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Common fixed point and fixed point results under *c*-distance in cone metric spaces

Zaid Mohammed Fadail[1] , *Abd Ghafur Bin Ahmad*[2] , *Stojan Radenovi*[3]

[1]School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor Darul Ehsan, Malaysia.
[2]School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor Darul Ehsan, Malaysia.
[3]University of Belgrade, Faculty of Mechanical Engineering, Kraljice Marije 16, 11120 Beograd, Serbia.

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Abstract: In this paper, we extend the some results of Abbas and Jungck (2008) and Jungck *et al*. (2009) under the concept of a *c*-distance in cone metric spaces. Also, we improve the recent result of Wang and Guo (2011) by omitting the assumption of normality for cones and using the concept of weakly compatible for mappings. In particular, we give some examples for which our result successfully detects a common fixed point in contrast to the some results of Abbas and Jungck, Jungck *et al*. and Wang and Guo which are not applied to show the existence of a common fixed point.

Keywords: Common fixed points, Fixed points, Cone metric spaces, *c*-distance.

1 Introduction

Let X be an arbitrary nonempty set and $f : X \to X$ be a mapping. A fixed point for f is a point $x \in X$ such that $fx = x$. Fixed point theory, one of the very active research areas in mathematics as well as quantitative sciences, focuses on maps and abstract spaces for which one can assure the existence and/or uniqueness of fixed points when they are put together. Many useful mathematical facts can be expressed by assertions that say that certain mappings have fixed points. It has applications in many fields such as chemistry, biology, statistics, economics, computer science and engineering (see, e.g., [1,2,3,4]).

The study of common fixed points of mappings satisfying certain contractive conditions has been at the centre of vigorous research activity. In 1976, Jungck [11], proved a common fixed point theorem for commuting maps, generalizing the Banach contraction principle. Jungck [5,6] defined a pair of self mappings to be weakly compatible if they commute at their coincidence points. In recent years, several authors have obtained coincidence point results for various classes of mappings on a metric space, utilizing these concepts. For a survey of coincidence point theory, its applications, comparison of different contractive conditions and related results, we refer to [7,8,9,10].

In 2008, Abbas and Jungck [14], have studied common fixed point results for noncommuting mappings without continuity in cone metric space with normal cone. The concept of a cone metric space was introduced in the work of Huang and Zhang [13] which is more general than the concept of a metric space. In 2009, Jungck et al. [15] generalized the results of Abbas and Jungck [14] by omitting the assumption of normality in the results. For more details about common fixed point results on cone metric spaces see [25,26,27,28,29]

In 2011, Cho et al. [16] introduced a new concept of a *c*-distance in cone metric spaces which is a generalization of ω-*distance* of Kada et al.[12] with some properties and proved some fixed point theorems in ordered cone metric spaces. For recent results under *c*-distance see [20,21,22,23]

Wang and Guo [17] proved the following common fixed result under *c*-distance in cone metric space. They

* Corresponding author e-mail: zaid_fatail@yahoo.com

assumed that the cone P is normal and they did not require that f and g are weakly compatible.

Theorem 1.1 ([17]) *Let (X,d) be a cone metric space and P is a normal cone with normal constant K where q is a c-distance on X. Let $a_i \in (0,1)(i=1,2,3,4)$ be constants with $a_1+2a_2+a_3+a_4<1$, let $f:X\longrightarrow X$ and $g:X\longrightarrow X$ be two mappings satisfying the condition*

$$q(fx,fy)\preceq a_1kq(gx,gy)+a_2q(gx,fy)+a_3q(gx,fx)+a_4q(gy,fy)$$

for all $x,y\in X$. Suppose that $f(X)\subseteq g(X)$ and $g(X)$ is a complete subspace of X. If f and g satisfy

$$\inf\{\|q(fx,y)\|+\|q(gx,y)\|+\|q(gx,fx)\|:x\in X\}>0$$

for all $y\in X$ with $y\neq fy$ or $y\neq gy$, then f and g have a common fixed point in X.

In above theorem, even it was without assumption of weakly compatible for mappings, but we can not apply it in the case of nonnormal cones. The aim of this paper is to contribute in the study of common fixed points for self mappings on the part of c-distance in cone metric spaces. In this paper, we improve Theorem 1.1 by removing normality condition in it formulation but we assume that the mappings are weakly compatible. Also, our theorem extends some results of Abbas and Jungck [14] and Jungck et al. [15] under c-distance in cone metric spaces by omitting the assumption of normality in the results. Some examples to support our results are given.

2 Preliminaries

Let E be a real Banach space and θ denote the zero element in E. A cone P is a subset of E such that

1. P is nonempty set closed and $P\neq\{\theta\}$,
2. If a,b are nonnegative real numbers and $x,y\in P$ then $ax+by\in P$,
3. $x\in P$ and $-x\in P$ implies $x=\theta$.

For any cone $P\subset E$, the partial ordering $\preceq$ with respect to P is defined by $x\preceq y$ if and only if $y-x\in P$. The notation of $\prec$ stand for $x\preceq y$ but $x\neq y$. Also, we used $x\ll y$ to indicate that $y-x\in \operatorname{int}P$, where $\operatorname{int}P$ denotes the interior of P. A cone P is called normal if there exists a number K such that

$$\theta\preceq x\preceq y\Longrightarrow \|x\|\leq K\|y\| \tag{2.1}$$

for all $x,y\in E$. Equivalently, the cone P is normal if

$$(\forall n)\, x_n\preceq y_n\preceq z_n \text{ and } \lim_{n\to\infty}x_n=\lim_{n\to\infty}z_n=x \text{ imply } \lim_{n\to\infty}y_n=x. \tag{2.2}$$

The least positive number K satisfying Condition 2.1 is called the normal constant of P.

Example 2.1 ([24]) Let $E=C^1_{\mathbb{R}}[0,1]$ with $\|x\|=\|x\|_\infty+\|x'\|_\infty$ and $P=\{x\in E:x(t)\geq 0\}$. This cone is nonnormal. Consider, for example, $x_n(t)=\frac{t^n}{n}$ and $y_n(t)=\frac{1}{n}$. Then $\theta\preceq x_n\preceq y_n$, and $\lim_{n\to\infty}y_n=\theta$, but $\|x_n\|=\max_{t\in[0,1]}|\frac{t^n}{n}|+\max_{t\in[0,1]}|t^{n-1}|=\frac{1}{n}+1>1$; hence x_n does not converge to zero. It follows by Condition 2.2 that P is a nonnormal cone.

Definition 2.1 *([13]) Let X be a nonempty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone P. Suppose that the mapping $d:X\times X\longrightarrow E$ satisfies the following condition:*

1. *$\theta\preceq d(x,y)$ for all $x,y\in X$ and $d(x,y)=\theta$ if and only if $x=y$,*
2. *$d(x,y)=d(y,x)$ for all $x,y\in X$,*
3. *$d(x,y)\preceq d(x,y)+d(y,z)$ for all $x,y,z\in X$.*

Then d is called a cone metric on X and (X,d) is called a cone metric space.

Example 2.2 ([13]) Let $E=\mathbb{R}^2$ and $P=\{(x,y)\in E:x,y\geq 0\}$. Let $X=\mathbb{R}$ and define a mapping $d:X\times X\longrightarrow E$ by $d(x,y)=(|x-y|,\alpha|x-y|)$ for all $x,y\in X$ where $\alpha\geq 0$ is a constant. Then (X,d) is a cone metric space (The cone P is normal with normal constant $K=1$).

Definition 2.2 *([13]) Let (X,d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x\in X$.*

1. *For all $c\in E$ with $\theta\ll c$, if there exists a positive integer N such that $d(x_n,x)\ll c$ for all $n>N$, then x_n is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n\longrightarrow x$.*
2. *For all $c\in E$ with $\theta\ll c$, if there exists a positive integer N such that $d(x_n,x_m)\ll c$ for all $n,m>N$, then $\{x_n\}$ is called a Cauchy sequence in X.*
3. *A cone metric space (X,d) is called complete if every Cauchy sequence in X is convergent.*

Lemma 2.1 ([15])

1. *If E be a real Banach space with a cone P and $a\preceq\lambda a$ where $a\in P$ and $0\leq\lambda<1$, then $a=\theta$.*
2. *If $c\in \operatorname{int}P$, $\theta\preceq a_n$ and $a_n\longrightarrow\theta$, then there exists a positive integer N such that $a_n\ll c$ for all $n\geq N$.*

Definition 2.3 *([16]) Let (X,d) be a cone metric space. A function $q:X\times X\longrightarrow E$ is called a* c-distance *on X if the following conditions hold:*

(q1) $\theta\preceq q(x,y)$ for all $x,y\in X$,
(q2) $q(x,y)\preceq q(x,y)+q(y,z)$ for all $x,y,z\in X$,
(q3) for each $x\in X$ and $n\geq 1$, if $q(x,y_n)\preceq u$ for some $u=u_x\in P$, then $q(x,y)\preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y\in X$,
(q4) for all $c\in E$ with $\theta\ll c$, there exists $e\in E$ with $\theta\ll e$ such that $q(z,x)\ll e$ and $q(z,y)\ll e$ imply $d(x,y)\ll c$.

Example 2.3 Let $E=\mathbb{R}^2$ and $P=\{(x,y)\in E:x,y\geq 0\}$. Let $X=[0,\infty)$ and define a mapping $d:X\times X\longrightarrow E$ by $d(x,y)=(|x-y|,|x-y|)$ for all $x,y\in X$. Then (X,d) is a cone metric space. Define a mapping $q:X\times X\longrightarrow E$ by $q(x,y)=(y,y)$ for all $x,y\in X$. Then q is a c-distance on X (see [18,19]).

Example 2.4 Let $X = [0,1]$, $E = C^1_{\mathbb{R}}[0,1]$ (the set of all real valued functions on X which also have continuous derivatives on X), $P = \{\varphi \in E : \varphi(t) \geq 0\}$. A cone metric d on X is defined by $d(x,y)(t) := |x-y| \cdot \phi(t)$ where $\phi \in P$ is an arbitrary function (e.g., $\phi(t) = e^t$). This cone is nonnormal (see Example 2.1). It is easy to see that (X,d) is a complete cone metric space. Define a mapping $q : X \times X \longrightarrow E$ by $q(x,y)(t) := y \cdot e^t$ for all $x,y \in X$. It is easy to see that q is a c-distance on X.

The following lemma is useful in our work.

Lemma 2.2([16]) *Let (X,d) be a cone metric space and q is a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x,y,z \in X$. Suppose that u_n is a sequences in P converging to θ. Then the following hold:*

1. *If $q(x_n,y) \preceq u_n$ and $q(x_n,z) \preceq u_n$, then $y = z$.*
2. *If $q(x_n,y_n) \preceq u_n$ and $q(x_n,z) \preceq u_n$, then $\{y_n\}$ converges to z.*
3. *If $q(x_n,x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*
4. *If $q(y,x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.*

Remark 2.1(*[16]*)

1. *$q(x,y) = q(y,x)$ does not necessarily for all $x,y \in X$.*
2. *$q(x,y) = \theta$ is not necessarily equivalent to $x = y$ for all $x,y \in X$.*

Recall the following definitions:

Definition 2.4 *An element $x \in X$ is called*

1. *a coincidence point of mappings $f : X \longrightarrow X$ and $g : X \longrightarrow X$ if $w = gx = fx$ and w is called a point of coincidence.*
2. *a common fixed point of mappings $f : X \longrightarrow X$ and $g : X \longrightarrow X$ if $x = gx = fx$.*

Definition 2.5 *The mappings $f : X \longrightarrow X$ and $g : X \longrightarrow X$ are called weakly compatible if $gfx = fgx$ whenever $gx = fx$.*

Theorem 2.1([14]) *Let (X,d) be a cone metric space, and P is a normal cone with normal constant K. Let $f : X \longrightarrow X$ and $\mathrm{g} : X \longrightarrow X$ be two self mappings satisfying the contractive condition*

$$d(fx, fy) \preceq kd(\mathrm{g}x,\mathrm{g}y) \tag{2.3}$$

for all $x,y \in X$,where $k \in [0,1)$ is a constant. If $f(X) \subseteq \mathrm{g}(X)$ and $\mathrm{g}(X)$ is a complete subset of X, then f and g have a unique point of coincidence w in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

3 Common fixed Point results

In this section, we will prove common fixed point theorem under the concept c-distance on cone metric spaces with out assumption of normality cones. The only assumptions are that, the mappings are weakly compatible and the cone P is solid, that is int$P \neq \emptyset$.

Theorem 3.1 *Let (X,d) be a cone metric space over a solid cone P and q is a c-distance on X. Let $f : X \longrightarrow X$ and $\mathrm{g} : X \longrightarrow X$ be two self mappings satisfies the contractive condition*

$$q(fx, fy) \preceq a_1q(\mathrm{g}x,\mathrm{g}y) + a_2q(\mathrm{g}x, fx) + a_3q(\mathrm{g}y, fy) + a_4q(\mathrm{g}x, fy),$$

for all $x,y \in X$, where $a_i, i = 1,2,3,4$ are none negative real numbers such that $a_1 + a_2 + a_3 + 2a_4 < 1$. If $f(X) \subseteq \mathrm{g}(X)$ and $\mathrm{g}(X)$ is a complete subspace of X, then f and g have a coincidence point x^* in X. Further, if $w = \mathrm{g}x^* = fx^*$ then $q(w,w) = \theta$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $\mathrm{g}x_1 = fx_0$. This can be done because $f(X) \subseteq \mathrm{g}(X)$. Continuing this process we obtain a sequence $\{x_n\}$ in X such that $\mathrm{g}x_{n+1} = fx_n$. Then we have

$$\begin{aligned}
q(\mathrm{g}x_n, \mathrm{g}x_{n+1}) &= q(fx_{n-1}, fx_n) \\
&\preceq a_1q(\mathrm{g}x_{n-1}, \mathrm{g}x_n) + a_2q(\mathrm{g}x_{n-1}, fx_{n-1}) + a_3q(\mathrm{g}x_n, fx_n) \\
&\quad + a_4q(\mathrm{g}x_{n-1}, fx_n) \\
&= a_1q(\mathrm{g}x_{n-1},\mathrm{g}x_n) + a_2q(\mathrm{g}x_{n-1},\mathrm{g}x_n) + a_3q(\mathrm{g}x_n,\mathrm{g}x_{n+1}) \\
&\quad + a_4q(\mathrm{g}x_{n-1},\mathrm{g}x_{n+1}) \\
&\preceq a_1q(\mathrm{g}x_{n-1},\mathrm{g}x_n) + a_2q(\mathrm{g}x_{n-1},\mathrm{g}x_n) + a_3q(\mathrm{g}x_n,\mathrm{g}x_{n+1}) \\
&\quad + a_4(q(\mathrm{g}x_{n-1},\mathrm{g}x_n) + q(\mathrm{g}x_n,\mathrm{g}x_{n+1})).
\end{aligned}$$

So,

$$\begin{aligned}
q(\mathrm{g}x_n,\mathrm{g}x_{n+1}) &\preceq \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(\mathrm{g}x_{n-1},\mathrm{g}x_n) \\
&= hq(\mathrm{g}x_{n-1},\mathrm{g}x_n) \\
&\preceq h^2q(\mathrm{g}x_{n-2},\mathrm{g}x_{n-1}) \\
&\quad . \\
&\quad . \\
&\quad . \\
&\preceq h^nq(\mathrm{g}x_0,\mathrm{g}x_1),
\end{aligned}$$

where $h = \frac{a_1+a_2+a_4}{1-a_3-a_4} < 1$.

Note that

$$q(\mathrm{g}x_n,\mathrm{g}x_{n+1}) = q(fx_{n-1}, fx_n) \preceq hq(\mathrm{g}x_{n-1},\mathrm{g}x_n). \tag{3.1}$$

Let $m > n \geq 1$. Then it follows that

$$\begin{aligned}
q(\mathrm{g}x_n,\mathrm{g}x_m) &\preceq q(\mathrm{g}x_n,\mathrm{g}x_{n+1}) + q(\mathrm{g}x_{n+1},\mathrm{g}x_{n+2}) + ... + q(\mathrm{g}x_{m-1},\mathrm{g}x_m) \\
&\preceq (h^n + h^{n+1} + ... + h^{m-1})q(\mathrm{g}x_0,\mathrm{g}x_1) \\
&\preceq \frac{h^n}{1-h} q(\mathrm{g}x_0,\mathrm{g}x_1).
\end{aligned}$$

Thus, Lemma 2.2 (3) shows that $\{\mathrm{g}x_n\}$ is a Cauchy sequence in X. Since $\mathrm{g}(X)$ is complete, there exists $x^* \in X$ such that $\mathrm{g}x_n \longrightarrow \mathrm{g}x^*$ as $n \longrightarrow \infty$. By q3 we have:

$$q(\mathrm{g}x_n,\mathrm{g}x^*) \preceq \frac{h^n}{1-h} q(\mathrm{g}x_0,\mathrm{g}x_1). \tag{3.2}$$

On the other hand and by using (3.1):

$$\begin{aligned} q(gx_n, fx^*) &= q(fx_{n-1}, fx^*) \\ &\preceq hq(gx_{n-1}, gx^*) \\ &\preceq h\frac{h^{n-1}}{1-h}q(gx_0, gx_1) \\ &= \frac{h^n}{1-h}q(gx_0, gx_1). \end{aligned} \quad (3.3)$$

Thus, Lemma 2.2 (1), (3.2) and (3.3) show that $gx^* = fx^*$. Therefore, x^* is a coincidence point of f and g and w is a point of coincidence of f and g where $w = gx^* = fx^*$ for some x^* in X.

Suppose that $w = gx^* = fx^*$. Then we have

$q(w,w) = q(fx^*, fx^*)$
$\preceq a_1\ q(gx^*, gx^*) + a_2q(gx^*, fx^*) + a_3q(gx^*, fx^*) + a_4q(gx^*, fx^*)$
$= a_1q(w,w) + a_2q(w,w) + a_3q(w,w) + a_4q(w,w)$
$= (a_1 + a_2 + a_3 + a_4)q(w,w)$.

Since $(a_1 + a_2 + a_3 + a_4) < 1$, Lemma 2.1 (1) shows that $q(w,w) = \theta$.

Finally, suppose there is another point of coincidence u of f and g such that $u = fy^* = gy^*$ for some y^* in X. Then we have

$q(w,u) = q(fx^*, fy^*)$
$\preceq a_1q(gx^*, gy^*) + a_2q(gx^*, fx^*) + a_3q(gy^*, fy^*) + a_4q(gx^*, fy^*)$
$= a_1q(w,u) + a_2q(w,w) + a_3q(u,u) + a_4q(w,u)$
$= a_1q(w,u) + a_4q(w,u)$
$= (a_1 + a_4)q(w,u)$.

Since $(a_1 + a_4) < 1$, Lemma 2.1 (1) shows that $q(w,u) = \theta$. Also, we have $q(w,w) = \theta$. Thus, Lemma 2.2 (1) shows that $w = u$. Therefore, w is the unique point of coincidence. Now, let $w = gx^* = fx^*$. Since f and g are weakly compatible, we have

$$gw = ggx^* = gfx^* = fgx^* = fw.$$

Hence, gw is a point of coincidence. The uniqueness of the point of coincidence implies that $gw = gx^*$. Therefore, $w = gw = fw$. Hence, w is the unique common fixed point of f and g.

The following corollaries can be obtained as consequences of Theorem 3.1 which are the extension of some results of Abbas and Jungck [14] and Jungck [15] under the concept of c-distance.

Corollary 3.1 *Let (X,d) be a cone metric space over a solid cone P and q is a c-distance on X. Let $f : X \longrightarrow X$ and $g : X \longrightarrow X$ be two self mappings satisfies the contractive condition*

$$q(fx, fy) \preceq kq(gx, gy),$$

for all $x,y \in X$, where $k \in [0,1)$ is a constant. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a coincidence point x^* in X. Further, if $w = gx^* = fx^*$ then $q(w,w) = \theta$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Corollary 3.2 *Let (X,d) be a cone metric space over a solid cone P and q is a c-distance on X. Let $f : X \longrightarrow X$ and $g : X \longrightarrow X$ be two self mappings satisfies the contractive condition*

$$q(fx, fy) \preceq kq(gx, fx) + lq(gy, fy),$$

for all $x,y \in X$, where k,l are none negative real numbers such that $k + l < 1$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a coincidence point x^* in X. Further, if $w = gx^* = fx^*$ then $q(w,w) = \theta$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Corollary 3.3 *Let (X,d) be a cone metric space over a solid cone P and q is a c-distance on X. Let $f : X \longrightarrow X$ and $g : X \longrightarrow X$ be two self mappings satisfies the contractive condition*

$$q(fx, fy) \preceq k(q(gx, fx) + q(gy, fy)),$$

for all $x,y \in X$, where $k \in [0, \frac{1}{2})$ is a constant. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a coincidence point x^* in X. Further, if $w = gx^* = fx^*$ then $q(w,w) = \theta$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Now, we present two examples to illustrate our results. In first example (the case of a normal cone), the conditions of Corollary 3.1 are fulfilled, but Theorem 2.1 of Abbas and Jungck ([14], Theorem 2.1) and the result of Jungck et al. ([15],Corollary 2.3) cannot be applied. The second example (the case of a nonnormal cone), the conditions of 3.1 are fulfilled, but Theorem 1.1 of Wang and Guo ([17], Theorem 2.2) cannot be applied.

Example 3.1 (the case of a normal cone) Consider Example 2.3. Define the mappings $f : X \longrightarrow X$ by

$$f(x) = \begin{cases} \frac{4x}{3}, & \text{if} \quad x \neq \frac{1}{2}, \\ \frac{1}{3}, & \text{if} \quad x = \frac{1}{2}. \end{cases}$$

and $g : X \longrightarrow X$ by $gx = 2x$ for all $x \in X$. Clear that $f(X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Since

$$\begin{aligned} d(f(1), f(\frac{1}{2})) &= d(\frac{4}{3}, \frac{1}{3}) \\ &= (1,1) \\ &= d(2,1) \\ &= d(g(1), g(\frac{1}{2})), \end{aligned}$$

there is no $k \in [0,1)$ such that $d(fx, fy) \preceq kd(gx, gy)$ for all $x, y \in X$. Hence, Theorem 2.1 of Abbas and Jungck ([14], Theorem 2.1) can not apply to this example on cone metric space. To check this example on c-distance, we have:

1. If $x = y = \frac{1}{2}$, then we have $q(f(\frac{1}{2}), f(\frac{1}{2})) = (f(\frac{1}{2}), f(\frac{1}{2})) = (\frac{1}{3}, \frac{1}{3}) \preceq \frac{4}{5}(1,1) = k(g(\frac{1}{2}), g(\frac{1}{2}))$ with $k = \frac{4}{5}$.
2. If $x \neq y \neq \frac{1}{2}$, then we have $q(fx, fy) = (\frac{4y}{3}, \frac{4y}{3}) = (\frac{2}{3}2y, \frac{2}{3}2y) = \frac{2}{3}(2y, 2y) \preceq \frac{4}{5}(2y, 2y) = kq(gx, gy)$ with $k = \frac{4}{5}$.
3. If $x = \frac{1}{2}, y \neq \frac{1}{2}$, then we have $q(f(\frac{1}{2}), fy) = (\frac{4y}{3}, \frac{4y}{3}) = (\frac{2}{3}2y, \frac{2}{3}2y) = \frac{2}{3}(2y, 2y) \preceq \frac{4}{5}(2y, 2y) = kq(gx, gy)$ with $k = \frac{4}{5}$.
4. If $x \neq \frac{1}{2}, y = \frac{1}{2}$, then we have $q(fx, f(\frac{1}{2})) = (f(\frac{1}{2}), f(\frac{1}{2})) = (\frac{1}{3}, \frac{1}{3}) \preceq \frac{4}{5}(1,1) = k(g(\frac{1}{2}), g(\frac{1}{2}))$ with $k = \frac{4}{5}$.

Hence, $q(fx, fy) \preceq kq(gx, gy)$ for all $x, y \in X$ where $k = \frac{4}{5} \in [0,1)$. Also f and g are weakly compatible at $x = 0$. Therefore, all conditions of Corollary 3.1 are satisfied. Hence, f and g have a unique common fixed point $x = 0$ and $f(0) = g(0) = 0$ with $q(0,0) = 0$.

Example 3.2 (the case of a nonnormal cone) Consider Example 2.4. Define the mappings $f : X \longrightarrow X$ by $fx = \frac{x^2}{4}$ and $g : X \longrightarrow X$ by $gx = \frac{x}{2}$ for all $x \in X$. Clear that $f(X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Note that, we (because of nonnormality of the cone) can not apply Theorem 1.1 of Wang and Guo ([17], Theorem 2.2). But we can apply our theorem, observe that:

$$\begin{aligned}
q(fx, fy)(t) &= fy \cdot e^t \\
&= \frac{y^2}{4} \cdot e^t \\
&= \frac{y}{2}\frac{y}{2} \cdot e^t \\
&\preceq \frac{1}{2}\frac{y}{2} \cdot e^t \\
&= \frac{1}{2}(\frac{y}{2} \cdot e^t) \\
&= \frac{1}{2}(gy \cdot e^t) \\
&= a_1 q(gx, gy)(t) \\
&\preceq a_1 q(gx, gy)(t) + a_2 q(gx, fx)(t)
\end{aligned}$$
$$+ a_3 q(gy, fy)(t) + a_4 q(gx, fy)(t),$$

where $a_1 = \frac{1}{2}, a_2 = a_3 = \frac{1}{8}, a_4 = \frac{1}{16}$ and $a_1 + a_2 + a_3 + 2a_4 = \frac{7}{8} < 1$. Also, f and g are weakly compatible at $x = 0$. Therefore, all conditions of Theorem 3.1 are satisfied. Hence, f and g have a unique common fixed point $x = 0$ and $f(0) = g(0) = 0$ with $q(0,0) = 0$.

Finally, we have the following result (immediately consequence of Theorem 3.1).

Theorem 3.2 *Let (X,d) be a complete cone metric space over a solid cone P and q is a c-distance on X. Let $f : X \longrightarrow X$ be a self mapping satisfies the contractive condition*

$$q(fx, fy) \preceq a_1 q(x,y) + a_2 q(x, fx) + a_3 q(y, fy) + a_4 q(x, fy),$$

for all $x, y \in X$, where $a_i, i = 1,2,3,4$ are none negative real numbers such that $a_1 + a_2 + a_3 + 2a_4 < 1$. Then f has a fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$ then $q(v,v) = \theta$. The fixed point is unique.

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References

[1] K.C. Border, Fixed Point Theorems with Applications to Economics and Game Theory , Cambridge University Press, New York, 1985.

[2] A. Cataldo, E.A. Lee, X. Liu, E. D. Matsikoudis, H. Zheng, A constructive Fixed point theorem and the feedback semantics of timed systems, Tech. Report UCB/EECS-2006-4, EECS Dept., University of California, Berkeley, (2006).

[3] A. Hyvärinen, Fast and Robust Fixed-Point Algorithms for Independent Component Analysis, IEEE Transactions on Neural Networks, **10**, (1999) no:3, 626-634.

[4] A. Noumsi, S. Derrien, P. Quinton, Acceleration of a Content Based Image Retrieval Application on the RDISK Cluster, IEEE International Parallel and Distributed Processing Symposium, April 2006.

[5] G. Jungck, Common fixed points for noncontinuous nonself maps on non-metric spaces, Far East J. Math. Sci. (FJMS) **4** (1996) 199-215.

[6] G. Jungck, B.E. Rhoades, Fixed point for set valued functions without continuity, Indian J. Pure Appl. Math. **29** (3) (1998) 227-238.

[7] R.P. Pant, Common fixed points of noncommuting mappings, J. Math. Anal. Appl. 188 (1994) 436-440.

[8] B.E. Rhoades, A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. **26** (1977) 257-290.

[9] I. Beg, M. Abbas, Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition, Fixed Point Theory Appl. **2006** (2006) 1-7, Article ID 74503.

[10] G. Jungck, Common fixed points for commuting and compatible maps on compacta, Proc. Amer. Math. Soc. **103** (1988) 977-983.

[11] G. Jungck, Commuting maps and fixed points, Amer. Math. Monthly **83** (1976) 261-263.
[12] O. Kada, T. Suzuki and W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, Math. Japon. **44** (1996) 381-391.
[13] L.-G. Huang and X. Zhang, Cone Metric Spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. **332** (2007) 1468-1476.
[14] M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric space, J. Math. Anal. Appl. **341** (2008) 416-420.
[15] G. Jungck, S. Radenović, S. Radojević and V. Rakočević, Common fixed point theorems for weakly compatible pairs on cone metric spaces. Fixed Point Theory and Applications,**2009**, Article ID 643840, 13 pages, 2009.
[16] Y.J. Cho, R. Saadati and S. Wang, Common fixed point theorems on generalized distance in ordered cone metric spaces, Comput. Math. Appl. **61** (2011) 1254-1260.
[17] S. Wang and B. Guo, Distance in cone metric spaces and common fixed point theorems, Applied Mathematical Letters. **24** (2011) 1735-1739.
[18] Z. M. Fadail and A. G. B. Ahmad, Coupled fixed point theorems of single-valued mapping for *c*-distance in cone metric spaces, Journal of Applied Mathematics, **2012**, Article ID 246516, 20 pages, 2012.
[19] Z. M. Fadail and A. G. B. Ahmad, Common coupled fixed point theorems of single-valued mapping for *c*-distance in cone metric spaces, Abstract and Applied Analysis, **2012**, Article ID 901792, 24 pages, 2012.
[20] M. Đorđević, D. Đorić, Z. Kadelburg, S. Radenović and D. Spasić, Fixed point results under c-distance in tvs-cone metric spaces, Fixed Point Theory and Applications 2011, 2011:29, doi:10.1186/1687/1812/2011/29.
[21] Z. M. Fadail, A. G. B. Ahmad and Z. Golubović, Fixed point theorems of single-valued mapping for *c*-distance in cone metric spaces, Abstract and Applied Analysis, **2012**, Article ID 826815, 11 pages, 2012.
[22] Z. M. Fadail, A. G. B. Ahmad and Ljiljana Paunović, New fixed point results of single-valued mapping for *c*-distance in cone metric spaces, Abstract and Applied Analysis, **2012**, Article ID 639713, 12 pages, 2012.
[23] W. Shatanawi, E. Karapinar and H. Aydi, Coupled coincidence points in partially ordered cone metric spaces with *c*-distance, Journal of Applied Mathematics, **2012**, Article ID 312078, 15 pages, 2012.
[24] A. G. B. Ahmad, Z. M. Fadail, M. Abbas, Z. Kadelburg and S. Radenović, Some fixed and periodic points in abstract metric spaces, Abstract and Applied Analysis, **2012**, Article ID 908423, 15 pages, 2012.
[25] M. Abbas and B.E. Rhoades, Fixed and periodic point results in cone metric spaces, Appl. Math. Lett. **22** (2009) 511-515.
[26] A. Azam and M. Arshad, Common fixed points of generalized contractive maps in cone metric space, Bull. Iranian Math. Soc. **35** (2) (2009) 225-264.
[27] D. Ilić and V. Rakočević, Common fixed point for maps on cone metric space, J. Math. Anal. Appl. **341** (2008) 876-882.
[28] Z. Kadelburg, S. Radenović and V. Rakočević, Topological vector spaces-valued cone metric spaces and fixed point theorems, Fixed Point Theory and Applications, **2010**, Article ID 170253, 17 pages, doi: 10.1155/2009/170253
[29] Z. Kadelburg and S. Radenović, Some common fixed point results in non-normal cone metric spaces, Journal of Nonlinear Sciences and Applications, **3** (2010) no.3, 193-202.

Zaid Mohammed Fadail is a Ph.D Student at the School of Mathematical Sciences, Faculty of Science and Technology, the National University of Malaysia. His interest is on functional analysis and Topology and group theory.

Abd Ghafur Bin Ahmad is an associate professor at the School of Mathematical Sciences, Faculty of Science and Technology, the National University of Malaysia. His interest is on combinatorial algebra theory especially in geometric and functional group theory. He has been selected as referee for several journals such as Bulletin of Malaysian Mathematical Society, Malaysian Journal of Mathematical Sciences, Sains Malaysiana, Matematika, Fuzzy Set and System and Journal of Quality and Measurement.

Stojan Radenovi has a position of full professor at Department of Mathematics, Faculty of Mechanical Engineering, University of Belgrade, Serbia. He is referee for several journals: Kragujevac J. of Mathematics, Matemati?cki Vesnik, Computers and Mathematics with Applications, Ap- plied Mathematics Letters, Nonlinear Analysis, Fixed Point Theory and Ap- plications, Fixed Point Theory, Journal of Applied Mathematics and others. Research interests are in Functional Analysis, especially in the theory of Lo- cally Convex Spaces and Nonlinear Analysis, especially in the theory of fixed point in Abstract Metric Spaces and Metric Spaces.