Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.12785/jsap/020302

Analyzing Skewed Data with the Epsilon Skew Gamma Distribution

Ebtisam Abdulah[1,*] *and Hassan Elsalloukh*[2]

[1] Department of Applied Science, University of Arkansas at Little Rock, Arkansas 72204, USA
[2] Department of Mathematics and Statistics, University of Arkansas at Little Rock, Arkansas 72204, USA

Received: 2 Apr. 2013, Revised: 26 Jun. 2013, Accepted: 29 Jun. 2013
Published online: 1 Nov. 2013

Abstract: A new distribution, the Epsilon Skew Gamma (ESΓ) distribution, which was first introduced by Abdulah [1], is used on a near Gamma data. We first redefine the ESΓ distribution, its properties, and characteristics, and then we estimate its parameters using the maximum likelihood and moment estimators. We finally use these estimators to fit the data with the ESΓ distribution.

Keywords: Skewed Data, Epsilon Skew, Gamma Distribution

1 Introduction

The statistical analysis requires knowing of the probability model or distribution, so that in real life applications, we seek to have distributions for analyzing skewed data and involving tail behavior. Many types of distributions are classified as either skewed (positive or negative) or kurtotic (heavy or thin tailed) relative to a normal distribution. In this century, there has been increasing interest in building and modeling skewed parametric families of distributions that allow variety in the shape and tail behavior in order to measure the divergence and central tendency in case studies, particularly, when modeling data that have high influence on the curve performance to the right or left side of the distribution. Many studies have been proposed to introduce skew-symmetric distributions which can account for both skewness and kurtotic, see e.g., Jones and Faddy [18] who extended the skewness concept to the t-distribution. Azzalini [9] and Arellano [6] introduced general family of skewed distributions and derived several skew-symmetric distributions, like skew exponential power (SEP), skew t, skew logistic, and skew-symmetrized gamma distributions. Gupta [17] derived pdfs for several skew-symmetric distributions and studied some of its properties, in particular, skew normal, skew uniform, skew t, skew Cauchy, skew Laplace, and skew logistics. All the researches mentioned above and many others, like Arnold and Beaver [7], Wahed [21], and Ali [4] have adopted the principle of Azzalini [8] work in construction skewed-symmetric distributions by using the following pdf

$$f(x) = 2\phi(x)\psi(\alpha x) \qquad -\infty < x < \infty\,, \tag{1}$$

where $\phi(x)$ and $\psi(x)$ denote the pdf and cdf of the standard normal distribution, respectively, and α is any real number. Some authors have constructed models with stronger degree of asymmetry by adding another skewness parameter ε. Mudholkar [20] presented the Epsilon Skew Normal (ESN) distribution with a skewness parameter $|\varepsilon| < 1$. The ESN and Epsilon Skew Laplace (ESL) Elsalloukh [12] and Almousawi [5], are two special cases of the Epsilon Skew exponential power (ESEP) distribution which was developed by Elsalloukh [11] and [13]. Some proposals have been focused on some kind of skewed model which arising from symmetric reflected distributions, such as reflected Γ Borghi [10], double Weibull, and reflected beta prime distributions. Ali [3] used Azzalini's equation (1) to construct skew-symmetric distribution taking pdf as a Laplace kernel and cdf comes from either Laplace, reflected Γ, double Weibull, reflected Pareto, reflected beta prime, and reflected uniform distributions. Some statisticians added bimodality features to the skewed distributions. Gómez [15] defined the uni-bimodal skew flexible normal (SFN) distribution by including an additional parameter δ to (1) and highlighted, when $\delta < 0$, the distribution becomes a bimodal. Eugenea [14] studied the properties of a special class from a general family, the logit beta random variable, which was called Beta-Normal (BN)

* Corresponding author e-mail: ekabdulah@ualr.edu

with four parameters $(\alpha, \beta, \mu, \sigma)$, where α and β two shape parameters. The distribution is bimodal when both α and β are less than 0.214. Recently, reflected inverted Γ distribution was extended to Epsilon Skew Inverted Gamma ESIΓ, Abdulah [2] which can compile three attractive features, skewness, kurtosis, and bimodality. In this paper we introduce the ESΓ distribution with four parameters. The importance of this distribution comes; it can produce several other important distributions as special cases like, exponential, ESL, χ^2, and reflected Γ distributions. Additionally, the ESΓ is efficient model for fitting not only skewed, peaked or flat-tailed data, but also skewed, peaked or flat with bimodal features for data that comes from two different populations. The remaining part of the paper is structured as follows. Section 2 includes definitions and basic properties of the ESΓ. In Sections 3 and 4, the estimation for the model parameters are estimated using the MLE and MME methods, respectively. In Sections 5 and 6, the moment generating and characteristic functions for ESΓ are derived, respectively. Section 7 shows how the model can be applied in two examples. We conclude by presenting the results of the skew symmetric class, ESΓ distribution, in Section 8.

2 Definition and Basic Properties of the ESΓ Distribution

Proposition 1. *Elsalloukh [11] If $Z \sim ESEP\ (\theta, \beta, k, \varepsilon)$ is a random variable, then the random variable $X = \left(\frac{(Z-\theta)}{\varepsilon_i \beta}\right)^k$ is a standardized ESΓ, that is $X \sim ES\Gamma\ (0, 1, 1/k, \varepsilon_i)$. This is a transformation case, where $i = 1, 2$, $\varepsilon_1 = 1/(1-\varepsilon)$ for $x \geq 0$, and $\varepsilon_2 = 1/(1+\varepsilon)$ for $x < 0$.*

You can refer to Abdulah [1], where the proof was shown. Therefore, the pdf of ESΓ distribution can be obtained with the following definition.

Definition 1. *Abdulah [1] A random variable X is said to have an ESΓ distribution with parameters $\theta \in R$, $\beta > 0$, $k > 0$, and $|\varepsilon| < 1$ that are location, scale, shape, and skewness parameters, respectively, if it has the pdf*

$$f(x) = \frac{1}{2\Gamma(k)\beta^k} \begin{cases} \left(\frac{(x-\theta)}{(1-\varepsilon)}\right)^{(k-1)} e^{-\frac{(x-\theta)}{\beta(1-\varepsilon)}} & if\, x \geq \theta \\ \left(\frac{(\theta-x)}{(1+\varepsilon)}\right)^{(k-1)} e^{-\frac{(\theta-x)}{\beta(1+\varepsilon)}} & if\, x < \theta\,. \end{cases} \tag{2}$$

Note that when $\varepsilon = 0$, $X \sim$ symmetric reflected $\Gamma\ (\theta, \beta, k)$ distribution, when $\varepsilon > 0$, the distribution is skewed to the right viz the right tail is longer than the left tail, and when $\varepsilon < 0$, the distribution is skewed to the left and the left tail is longer than the right tail.

Proposition 2. *Abdulah [1] If $X \sim ES\Gamma(\theta, \beta, k, \varepsilon)$, then the cumulative distribution, $F(x)$, function of X is*

$$F(x) = \begin{cases} 1 - \frac{(1-\varepsilon)}{2\Gamma(k)}\Gamma(k, g(x)) & for\, x \geq \theta \\ \frac{(1+\varepsilon)}{2\Gamma(k)}\Gamma(k, h(x)) & for\, x < \theta\,. \end{cases}$$

Figure 1 shows ESΓ with different values for ε, Figure 2 shows the cdf of ESΓ with $\varepsilon = 0.3$, and Figure 3 shows the difference between ESΓ and reflected Γ distributions. Note that when $k = 1$, (2) becomes ESL as defined in Elsalloukh [13], [12], and Almousawi [5].

3 Central Moments and First Four Moments for the ESΓ Distribution

This section is devoted to characterize the features of ESΓ distribution through the central moments and first four moments by using the following proposition.

Proposition 3. *Abdulah [1] If $X \sim ES\Gamma(\theta, \beta, k, \varepsilon)$, then the central moments, mean, variance, and skewness and kurtosis coefficients are, respectively*

1.
$$E(X-\theta)^n = \frac{\beta^n \Gamma(n+k)}{2\Gamma(k)} \left[(-1)^n (1+\varepsilon)^{n+1} + (1-\varepsilon)^{n+1}\right],$$

2.
$$E(X) = \theta - 2k\beta\varepsilon\,,$$

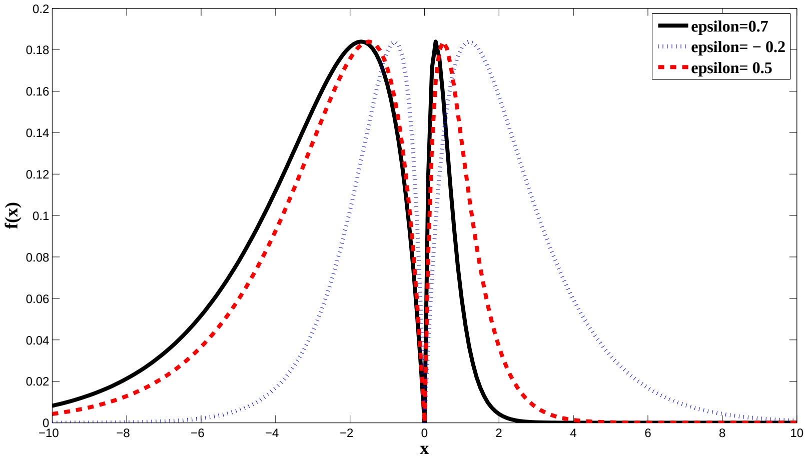

Fig. 1: ESΓ Density Functions for Different Values for ε.

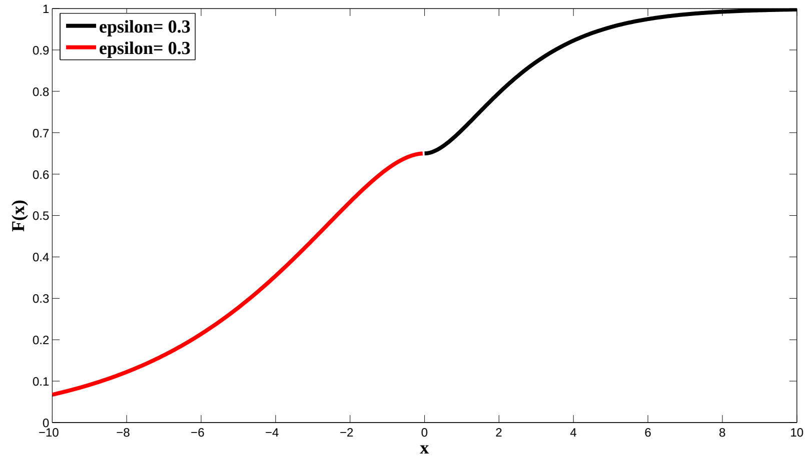

Fig. 2: CDF For ESΓ Density Functions with ε=0.3.

3.

$$Var(X) = \beta^2 k\left[(k+1)(1+3\varepsilon^2) - 4k\varepsilon^2\right],$$

4.

$$\lambda_1 = \left\{\frac{1}{\Gamma(k)}\left[\Gamma(k+2) + (3\Gamma(k+2) - 4k^2\Gamma(k))\varepsilon^2\right]\right\}^{-3/2} \left[\frac{-2k\varepsilon\Gamma(k) + (1+3\varepsilon^2)\Gamma(k+2) - 4\varepsilon(1+\varepsilon^2)\Gamma(k+3)}{\Gamma(k)}\right],$$

5.

$$\lambda_2 = \left\{\frac{1}{\Gamma(k)}\left[\Gamma(k+2) + (3\Gamma(k+2) - 4k^2\Gamma(k))\varepsilon^2\right]\right\}^{-2} \left[\frac{-2k\varepsilon\Gamma(k) + (1+3\varepsilon^2)\Gamma(k+2) - 4\varepsilon(1+\varepsilon^2)\Gamma(k+3) + 2\Gamma(k+4)(1+10\varepsilon^2+5\varepsilon^4)}{\Gamma(k)}\right]$$

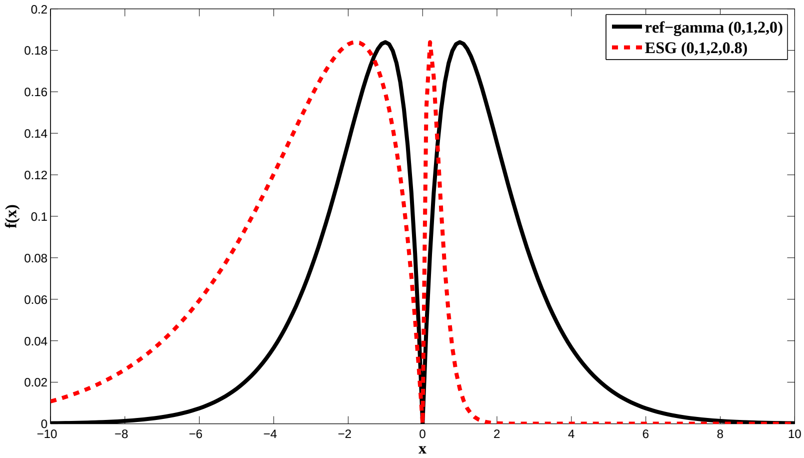

Fig. 3: ESΓ and Reflected Γ Density Functions.

The proof for all these moments was shown in Abdulah [1].

4 Maximum Likelihood Estimation for the ESΓ Parameters

In order to estimate and study the behavior of the parameters of the ESΓ distribution, we consider the likelihood equations which lead to the maximum likelihood estimators assuming the location parameter $\theta = 0$; this means we standardize the distribution by assuming $\theta = 0$ and treat the other parameters β, k, and ε as unknown.
Suppose that $X \sim \mathrm{ES}\Gamma(0, \beta, k, \varepsilon)$ be a random variable with a pdf given in (2), then the likelihood function is Abdulah [1]

$$L(\gamma) = \left(\frac{1}{2\Gamma(k)\beta^k}\right)^n \begin{cases} \prod_{i=1}^n \left(\frac{x_i^+}{(1-\varepsilon)}\right)^{(k-1)} e^{-\frac{\sum_{i=1}^n x_i^+}{\beta(1-\varepsilon)}} & \text{if } x \geq 0 \\ \prod_{i=1}^n \left(\frac{x_i^-}{(1+\varepsilon)}\right)^{(k-1)} e^{-\frac{\sum_{i=1}^n x_i^-}{\beta(1+\varepsilon)}} & \text{if } x < 0\,, \end{cases}$$

where $\gamma = (\beta, k, \varepsilon)$,

$$x_i^+ = \begin{cases} x_i & \text{if } x_i \geq 0 \\ 0 & \text{o/w}\,, \end{cases}$$

and

$$x_i^- = \begin{cases} -x_i & \text{if } x_i \leq 0 \\ 0 & \text{o/w}\,. \end{cases}$$

Then the log likelihood function is

$$\log L(\gamma) = -n\log(2) - n\log\Gamma(k) - nk\log(\beta) + (k-1)\sum_{i=1}^n \log\left(\frac{x_i^+}{(1-\varepsilon)}\right) - \frac{\sum_{i=1}^n x_i^+}{\beta(1-\varepsilon)}$$
$$+ (k-1)\sum_{i=1}^n \log\left(\frac{x_i^-}{(1+\varepsilon)}\right) - \frac{\sum_{i=1}^n x_i^-}{\beta(1+\varepsilon)}\,. \tag{3}$$

Differentiating (3) with respect to β leads to the MLE of β

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i^+(1+\hat{\varepsilon}) + \sum_{i=1}^n x_i^-(1-\hat{\varepsilon})}{n\hat{k}(1-\hat{\varepsilon}^2)}\,,$$

and the MLEs of k and ε are maximized numerically Abdulah [1].

5 Method of Moments Estimation(MME) for the ESΓ Parameters

Traditionally, researchers have used different methods to estimate the parameters particularly, when the number of the unknown distribution's parameters are increased. This stems from the MM estimators can provide significant initial values for finding the solutions of the likelihood equations and estimation by the MM can be reflecting a more precise approximation of the MLE estimates by the Newton-Raphson method. Since the ESΓ distribution consists of four parameters, two main scenarios are followed

1. Let θ and β be unknown and assume the parameters k and ε are known so that the MMEs for θ and β are, respectively

$$\tilde{\theta} = \bar{x} + \frac{2\sqrt{k}\varepsilon s}{\sqrt{(k+1)(1+3\varepsilon^2) - 4k\varepsilon^2}}$$

and

$$\tilde{\beta} = \frac{s}{\sqrt{k\left[(k+1)(1+3\varepsilon^2) - 4k\varepsilon^2\right]}} ,$$

where $\bar{x}$ and s^2 are the sample mean and variance, respectively Abdulah [1].

2. Let k and β be unknown and assume θ and ε are known, the moment estimators for β and k are, respectively suppose $\theta = 0$, since it is known, we have

$$\bar{x} = -2\tilde{k}\tilde{\beta}\varepsilon \Longrightarrow \tilde{k} = \frac{-\bar{x}}{2\tilde{\beta}\varepsilon} \tag{4}$$

and

$$\tilde{\beta} = \frac{\bar{x}}{2\varepsilon} - \frac{2\varepsilon\sum_{i=1}^{n} x_i^2}{2n\varepsilon\bar{x}(1+3\varepsilon^2)} \tag{5}$$

substitute (5) in (4) we obtain the MME of k Abdulah [1].

6 Moment Generating and Characteristic Functions of the ESΓ Distribution

The mgf of a random variable X is defined by

$$\mu_x(t) = E(e^{tx}), \quad \text{-}h < t < h, h > 0 .$$

Proposition 4. *Abdulah [1] If $X \sim ES\Gamma(0,\beta,k,\varepsilon)$, then the mgf of X can take the form*

$$\mu_x(t) = \frac{(1+\varepsilon)}{2(1+t\beta(1+\varepsilon))^k} + \frac{(1-\varepsilon)}{2(1-t\beta(1-\varepsilon))^k} . \tag{6}$$

The proof was given by Abdulah [1]. Note that when $\varepsilon = 0$, (6) becomes the mgf of the reflected Γ distribution, while when $k = 1$ the mgf of ESL is retrieved. Also, it can be shown when the mgf exists the rth derivative exists and the rth moment at $t = 0$ can be obtained.

Proposition 5. *Abdulah [1] If $X \sim ES\Gamma\ (0,\beta,k,\varepsilon)$, then the characteristic function of X is*

$$\phi_x(t) = \frac{(1+\varepsilon)}{2(1+it\beta(1+\varepsilon))^k} + \frac{(1-\varepsilon)}{2(1-it\beta(1-\varepsilon))^k} .$$

7 Applications

This section is devoted to apply two data sets of examples on the ESΓ and some other distributions that are interesting to us. In the first one, we try to fit the five models; regular Γ, reflected Γ, ESΓ, regular Weibull, and exponentiated exponential (EE) in testing a life for 23 ball bearings Lawless [19] and Gupta [16]. In the second example, we fit the models; regular Γ, reflected Γ, ESΓ, and beta normal distribution for the 252 adults numbers of the flour beetle, species Tribolium Castaneum Eugenea [14]. In the analysis for both examples, we depend on the log likelihood scores,

Table 1: Results of the Parameter MLEs and Corresponding Values of $\log L$, *AIC*, and *BIC* for the Five Fitted Distributions of Ball Bearings Data.

Distribution	scale	shape	skewness	$\log L$	*AIC*	*BIC*
regularΓ	0.0556	4.0196		-113.0274	230.1	232.3
reflected Γ	0.4715	1.6649	0	-31.8257	69.6	67.7
regular Weibull	0.0122	2.1050		-113.6887	231.3	233.6
EE	0.0314	5.2589		-112.9763	229.9	232.2
ESΓ	0.3251	2.5737	-0.249	-31.6234	69.2	67.3

information criteria such as Akaike information criterion *AIC* which is based on the number of parameters in the model, Bayesian information criterion *BIC*, and on the density plots for selecting the best fitting model.

Example (1)
The data for this example are taken from Lawless [19] book, page 99. The data are to test the endurance of deep-groove ball bearings and each observation represents the number of million cycles before the failure times. Parameters estimation for fitting the distributions, Γ, Weibull, and EE have been studied by Gupta et al. [16]. In this analysis, we check whether these five models, mentioned above, are reasonably good fitting for describing ball bearings data. Using SAS software, we estimate the parameters of the reflected Γ and ESΓ distributions, calculate the values of the log likelihood scores and their corresponding values of the *AIC*, and *BIC*. The models with smallest values of these criteria are picked it up as a best fitting model. The results in Table 1 present the negative log likelihood values, *AIC*, and *BIC*. It is clear that the ESΓ works reasonably as a best fit distributions for modeling the life tests of ball bearings data, since it has the largest likelihood values and lowest *AIC* and *BIC* values. Figure 4 shows the density plots of the fitted distributions reflected Γ and ESΓ with the histogram of the observations which displays the presentation of the ball bearings data. The plot depicts that the ESΓ distribution is the closest distribution to the histogram than the reflected Γ distribution.

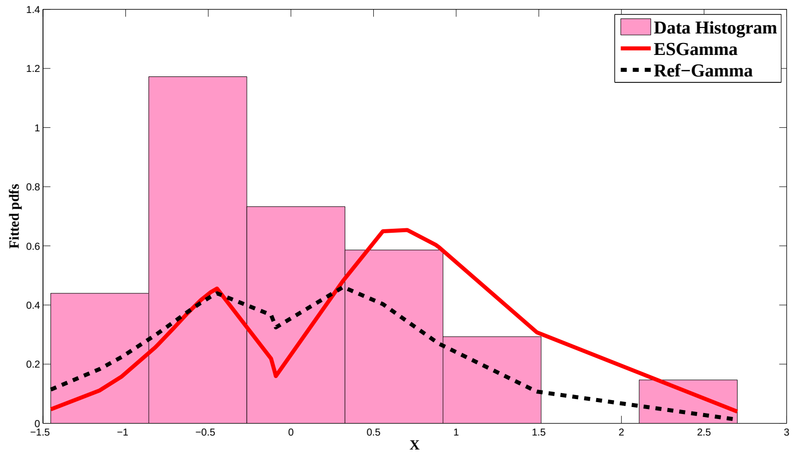

Fig. 4: Fitted Density Functions of the Distributions on the Histogram for Ball Bearings Data.

Example (2)
This example deals with the empirical data sets for growing populations of adult numbers for flour beetle, species Tribolium Castaneum. The data are listed in Eugenea [14] as the x value and observed frequency distributions for strains 2 of Tribolium Castaneum. These data were applied by Eugenea [14] on the Γ, Lagrange-Γ distributions, and beta-normal distribution which is characterized as a bimodal distribution. In this example, we conduct a brief comparison of the four models; regular Γ, reflected Γ, ESΓ, and beta-normal distributions. Using SAS, we estimate the parameters, log likelihood values, *AIC*, and *BIC* for each distribution and comparing the results which are summarized in Table 2 and Figure 5. It is clear that the reflected Γ distribution is a suitable as a better fit distribution to model the flour beetle data among the alternative distributions, since it has the lower values of *AIC*, and *BIC*. Also, the comparison of the

Table 2: Results of the Parameter MLEs and Corresponding Values of $\log L$, AIC, and BIC for the Four Fitted Distributions of Flour Beetle Data.

Distribution	scale	shape	skewness	$\hat{\mu}$	$\hat{\sigma}$	$\log L$	*AIC*	*BIC*
regularΓ	9.8983	8.2585	-	-	-	-1190.782	2387.6	2398.2
ref Γ	0.4857	1.6625	0	-	-	-335.904	715.8	722.9
ESΓ	0.4338	1.8875	-0.1178	-	-	-355.680	717.4	727.9
beta normal	-	shape a 0.160	shape b 0.160	61.29	7.42	-714.1	722.1	736.3

aforemention distributions is based on probability density plots with the histogram of the observations. The plot curves in Figure 5 show that the reflected Γ distribution are fitted to the sample data.

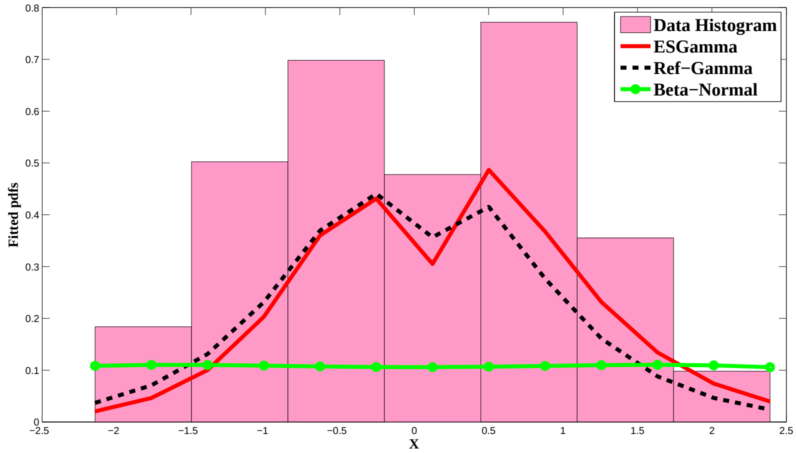

Fig. 5: Fitted Density Functions of the Distributions on the Histogram for Flour Beetle.

8 Conclusion

In the present study, we have focused on ESΓ distribution in the context of skewed, kurtotic, and bimodel literatures. We have derived the main properties, distribution function, and MLE and MME estimators. Furthermore, we have presented the model under two real data sets. In the first case the ESΓ and a reflected Γ distributions give better fit as compared to the regular Γ, Weibull, and EE while from the second case we compare the fits of symmetric and asymmetric Γ distributions to the flour beetle data, where the parameter $\varepsilon = -0.1178$ controls the left skewed and it was observed that the reflected gamma distribution is better to model the flour beetle. The ESΓ distribution can accommodate enough for modeling non-normal data.

References

[1] E. K. Abdulah and H. Elsalloukh, The Epsilon Skew Gamma Distribution, JSM Proceeding, Biometrics Section. San Diago California: American Statistical Association, (2012).

[2] E. K. Abdulah and H. Elsalloukh, The Epsilon Skew Inverted Gamma Distribution, JSM Proceeding, Biometrics Section. San Diago California: American Statistical Association, (2012).

[3] M. M. Ali, M. Pal and J. Woo, Skewed Reflected Distributions Generated by the Laplace Kernel, Austrian Journal of Statistics, **38**, 45-58 (2009).

[4] M. M. Ali and J. Woo and S. Nadarajah, Some Skew-Symmetric Inverse Reflected Distributions, Barazilian Journal of Probability and Statistics, **24**, 1-23 (2010).
[5] H. Almousawi, The Multivariate Epsilon Skew-Laplace Distribution, Ph.D. Thesis, USA, (2011).
[6] R. B. Arellano-Valle, H. W. Gómez and F. A. Quintana, Statistical Inference for A general Class of Asymmetric Distributions, Journal of Statistical Planning and Inference, **128**, 427-443 (2005).
[7] B. C. Arnold and R. J. Beaver, The Skew-Cauchy Distribution, Statistics and Probability Letters, **49**, 285-290 (2000).
[8] A. Azzalini, A Class of Distributions that Includes the Normal Ones, Scandinavian Journal of Statistics, **12**, 171-178 (1985).
[9] A. Azzalini and A. Capitanio, Distribution Generated by Perturbation of Symmetry with Emphasis on A multivariate Skew t Distribution, Journal Roy. Stat. Soc., B **65**, 367-389 (2003).
[10] O. Borghi, Sobre una Distribution de Frecuenoies, Trabajos de Estadistica, **16**, 171-192 (1965).
[11] H. Elsalloukh, The Epsilon-Skew-Exponential Power Distribution, Ph.D. Thesis, USA, (2004).
[12] H. Elsalloukh, The Epsilon-Skew Laplace Distribution, The 2008 Proceedings of the American Statistical Association, Biometrics Section, Denever Colorado, (2008).
[13] H. Elsalloukh and H. J. Guardiola and M. D. Young, The Epsilon-Skew Exponential Power Distribution Family, Far East Journal of Theoretical Statistics, **16**, 97-112 (2005).
[14] N. Eugenea and C. Leea and F. Famoyea, Beta-Normal Distribution and Its Applications, Communications in Statistics;Theory and Methods, **31**, 497-512 (2002).
[15] H. W. Gómez and D. Elal-Olivero and H. S. Salinas and H. Bolfarine, Bimodal Extension Based on the Skew-Normal Distribution with Application to Pollen Data, Environmetrics, **22**, 50-62(2011).
[16] R. D. Gupta and D. Kundu, Expoentiated Exponential Family: An Alternative to Gamma and Weibull Distributions, Biometrical Journal, **43**, 117-130 (2001).
[17] A. K. Gupta and F. C. Chang and W. J. Huang, Some Skew-Symmetric Models, Random Operators and Stochastic Equations, **10**, 133-140 (2002).
[18] M. C. Jones and M. J. Faddy, A skew Extension of the t-Distribution with Applications, Journal of the Royal Statistical Society, B **65**, 159-174 (2003).
[19] J. F. Lawless, Statistical Models and Methods for Lifetime Data, (Wiley, 2nd, New-Jersey, (2003).
[20] G. Mudholkar and A. D. Hutson, The Epsilon-Skew-Normal Distribution for Analyzing Near-Normal Data, Journal of Statistical Planning and Inference, **83**, 291-309 (2000).
[21] M. S. Wahed and M. M. Ali, The Skewed Logistic Distribution, Journal of Statistical Research, **35**, 71-80 (2001).