Applied Mathematics & Information Sciences Letters
An International Journal

http://dx.doi.org/10.12785/amisl/010306

Application of first integral method to some special nonlinear partial differential equations

N. Taghizadeh, M. Mirzazadeh and A. Samiei Paghaleh*

Department of Mathematics, Faculty of Mathematical Sciences, University of Guilan, P.O.Box 1914 Rasht, Iran

Received: 23 Mar. 2013, Revised: 27 Jul. 2013, Accepted: 29 Jul. 2013
Published online: 1 Sep. 2013

Abstract: In this present work, we explore the application of the first integral method to some special nonlinear partial differential equations. The power of this manageable method is confirmed by applying it for three selected nonlinear partial differential equations. By using this method, we found some exact solutions of the Tzitzeica-Dodd-Bullough equation and nonlinear Klein-Gordon equation with power law nonlinearity and generalized nonlinear heat conduction equation and the BBM-like B $(2,2)$ equation. The first integral method can be applied to nonintegrable equations as well as to integrable ones. This method is based on the theory of commutative algebra

Keywords: First integral method; Tzitzeica-Dodd-Bullough equation; Nonlinear Klein-Gordon equation with power law nonlinearity; Generalized nonlinear heat conduction equation; BBM-like B $(2,2)$ equation

1 Introduction

Exact solutions to nonlinear evolution equations play an important role in nonlinear physical science, since these solutions may well describe various natural phenomena, such as vibrations, solitons, and propagation with a finite speed. Recently many new approaches for finding the exact solutions to nonlinear evolution equations have been proposed, multiple exp-function method [1], ansatz method and topological solitons [2,3], transformed rational function method [4], tanh-function method [5,6], extended tanh-function method [7,8], first integral method [10,11,12,13,14,15,16] and so on. The first integral method is a powerful solution method for the computation of exact traveling wave solutions. This method is one of the most direct and effective algebraic methods for finding exact solutions of nonlinear partial differential equations (PDEs). The first integral method was first proposed by Feng [10]in solving Burgers-KdV equation which is based on the ring theory of commutative algebra. This method was further developed by many authors in [11,12,13,14,15,16]. The aim of this paper is to find exact solutions of the Tzitzeica-Dodd-Bullough equation and nonlinear Klein-Gordon equation with power law nonlinearity and generalized nonlinear heat conduction equation and the BBM-like B (2, 2) equation by using the first integral method. Nonlinear Klein-Gordon equation with power law nonlinearity play a significant role in many scientific applications such as solid state physics, nonlinear optics and quantum field theory and the Tzitzeica-Dodd -Bullough equation appear in problems varying from fluid flow to quantum field theory.

The paper is arranged as follows. In Section 2, we describe briefly the first integral method. In Sections 3-6, we apply this method to the Tzitzeica-Dodd-Bullough equation and nonlinear Klein-Gordon equation with power law nonlinearity and generalized nonlinear heat conduction equation and the BBM-like B (2, 2) equation.

2 First integral method

The main steps of the first integral method are summarized as follows.

Step 1. Consider a general nonlinear PDE in the form

$$H\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t \partial x}, \dots\right) = 0, \quad (1)$$

where H is a polynomial in u and its partial derivatives. Using a wave variable $\xi = k(x + ly - ct)$ so that

$$u(x,y,t) = f(\xi),$$

* Corresponding author e-mail: mirzazadehs2@guilan.ac.ir

© 2013 NSP
Natural Sciences Publishing Cor.

Eq. (1) can be converted to an ordinary differential equation (ODE) as

$$E\left(f(\xi),\frac{df(\xi)}{d\xi},\frac{d^2f(\xi)}{d\xi^2},\ldots\right)=0, \tag{2}$$

where E is a polynomial in $f=f(\xi)$. If all terms contain derivatives, then Eq. (2) is integrated where integration constants are considered zeros.

Step 2. Suppose the solution of ODE (2) can be written as follows:

$$u(x,y,t)=f(\xi)=X(\xi), \tag{3}$$

and furthermore, we introduce a new independent variable $Y=Y(\xi)$ such that

$$Y(\xi)=\frac{X'(\xi)}{X(\xi)}, \tag{4}$$

where prime denotes derivative with respect to ξ.

Step 3. Under the conditions of Step 2, Eq. (2) can be converted to a system of nonlinear ODEs as follows

$$\frac{dX(\xi)}{d\xi}=X(\xi)Y(\xi),$$
$$\frac{dY(\xi)}{d\xi}=\Phi(X(\xi),Y(\xi)). \tag{5}$$

If we can find the integrals to Eq. (5), then the general solutions to Eq. (5) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is neither a systematic theory that can tell us how to find its first integrals, nor a logical way for telling us what these first integrals are. We will apply the so-called Division Theorem to obtain one first integral to Eq. (5) which reduces Eq. (2) to a first order integrable ODE. An exact solution to Eq. (1) is then obtained by solving this equation.

Division Theorem 2.1. Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials in $C[w,z]$; and $P(w,z)$ is irreducible in $C[w,z]$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C[w,z]$ such that

$$Q(w,z)=P(w,z)G(w,z).$$

3 Tzitzeica-Dodd-Bullough equation

In this section we consider the Tzitzeica-Dodd-Bullough equation [17]

$$u_{xt}=e^{-u}+e^{-2u}. \tag{6}$$

Using the transformation

$$u(x,t)=\operatorname{arcsinh}\left(\frac{v^{-1}-v}{2}\right),\quad v(x,t)=e^{-u} \tag{7}$$

carries Eq. (6) into the ODE

$$-vv_{xt}+v_xv_t-v^3-v^4=0. \tag{8}$$

Using the wave variable $\xi=k(x-ct)$ carries Eq. (8) into the ODE

$$k^2c\left(vv''-(v')^2\right)-v^3-v^4=0.$$

Rewrite this equation as follows

$$k^2c\frac{\left(vv''-(v')^2\right)}{v^2}-v-v^2=0. \tag{9}$$

Let

$$v(\xi)=X(\xi),\quad Y(\xi)=\frac{X'(\xi)}{X(\xi)}. \tag{10}$$

Then from Eq. (9) we have

$$\frac{dY}{d\xi}=\frac{1}{k^2c}\left(X(\xi)+X^2(\xi)\right). \tag{11}$$

Suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of Eqs. (10) and (11), and

$$q(X,Y)=\sum_{i=0}^{m}a_i(X)Y^i$$

is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$q(X(\xi),Y(\xi))=\sum_{i=0}^{m}a_i(X(\xi))Y^i(\xi)=0, \tag{12}$$

where $a_i(X)$, $i=0,1,\ldots,m$, are polynomials of X and $a_m(X)\neq 0$. Eq. (12) is called the first integral to Eqs. (10) and (11). Due to the Division Theorem, there exists a polynomial $g(X)+h(X)Y$, in the complex domain $C[X,Y]$ such that

$$\frac{dq}{d\xi}=\frac{dq}{dX}\frac{dX}{d\xi}+\frac{dq}{dY}\frac{dY}{d\xi}$$
$$=(g(X)+h(X)Y)\sum_{i=0}^{m}a_i(X)Y^i. \tag{13}$$

Suppose that $m=1$, by comparing with the coefficients of Y^i, $i=2,1,0$, on both sides of (13), we have

$$X\frac{da_1(X)}{dX}=h(X)a_1(X), \tag{14}$$
$$X\frac{da_0(X)}{dX}=g(X)a_1(X)+h(X)a_0(X), \tag{15}$$
$$g(X)a_0(X)=a_1(X)\left(\frac{1}{k^2c}(X+X^2)\right). \tag{16}$$

Since $a_i(X)$, $i=0,1$, are polynomials, then from (14) we deduce that $a_1(X)$ is constant and $h(X)=0$. For simplicity, take $a_1(X)=1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(a_0(X))=\deg(g(X))=1$. Suppose that

$$g(X)=A_0+A_1X,\quad a_0(X)=B_0+B_1X, \tag{17}$$

where $A_1\neq 0$, $B_1\neq 0$. Substituting (17) into (15), we obtain $A_0=0$, $A_1=B_1$.

© 2013 NSP
Natural Sciences Publishing Cor.

Substituting $a_0(X)$ and $g(X)$ into (16) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = 0, \;\; B_0 = \pm\frac{1}{k\sqrt{c}}, \;\; A_1 = B_1 = \pm\frac{1}{k\sqrt{c}}, \tag{18}$$

where c and k are arbitrary constants. Then we have

$$\begin{aligned}\frac{dX(\xi)}{d\xi} &= X(\xi)Y(\xi) = -X(\xi)a_0(X(\xi)) \\ &= \mp\frac{1}{k\sqrt{c}}\left(X(\xi)+X^2(\xi)\right).\end{aligned} \tag{19}$$

Solving Eq. (19), we obtain

$$\begin{aligned}X_1(\xi) &= -\frac{1}{1-e^{\frac{1}{k\sqrt{c}}(\xi+\xi_0)}}, \\ X_2(\xi) &= -\frac{1}{1-e^{-\frac{1}{k\sqrt{c}}(\xi+\xi_0)}},\end{aligned} \tag{20}$$

where ξ_0 is integration constant. Then

$$\begin{aligned}v_1(x,t) &= -\frac{1}{1-e^{\frac{1}{k\sqrt{c}}(k(x-ct)+\xi_0)}}, \\ v_2(x,t) &= -\frac{1}{1-e^{-\frac{1}{k\sqrt{c}}(k(x-ct)+\xi_0)}}.\end{aligned} \tag{21}$$

Using (7), we have exact solutions of the Tzitzeica-Dodd-Bullough equation in the following form

$$u_1(x,t) = \operatorname{arcsinh}\left(\frac{1-\left(1-e^{\frac{1}{k\sqrt{c}}(k(x-ct)+\xi_0)}\right)^2}{2\left(1-e^{\frac{1}{k\sqrt{c}}(k(x-ct)+\xi_0)}\right)}\right) \tag{22}$$

and

$$u_2(x,t) = \operatorname{arcsinh}\left(\frac{1-\left(1-e^{-\frac{1}{k\sqrt{c}}(k(x-ct)+\xi_0)}\right)^2}{2\left(1-e^{-\frac{1}{k\sqrt{c}}(k(x-ct)+\xi_0)}\right)}\right). \tag{23}$$

4 Nonlinear Klein-Gordon equation with power law nonlinearity

Let us consider the nonlinear Klein-Gordon equation with power law nonlinearity [18]

$$u_{tt} - a^2u_{xx} + \alpha u - \beta u^n + \gamma u^{2n-1} = 0, \quad n > 1. \tag{24}$$

We use the wave transformation

$$u(x,t) = U(\xi), \quad \xi = k(x-ct). \tag{25}$$

Substituting (25) into (24), we obtain ordinary differential equation:

$$k^2(c^2-a^2)\frac{d^2U}{d\xi^2} + \alpha U - \beta U^n + \gamma U^{2n-1} = 0. \tag{26}$$

Due to the difficulty in obtaining the first integral of Eq. (26), we propose a transformation denoted by $U = V^{\frac{1}{n-1}}$. Then Eq. (26) is converted to

$$\begin{aligned}&k^2(n-1)(c^2-a^2)VV'' + k^2(2-n)(c^2-a^2)(V')^2 \\ &\quad + (n-1)^2\alpha V^2 - (n-1)^2\beta V^3 \\ &\quad + (n-1)^2\gamma V^4 = 0.\end{aligned} \tag{27}$$

Rewrite this equation as follows

$$\begin{aligned}&k^2(n-1)(c^2-a^2)(VV''-(V')^2) + k^2(c^2-a^2)(V')^2 \\ &\quad + (n-1)^2\alpha V^2 - (n-1)^2\beta V^3 \\ &\quad + (n-1)^2\gamma V^4 = 0.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&k^2(n-1)(c^2-a^2)\frac{VV''-(V')^2}{V^2} + k^2(c^2-a^2)(\frac{V'}{V})^2 \\ &\quad + (n-1)^2\alpha - (n-1)^2\beta V \\ &\quad + (n-1)^2\gamma V^2 = 0.\end{aligned} \tag{28}$$

Let

$$v(\xi) = X(\xi), \quad Y(\xi) = \frac{X'(\xi)}{X(\xi)}. \tag{29}$$

Then from Eq. (28) we have

$$\begin{aligned}\frac{dY}{d\xi} &= \frac{n-1}{k^2(c^2-a^2)}\left(\beta X(\xi) - \gamma X^2(\xi) - \alpha\right) \\ &\quad - \frac{1}{n-1}Y^2.\end{aligned} \tag{30}$$

Suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of Eqs. (29) and (30), and

$$q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$$

is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$q(X(\xi),Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi))Y^i(\xi) = 0, \tag{31}$$

where $a_i(X)$, $i = 0,1,\ldots,m$, are polynomials of X and $a_m(X) \neq 0$. Eq. (31) is called the first integral to Eqs. (29) and (30). According to the Division Theorem, there exists a polynomial $g(X)+h(X)Y$, in the complex domain $C[X,Y]$ such that

$$\begin{aligned}\frac{dq}{d\xi} &= \frac{dq}{dX}\frac{dX}{d\xi} + \frac{dq}{dY}\frac{dY}{d\xi} \\ &= (g(X)+h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i.\end{aligned} \tag{32}$$

© 2013 NSP
Natural Sciences Publishing Cor.

Suppose that $m=1$, by comparing with the coefficients of $Y^i,\ i=2,1,0$, on both sides of (32), we have

$$X\frac{da_1(X)}{dX} - \frac{1}{n-1}a_1(X) = h(X)a_1(X), \tag{33}$$

$$X\frac{da_0(X)}{dX} = g(X)a_1(X)+h(X)a_0(X), \tag{34}$$

$$g(X)a_0(X) = \frac{(n-1)a_1(X)}{k^2(c^2-a^2)}\left(\beta X-\gamma X^2-\alpha\right). \tag{35}$$

Since $a_i(X),\ i=0,1$, are polynomials, then from (33) we deduce that $a_1(X)$ is constant and $h(X) = -\frac{1}{n-1}$. For simplicity, take $a_1(X)=1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that
$\deg(a_0(X)) = \deg(g(X)) = 1$.
Suppose that

$$g(X) = A_0+A_1X,\quad a_0(X) = B_0+B_1X, \tag{36}$$

where $A_1\neq 0,\quad B_1\neq 0$. Substituting (36) into (34), we have $A_0 = \frac{1}{n-1}B_0,\ A_1 = \frac{n}{n-1}B_1$.
Substituting $a_0(X)$ and $g(X)$ into (35) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$\begin{aligned}
A_0 &= \mp\frac{1}{k}\sqrt{\frac{\alpha}{a^2-c^2}},\qquad B_0 = \mp\frac{n-1}{k}\sqrt{\frac{\alpha}{a^2-c^2}},\\
A_1 &= \pm\frac{n\beta}{k(n+1)\alpha}\sqrt{\frac{\alpha}{a^2-c^2}},\\
B_1 &= \pm\frac{(n-1)\beta}{k(n+1)\alpha}\sqrt{\frac{\alpha}{a^2-c^2}},\\
\alpha &= \frac{n\beta^2}{(n+1)^2\gamma},
\end{aligned} \tag{37}$$

where $\alpha,\ \beta,\ \gamma,\ k$ and c are arbitrary constants. Then we have

$$\begin{aligned}
\frac{dX(\xi)}{d\xi} &= X(\xi)Y(\xi) = -X(\xi)a_0(X(\xi))\\
&= \mp\frac{n-1}{k}\sqrt{\frac{\alpha}{a^2-c^2}}(X(\xi)\\
&\quad -\frac{\beta}{(n+1)\alpha}X^2(\xi)).
\end{aligned} \tag{38}$$

Solving Eq. (38), we obtain

$$X_1(\xi) = \frac{\alpha(n+1)}{\beta}\frac{e^{\frac{n-1}{k}\sqrt{\frac{\alpha}{a^2-c^2}}(\xi+\xi_0)}}{1+e^{\frac{n-1}{k}\sqrt{\frac{\alpha}{a^2-c^2}}(\xi+\xi_0)}} \tag{39}$$

and

$$X_2(\xi) = \frac{\alpha(n+1)}{\beta}\frac{e^{-\frac{n-1}{k}\sqrt{\frac{\alpha}{a^2-c^2}}(\xi+\xi_0)}}{1+e^{-\frac{n-1}{k}\sqrt{\frac{\alpha}{a^2-c^2}}(\xi+\xi_0)}},$$

where ξ_0 is integration constant. Then

$$V_1(x,t) = \frac{\alpha(n+1)}{\beta}\frac{e^{\frac{n-1}{k}\sqrt{\frac{\alpha}{a^2-c^2}}(k(x-ct)+\xi_0)}}{1+e^{\frac{n-1}{k}\sqrt{\frac{\alpha}{a^2-c^2}}(k(x-ct)+\xi_0)}} \tag{40}$$

and

$$V_2(x,t) = \frac{\alpha(n+1)}{\beta}\frac{e^{-\frac{n-1}{k}\sqrt{\frac{\alpha}{a^2-c^2}}(k(x-ct)+\xi_0)}}{1+e^{-\frac{n-1}{k}\sqrt{\frac{\alpha}{a^2-c^2}}(k(x-ct)+\xi_0)}}.$$

Thus, we have the exact solutions of the nonlinear Klein-Gordon equation with power law nonlinearity in the following form

$$u_1(x,t) = \left\{\frac{\alpha(n+1)}{\beta}\frac{e^{\frac{n-1}{k}\sqrt{\frac{\alpha}{a^2-c^2}}(k(x-ct)+\xi_0)}}{1+e^{\frac{n-1}{k}\sqrt{\frac{\alpha}{a^2-c^2}}(k(x-ct)+\xi_0)}}\right\}^{\frac{1}{n-1}} \tag{41}$$

and

$$u_2(x,t) = \left\{\frac{\alpha(n+1)}{\beta}\frac{e^{-\frac{n-1}{k}\sqrt{\frac{\alpha}{a^2-c^2}}(k(x-ct)+\xi_0)}}{1+e^{-\frac{n-1}{k}\sqrt{\frac{\alpha}{a^2-c^2}}(k(x-ct)+\xi_0)}}\right\}^{\frac{1}{n-1}}. \tag{42}$$

5 A generalized form of the nonlinear heat equation

We next consider a generalized form of the nonlinear heat conduction equation [19]

$$u_t - a(u^n)_{xx} - u + u^n = 0. \tag{43}$$

Using the wave variable $\xi = k(x-ct)$ carries Eq. (43) to

$$-kcu' - ak^2(u^n)'' - u + u^n = 0. \tag{44}$$

Due to the difficulty in obtaining the first integral of Eq. (44), we propose a transformation $u = V^{-\frac{1}{n-1}}$. Then Eq. (44) is converted to

$$\begin{aligned}
&kc(n-1)V^2V' + ak^2n(1-2n)(V')^2 + ak^2n(n-1)VV''\\
&\quad - (n-1)^2V^3 + (n-1)^2V^2 = 0.
\end{aligned} \tag{45}$$

Rewrite this equation as follows

$$\begin{aligned}
&kc(n-1)V^2V' + ak^2n(n-1)(VV''-(V')^2) - ak^2n^2(V')^2\\
&\quad - (n-1)^2V^3 + (n-1)^2V^2 = 0.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&kc(n-1)V' + ak^2n(n-1)\left(\frac{VV''-(V')^2}{V^2}\right) - ak^2n^2\left(\frac{V'}{V}\right)^2\\
&\quad - (n-1)^2V + (n-1)^2 = 0.
\end{aligned} \tag{46}$$

© 2013 NSP
Natural Sciences Publishing Cor.

Let

$$v(\xi) = X(\xi), \quad Y(\xi) = \frac{X'(\xi)}{X(\xi)}. \tag{47}$$

Then from Eq. (46) we have

$$\frac{dY}{d\xi} = -\frac{c}{akn}X(\xi)Y(\xi) + \frac{n-1}{ak^2n}(X(\xi)-1) + \frac{n}{n-1}Y^2. \tag{48}$$

Suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of Eqs. (47) and (48), and

$$q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$$

is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$q(X(\xi),Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi))Y^i(\xi) = 0, \tag{49}$$

where $a_i(X)$, $i = 0,1,\ldots,m$, are polynomials of X and $a_m(X) \neq 0$. According to the Division Theorem, there exists a polynomial $g(X)+h(X)Y$, in the complex domain $C[X,Y]$ such that

$$\frac{dq}{d\xi} = \frac{dq}{dX}\frac{dX}{d\xi} + \frac{dq}{dY}\frac{dY}{d\xi} = (g(X)+h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i. \tag{50}$$

Suppose that $m = 1$, by comparing with the coefficients of Y^i, $i = 2,1,0$, on both sides of (50), we have

$$\frac{Xda_1(X)}{dX} + \frac{n}{n-1}a_1(X) = h(X)a_1(X), \tag{51}$$

$$\frac{Xda_0(X)}{dX} - \frac{cX}{akn}a_1(X) = g(X)a_1(X)+h(X)a_0(X), \tag{52}$$

$$g(X)a_0(X) = \frac{(n-1)a_1(X)}{ak^2n}(X-1). \tag{53}$$

Since $a_i(X)$, $i = 0,1$, are polynomials, then from (51) we deduce that $a_1(X)$ is constant and $h(X) = \frac{n}{n-1}$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that
$\deg(a_0(X)) = 1, \quad \deg(g(X)) = 0$.
Suppose that

$$g(X) = A_0, \quad a_0(X) = B_0 + B_1X, \tag{54}$$

where $A_0 \neq 0, \ B_1 \neq 0$.
Substituting (54) into (52), we obtain
$A_0 = -\frac{n}{n-1}B_0, \quad c = \frac{akn}{1-n}B_1$.
Substituting $a_0(X)$ and $g(X)$ into (53) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = \frac{1}{k\sqrt{a}}, \ B_0 = -\frac{n-1}{nk\sqrt{a}}, \ B_1 = \frac{n-1}{nk\sqrt{a}}, \ c = -\sqrt{a} \tag{55}$$

and

$$A_0 = -\frac{1}{k\sqrt{a}}, \ B_0 = \frac{n-1}{nk\sqrt{a}}, \ B_1 = -\frac{n-1}{nk\sqrt{a}}, \ c = \sqrt{a}, \tag{56}$$

where a and k are arbitrary constants. By using (55), we obtain

$$\frac{dX(\xi)}{d\xi} = X(\xi)Y(\xi) = -X(\xi)a_0(X(\xi)) = \frac{n-1}{nk\sqrt{a}}(X(\xi)-X^2(\xi)). \tag{57}$$

Solving Eq. (57), we obtain

$$X_1(\xi) = \frac{e^{\frac{n-1}{nk\sqrt{a}}(\xi+\xi_0)}}{1+e^{\frac{n-1}{nk\sqrt{a}}(\xi+\xi_0)}}, \tag{58}$$

where ξ_0 is integration constant. Then

$$V_1(\xi) = \frac{e^{\frac{n-1}{nk\sqrt{a}}(k(x-ct)+\xi_0)}}{1+e^{\frac{n-1}{nk\sqrt{a}}(k(x-ct)+\xi_0)}}. \tag{59}$$

Thus, we have the exact solution of the generalized nonlinear heat conduction equation in the following form

$$u_1(x,t) = \left\{ \frac{e^{\frac{n-1}{nk\sqrt{a}}(k(x+\sqrt{a}t)+\xi_0)}}{1+e^{\frac{n-1}{nk\sqrt{a}}(k(x+\sqrt{a}t)+\xi_0)}} \right\}^{-\frac{1}{n-1}}. \tag{60}$$

Similarly, in the case of (56), from (47), we obtain

$$\frac{dX(\xi)}{d\xi} = X(\xi)Y(\xi) = -X(\xi)a_0(X(\xi)) = -\frac{n-1}{nk\sqrt{a}}(X(\xi)-X^2(\xi)), \tag{61}$$

and then the exact solution of the generalized nonlinear heat conduction equation can be written as

$$u_2(x,t) = \left\{ \frac{e^{-\frac{n-1}{nk\sqrt{a}}(k(x-\sqrt{a}t)+\xi_0)}}{1+e^{-\frac{n-1}{nk\sqrt{a}}(k(x-\sqrt{a}t)+\xi_0)}} \right\}^{-\frac{1}{n-1}}. \tag{62}$$

6 The BBM-like B (2, 2) equation

In this section we study the BBM-like B (2, 2) equation

$$u_t + (u^2)_x - (u^2)_{xxt} = 0. \tag{63}$$

We use the wave transformation

$$u(x,t) = u(\xi), \quad \xi = k(x-ct) \tag{64}$$

where k and c are constants and $u(\xi)$ is real function.
Substituting (64) into (63), we obtain ordinary differential equation:

$$-cu' + (u^2)' + k^2c(u^2)''' = 0. \tag{65}$$

© 2013 NSP
Natural Sciences Publishing Cor.

Integrating Eq. (65) respect to ξ, then we have

$$-cu+u^2+2k^2c(u')^2+2k^2cuu''=R, \qquad (66)$$

where R is integration constant.
If we let $X=u,\ \ Y=\frac{du}{d\xi}$, the Eq. (66) is equivalent to the two dimensional autonomous system

$$\frac{dX(\xi)}{d\xi}=Y,$$
$$\frac{dY(\xi)}{d\xi}=\frac{1}{2k^2cX}\left(cX-X^2-2k^2cY^2+R\right). \qquad (67)$$

Making the following transformation

$$d\eta=\frac{d\xi}{2k^2cX}, \qquad (68)$$

then system (68) becomes

$$\frac{dX}{d\eta}=2k^2cXY,$$
$$\frac{dY}{d\eta}=cX-X^2-2k^2cY^2+R. \qquad (69)$$

Now, we are applying the Division Theorem to seek the first integral to system (69). Suppose that $X=X(\eta),\ \ Y=Y(\eta)$ are the nontrivial solutions to (69), and

$$q(X,Y)=\sum_{i=0}^{m}a_i(X)Y^i$$

is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$q(X(\eta),Y(\eta))=\sum_{i=0}^{m}a_i(X(\eta))Y^i(\eta)=0, \qquad (70)$$

where $a_i(X),\ \ i=0,1,\ldots,m$, are polynomials of X and $a_m(X)\neq 0$. Suppose that $m=1$ in (70). According to the Division Theorem, there exists a polynomial $g(X)+h(X)Y$, in the complex domain $C[X,Y]$ such that

$$\begin{aligned}\frac{dq}{d\eta}&=\frac{dq}{dX}\frac{dX}{d\eta}+\frac{dq}{dY}\frac{dY}{d\eta}\\&=(\sum_{i=0}^{1}a_i'(X)Y^i)(2k^2cXY)+(\sum_{i=0}^{1}ia_i(X)Y^{i-1})(cX-X^2\\&-2k^2cY^2+R)=(g(X)+h(X)Y)\sum_{i=0}^{1}a_i(X)Y^i, \qquad (71)\end{aligned}$$

where prime denotes differentiation with respect to the variable X. By comparing with the coefficients of $Y^i,\ i=2,1,0$, of both sides of (71), we have

$$2k^2cX\frac{da_1(X)}{dX}=h(X)a_1(X), \qquad (72)$$
$$2k^2cX\frac{da_0(X)}{dX}=g(X)a_1(X)+h(X)a_0(X), \qquad (73)$$
$$g(X)a_0(X)=a_1(X)(cX-X^2+R). \qquad (74)$$

Since $a_i(X),\ i=0,1$, are polynomials, then from (72) we deduce that $a_1(X)$ is constant and $h(X)=-2k^2c$. For simplicity, take $a_1(X)=1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that
$\deg(a_0(X))=\deg(g(X))=1$.
Suppose that

$$g(X)=A_0+A_1X,\quad a_0(X)=B_0+B_1X, \qquad (75)$$

where $A_1\neq 0,\quad B_1\neq 0$. Substituting (75) into (73), we obtain

$$g(X)=2k^2cB_0+4k^2cB_1X.$$

Substituting $a_0(X)$ and $g(X)$ into (74) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$B_0=\pm\frac{\sqrt{-c}}{3k},\quad B_2=\pm\frac{1}{2k\sqrt{-c}},\quad R=-\frac{2}{9}c^2, \qquad (76)$$

where k and c are arbitrary constants.
Using the conditions (76) in (70), we obtain

$$Y\pm(\frac{\sqrt{-c}}{3k}+\frac{1}{2k\sqrt{-c}}X)=0. \qquad (77)$$

Combining this first integral with (67), the second order differential Eq. (66) can be reduced to

$$\frac{du}{d\xi}=\mp(\frac{\sqrt{-c}}{3k}+\frac{1}{2k\sqrt{-c}}u). \qquad (78)$$

Solving Eq. (78) and changing to the original variables, we obtain the exact solutions to the BBM-like B (2, 2) equation in the following form

$$u(x,t)=\frac{2c}{3}-2ce^{\pm\frac{1}{2k\sqrt{-c}}(k(x-ct)+\xi_0)}, \qquad (79)$$

where ξ_0 is an arbitrary constant.

7 Conclusion

The first integral method has been proposed and applied to exact solutions of the Tzitzeica-Dodd-Bullough equation, nonlinear Klein-Gordon equation with power law nonlinearity, generalized nonlinear heat conduction equation and the BBM-like B (2, 2) equation. The first integral method is effective in searching exact solutions of nonlinear partial differential equations. The method proposed in this paper can also be extended to solve some nonlinear evolution equations in mathematical physics.

References

[1] W. X. Ma, T. W. Huang, Y. Zhang Phys. Scr., **82**, 065003 (2010).

© 2013 NSP
Natural Sciences Publishing Cor.

[2] A. Biswas, Phys. Lett. A **372**, 4601-4602 (2008).
[3] A. Biswas, Commun. Nonlinear Sci. Numer. Simulat., **14**, 2845-2847 (2009).
[4] W. X. Ma, J. H. Lee, Chaos Solitons Fract. **42**, 1356-1363 (2009).
[5] W. Malfliet, W. Hereman, Phys. Scr., **54**, 563-568 (1966).
[6] W. Malfliet, W. Hereman, Phys. Scr., **54**, 569-575 (1966).
[7] W. X. Ma, B. Fuchssteiner, Int. J. Nonlinear Mech., **31**, 329-338 (1996).
[8] E. Fan, Phys. Lett. A **277**, 212-218 (2000).
[9] Z. S. Feng, J. Phys. A. **35**, 343-349 (2002).
[10] I. Aslan, Appl. Math. Comput., **217**, 8134-8139 (2011).
[11] I. Aslan, Math. Meth. Appl. Sci., **35**, 716-722 (2012).
[12] B. Lu, J. Math. Anal. Appl., **395**, 684-693 (2012).
[13] A. Bekir, O. Unsal, Pramana J. Phys., **79**, 3-17 (2012).
[14] F. Tascan, A. Bekir, M. Koparan, Commun. Non. Sci. Numer. Simul., **14**, 1810-1815 (2009).
[15] I. Aslan, Acta Physics Polonica, A **123**, 16-20 (2013).
[16] N. Taghizadeh, M. Mirzazadeh, F. Tascan, Appl. Math. Lett., **25**, 798-802 (2012).
[17] A. M. Wazwaz, Chaos, Solitons Fract., **25**, 55-63 (2005).
[18] A. M. Wazwaz, Chaos, Solitons Fract., **28**, 1005-1013 (2006).
[19] A. M. Wazwaz, Appl. Math. Comput., **169**, 321-338 (2005).
[20] S. Yadong, Chaos, Solitons and Fract., **25**, 1083-1091 (2005).

Nasir Taghizadeh is Professor of Pure Mathematics and Faculty of Mathematical sciences, guilan university, guilan, Iran. His research interests are theory of partial differential equations. He has published 62 papers.

Mohammad Mirzazadeh is Assistant Professor of Pure Mathematics and Faculty of Mathematical sciences, guilan university, guilan, Iran. His research interests are theory of partial differential equations.

Ameneh Samiei Paghaleh is currently a Ph.D. student at Guilan University, Guilan, Iran. His research interests include exact solution of nonlinear differential equations, differential geometry.

© 2013 NSP
Natural Sciences Publishing Cor.