introduced model for multichoice games is different to any existing one. Because of the advantages of the value proposed by van den Nouweland et al. [2], the given coalitional value considers the player participation levels, and endows the different weights with respect to the different participation levels.

However, we mainly study one coalitional value for multichoice games with a coalition strucutre, and it will be interesting to discuss other payoff indices. Furthermore, we shall study the payoff indices of multichoice games under precedence constraints by using graph theory [27, 28, 29, 30, 31, 32].

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