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Explicit Formulas for Some Generalized Polynomials

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Abstract: Using notions of composita and composition of generating functions, we establish some explicit formulas for the Generalized Hermite polynomials, the Generalized Humbert polynomials, the Lerch polynomials, and the Mahler polynomials.

Keywords: Composita, generating function, Generalized Hermite polynomials, Generalized Humbert polynomials, Lerch polynomials, Mahler polynomials.

1 Introduction

The use of polynomials in other areas of mathematics is very impressive: they are important to continued fractions, operator theory, analytic functions, interpolation, approximation theory, numerical analysis, electrostatics, statistical quantum mechanics, special functions, number theory, combinatorics, stochastic processes, sorting and data compression, and etc.

Interesting results in the field of obtaining explicit formulas for polynomials can be found in some recent works by Srivastava [1, 2], Cenkci [3], Boyadzhiev [4] and Kruchinin [5].

In this paper we use the method of obtaining expressions for polynomials based on the composition of generating functions, which was presented by the authors at the 10th International Conference of Numerical Analysis and Applied Mathematics [6].

The generating functions have important roles in many branches of mathematics and mathematical physics. Numerous investigations related to the generating functions for many polynomials can be found in many books and articles (see, for example, [7–15]

The main purpose of this paper is to obtain explicit formulas for the Generalised Hermite polynomials, the Generalised Humbert polynomials, the Lerch polynomials, and the Mahler polynomials.

2 Preliminary

In this section, we introduce some basic definitions, operations and notation we need.

In the paper [16] authors introduced the notion of the *composita* of a given ordinary generating function $F(t) = \sum_{n>0} g(n)t^n$.

Suppose $F(t) = \sum_{n>0} f(n)t^n$ is the generating function, in which there is no free term $f(0) = 0$. From this generating function we can write the following condition

$$[F(t)]^k = \sum_{n>0} F(n,k)t^n. \tag{1}$$

The expression $F(n,k)$ is the *composita* and it is denoted by $F^{\Delta}(n,k)$.

For more information one can see some related works [5, 17, 18].

Next we show some operations, and rules with compositae.

1. Suppose $F(t) = \sum_{n>0} f(n)t^n$, $B(t) = \sum_{n\geqslant 0} b(n)t^n$ are generating functions, and $F^{\Delta}(n,k)$ is the composita of $F(t)$. Then for the composition of generating functions $A(t) = B(F(t))$ the following condition holds

$$a(n) = \sum_{k=1}^{n} F^{\Delta}(n,k)b(k), \qquad a(0) = b(0), \tag{2}$$

where $A(t) = \sum_{n\geq 0} a(n)t^n$.

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2.Suppose $F(t)=\sum_{n>0} f(n)t^n$, $G(t)=\sum_{n>0} g(n)t^n$ are generating functions, and $F^{\Delta}(n,k)$, $G^{\Delta}(n,k)$ are their compositae, respectively. Then for the generating function $A(t)=F(t)+G(t)$ the composita is equal to

$$A^{\Delta}(n,k)=F^{\Delta}(n,k)+G^{\Delta}(n,k)+ \\ +\sum_{j=1}^{k-1}\binom{k}{j}\sum_{i=j}^{n-k+j}F^{\Delta}(i,j)G^{\Delta}(n-i,k-j). \quad (3)$$

3.Suppose $F(t)=\sum_{n>0} f(n)t^n$, $G(t)=\sum_{n>0} g(n)t^n$ are generating functions, and $F^{\Delta}(n,k)$, $G^{\Delta}(n,k)$ are their compositae, respectively. Then for the composition of generating functions $A(t)=G(F(t))$ the composita is equal to

$$A^{\Delta}(n,k)=\sum_{m=k}^{n}F^{\Delta}(n,m)G^{\Delta}(m,k). \quad (4)$$

3 Generalized Hermite polynomials

In this section we consider the generalization of the Hermite polynomials and obtain some interesting identities for these polynomials.

The Generalised Hermite polynomials are polynomials that arise in many different fields, for instance in quantum mechanics, optical systems, kinetic theory of gases, theories of fluctuations [19–21].

There exist vast investigations concerned with Hermite polynomials, for example, Dattoli [22], Subuhi Khan et al. [23, 24] study summation formulae; Brafman [25], Lahiri [26], Gould and Hopper [27], Dattoli [28] study the generalization of Hermite polynomials.

Using the notion of the composita and the generating functions for the Generalized Hermite polynomials, we get explicit representations.

The Gould-Hopper generalized Hermite polynomials are defined by the following generating function

$$\sum_{n\geq 0} g_n^m(x,h)\frac{t^n}{n!}=\exp(xt+ht^m), \quad (5)$$

where m is a positive integer.

First, we obtain the composita of the generating function

$$F(x,m,h,t)=(xt+ht^m)$$

as the coefficients with respect to t^n in $F^k(x,m,h,t)$, where $m\geq 1$ is integer and the other parameters are unrestricted.

Applying the binomial theorem, we have

$$F^k(x,m,h,t)=t^k(x+ht^{m-1})^k=t^k\sum_{j=0}^{k}\binom{k}{j}x^{k-j}h^jt^{j(m-1)}.$$

Substituting $n=j(m-1)+k$, we obtain the composita of $F(x,m,h,t)$

$$F^{\Delta}(n,k,x,m,h)=\begin{cases}\binom{k}{\frac{n-k}{m-1}}x^{k-\frac{n-k}{m-1}}h^{\frac{n-k}{m-1}}, & if\ \frac{n-k}{m-1}\in\mathbb{N},\\ 0, & otherwise,\end{cases} \quad (6)$$

where $n\geq k$.

Below we present few first terms of the composita in a triangular form for the case $m=3$

$$\begin{array}{ccccccccc} & & & & x & & & & \\ & & & 0 & & x^2 & & & \\ & & h & & 0 & & x^3 & & \\ & 0 & & 2hx & & 0 & & x^4 & \\ 0 & & 0 & & 3hx^2 & & 0 & & x^5 \end{array}$$

For case $m=1$, the composita is

$$F^{\Delta}(n,k,x,1,h)=\binom{k}{j}\delta_{n,k}x^{k-j}h^j$$

where $\delta_{n,k}$ is the Kronecker delta,

$$\delta_{n,k}=\begin{cases}1, & \text{if } n=k;\\ 0, & \text{if } n\neq k.\end{cases} \quad (7)$$

Therefore, according to (2), the expression for the Gould-Hopper generalized Hermite polynomials is

$$g_n^m(x,h)=n!\sum_{k=1}^{n}\frac{1}{k!}F^{\Delta}(n,k,x,m,h)$$

or making few operations, we get the Gould-Hopper explicit representation (cf. [27])

$$g_n^m(x,h)=n!\sum_{r=0}^{[\frac{n}{m}]}\frac{x^{n-mr}h^r}{r!(n-mr)!}. \quad (8)$$

Gould and Hopper also indicated another generalization of the Hermite polynomials by the generating function

$$\sum_{n\geq 0}H_n^r(x,a,p)\frac{t^n}{n!}=\left(1-\frac{t}{x}\right)^a exp\left(p\left(x^r-(x-t)^r\right)\right). \quad (9)$$

Let us consider the generating function as a product of two generating functions

$$\left(1-\frac{t}{x}\right)^a$$

and

$$exp\left(p\left(x^r-(x-t)^r\right)\right).$$

First we obtain the composita of $p\left(x^r-(x-t)^r\right)$. The coefficients of $(x-t)^r$ are equal to

$$(-1)^n\binom{r}{n}x^{r-n}.$$

Then the composita of the required function is

$$p^k \sum_{j=0}^{k} \binom{k}{j} (-1)^{n+j} \binom{jr}{n} x^{kr-n}.$$

Using (2), we get the expression for coefficients of the generating function

$$exp\,(p\,(x^r - (x-t)^r))$$

$$\sum_{k=0}^{n} \frac{p^k \left(\sum_{j=0}^{k} \binom{k}{j} (-1)^{n+j} \binom{jr}{n}\right) x^{kr-n}}{k!}.$$

Coefficients of the generating function

$$\left(1 - \frac{t}{x}\right)^a$$

are specified by the following expression

$$\binom{a}{n} (-1)^n x^{-n}.$$

Therefore, multiplying both expressions, we obtain the following formula for the generalized Hermite polynomials

$$H_n^r(x,a,p) = n! \sum_{i=0}^{n} \binom{a}{n-i} \sum_{k=0}^{i} p^k \sum_{j=0}^{k} \frac{(-1)^{n+j} \binom{jr}{i}}{(k-j)!j!} x^{kr-n} \tag{10}$$

or according to Gould and Hopper [27], the explicit formula for this case is

$$H_n^r(x,a,p) = (-1)^n n! \sum_{k=0}^{n} p^k \frac{x^{rk-n}}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{a+rj}{n}. \tag{11}$$

Now we consider the generalization to the multivariable case, which is introduced by Dattoli et. al in [28] and using the notion of the composita, we obtain the explicit representation. The multivariable generalized Hermite polynomials are defined by the following generating function

$$\exp(2xt - t^2 + 2yt^m - t^{2m}) = \sum_{n\geq 0} {}^{(m)}H_n(x,y) \frac{t^n}{n!}, \tag{12}$$

or by the ordinary Hermite polynomials

$${}^{(m)}H_n(x,y) = n! \sum_{n=0}^{[n/m]} \frac{H_{n-mr}(x) H_r(y)}{(n-mr)!r!}. \tag{13}$$

We start with calculation the composita of

$$G(y,m,t) = 2yt^m - t^{2m}.$$

Applying the binomial theorem, we have

$$G^k(y,m,t) = t^{mk}(2y - t^m)^k = t^{mk} \sum_{j=0}^{k} \binom{k}{j} (2y)^{k-j} (-1)^j t^{jm}.$$

Substituting $n = (k+j)m$, we obtain the composita of $G(y,m,t)$

$$G^{\Delta}(n,k,y,m) = \begin{cases} \binom{k}{\frac{n-km}{m}} (-1)^{\frac{n-km}{m}} (2y)^{2k-\frac{n}{m}}, & if \frac{n-mk}{m} \in \mathbb{N}, \\ 0, & otherwise. \end{cases} \tag{14}$$

According to (6), the composita of $F(x,t) = 2xt - t^2$ is

$$F^{\Delta}(n,k,2x,2,-1) = \binom{k}{n-k} (2x)^{2k-n} (-1)^{n-k}.$$

Using (3) and (2), we obtain the expression for the multivariable generalized Hermite polynomials

$${}^{(m)}H_n(x,y)t^n = \sum_{k=1}^{n} \frac{n!}{k!} \Big(F^{\Delta}(n,k,2x,2,-1) G^{\Delta}(n,k,y,m) +$$

$$+ \sum_{j=1}^{k-1} \binom{k}{j} \sum_{i=j}^{n-k+j} F^{\Delta}(i,j,2x,2,-1) G^{\Delta}(n-i,k-j,y,m) \Big) \tag{15}$$

where $n > 0$.

4 Generalized Humbert polynomials

In this section we apply the notion of composita to get explicit formulas for the generalized Humbert polynomials. In 1965, Gould [29] defined the generalized Humbert polynomial $P_n(m,x,y,p,C)$ by means of generating function

$$(C - mxt + yt^m)^p = \sum_{n\geq 0} P_n(m,x,y,p,C) t^n, \tag{16}$$

where $m \geq 1$ is integer and the other parameters are unrestricted.

Changing the parameters in (16) by appropriate way, one can obtain the generating functions for the following polynomials: the Gegenbauer polynomials, the Legendre polynomials, the Humbert polynomials, and many others.

Represent the generating function in the following form

$$(C - mxt + yt^m)^p = C^p \left(1 - \frac{1}{C}(mxt - yt^m)\right)^p.$$

According to (6), the composita of $\frac{1}{C}(mxt - yt^m)$ is

$$F^{\Delta}(n,k,x,m,h,C) = \frac{1}{C^k} \binom{k}{\frac{n-k}{m-1}} (mx)^{k-\frac{n-k}{m-1}} (-y)^{\frac{n-k}{m-1}},$$

where $\frac{n-k}{m-1} \in \mathbb{N}$ and $n \geq k$.

Coefficients of the generating function

$$(1-x)^p$$

are specified by the following expression

$$\binom{p}{n}(-1)^n.$$

Therefore, according to (2) and substituting $j=\frac{n-k}{m-1}$, we obtain

$$P_n(m,x,y,p,C)=\sum_{j=0}^{\left[\frac{n}{m}\right]}\binom{p}{n-(m-1)j}\frac{(-mx)^{n-mj}y^j}{C^{n-(m-1)j-p}}\times \times\binom{n-(m-1)j}{j}$$

or making few operations, we get the Gould explicit representation (cf. [29])

$$P_n(m,x,y,p,C)=\sum_{k=0}^{\left[\frac{n}{m}\right]}\binom{p}{k}\binom{p-k}{n-mk}\times \times C^{p-n+(m-1)k}y^k(-mx)^{n-mk}. \quad (17)$$

Next we obtain the explicit formula another way. We represent the generating function (16) as the following composition of generating functions

$$(1-mxt+yt^m)^p=C^p\exp\left(p\ln\left(1+\frac{1}{C}h(x,m,y,t)\right)\right),$$

where $h(x,m,t)=-mxt+yt^m$.

From (6), the composita of $\frac{1}{C}h(x,m,y,t)$ is equal to

$$\begin{cases}\frac{1}{C^k}\binom{k}{\frac{n-k}{m-1}}(-mx)^{k-\frac{n-k}{m-1}}y^{\frac{n-k}{m-1}}, & if\,\frac{n-k}{m-1}\in\mathbb{N},\\ 0, & otherwise.\end{cases}$$

According to Comtet [18], the following expression holds true:

$$\sum_{n\geq k}s(n,k)\frac{t^n}{n!}=\frac{1}{k!}\ln^k(1+t),$$

where $s(n,k)$ are the Stirling numbers of the first kind.

Then the composita of $\ln(1+t)$ is

$$\frac{k!}{n!}s(n,k). \quad (18)$$

Using (4) and substituting $i=\frac{n-j}{m-1}$, we obtain the composita of $p\ln\left(1+\frac{1}{C}h(x,m,,yt)\right)$

$$(p)^k\sum_{i=0}^{\left[\frac{n-k}{m-1}\right]}\frac{y^i(-mx)^{n-im}k!}{C^{n-i(m-1)}(n-i(m-1))!}\binom{n-i(m-1)}{i}\times \times s(n-i(m-1),k).$$

Therefore, according (2), the coefficients with respect to t for the generating function $C^p\exp\left(p\ln\left(1+\frac{1}{C}h(x,m,y,t)\right)\right)$ are determined by the expression

$$P_n(m,x,y,p,C)=\sum_{k=0}^{n}(p)^k\sum_{i=0}^{\left[\frac{n-k}{m-1}\right]}\frac{y^i(-mx)^{n-im}}{C^{n-i(m-1)-p}(n-i(m-1))!}\times \times\binom{n-i(m-1)}{i}s(n-i(m-1),k).$$

5 Lerch polynomials

The Lerch polynomials are defined by the following generating function (see, for details, [7,30])

$$(1-x\ln(1+t))^{-\lambda}=\sum_{n\geqslant 0}\Phi_n(x,\lambda)t^n.$$

From (18) the composita of $x\ln(1+t)$ is equal to

$$\frac{k!}{n!}s(n,k)x^k.$$

We know that for the generating function $\left(\frac{1}{1-x}\right)^\lambda$ the coefficients are determined by the expression

$$\binom{n+\lambda-1}{n}.$$

Therefore, according to (2), we obtain

$$\Phi_n(x,\lambda)=\sum_{k=0}^{n}\binom{k+\lambda-1}{k}\frac{k!}{n!}s(n,k)x^k. \quad (19)$$

6 Mahler polynomials

The Mahler polynomials are defined by the following generating function (see [8])

$$e^{x(1+t-e^t)}=\sum_{n\geqslant 0}s_n(x)\frac{t^n}{n!}.$$

According to Comtet [18], the following expression holds true:

$$\sum_{n\geq k}S(n,k)\frac{t^n}{n!}=\frac{1}{k!}(e^t-1)^k.$$

where $S(n,k)$ are the Stirling numbers of the second kind.

Then the composita of e^t-1 is

$$\frac{k!}{n!}S(n,k). \quad (20)$$

According to [16], the composita of $G(t)=t$ is

$$G^{\Delta}(n,k)=\delta_{n,k},$$

where $\delta_{n,k}$ – the Kronecker delta.

Using (3), the composita of the sum of the generating functions $G(t) = t$ and $F(t) = -(e^t - 1)$ is equal to

$$\delta_{n,k} + \sum_{j=1}^{k-1} \binom{k}{j} \sum_{i=j}^{n-k+j} \frac{j!}{i!} (-1)^j S(i,j) \delta_{n-i,k-j} + \frac{k!}{n!} S(n,k).$$

Since

$$\delta_{n-i,k-j} = \begin{cases} 1, & \text{if } n-i = k-j, \\ 0, & \text{otherwise,} \end{cases}$$

we get

$$\sum_{j=0}^{k} \binom{k}{j} \frac{j!(-1)^j}{(n-k+j)!} S(n-k+j,j).$$

Therefore, using (2), we obtain

$$s_n(x) = \sum_{k=0}^{n} x^k \left(\sum_{j=0}^{k} (-1)^j \binom{n}{k-j} S(n-k+j,j) \right).$$

For instance, the coefficients of the Mahler polynomials are considered as a triangle of coefficients of degrees x in the sequence $A137375$ [31].

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