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Common Fixed Points of Kannan Type Fuzzy Mappings on closed balls

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Abstract: We establish some common fixed point theorems for Kannan fuzzy mappings on closed balls in a complete metric space. Our investigation is based on the fact that fuzzy fixed point results can be obtained simply from the fixed point theorem of multivalued mappings with closed values. In real world problems there are various mathematical models in which the mappings are contractive on the subset of a space under consideration but not on the whole space itself. It seems that this technique of finding the fuzzy fixed points was ignored. Recently Azam et. al. [A. Azam, S. Hussain and M. Arshad, Common Fixed points of Chatterjea Type Fuzzy Mappings on closed balls, Neural Computing & Applications, DOI 10. 1007/s00521-012-0907-4] published a fixed points result for Chatterjea fuzzy locally contractive mappings on closed ball and our present work will supplement it. Our results generalize/improve several results of literature.

Keywords: Fuzzy fixed point; Kannan mapping; Contraction; Closed ball; Continuous mapping.

1 Introductions and Preliminaries

It is a well known fact that the results of fixed points are very useful for determining the existence and uniqueness of solutions to various mathematical models. Over the period of last forty years the theory of fixed points has been developed regarding the results, which are related to finding the fixed points of self and non-self nonlinear mappings. In 1922, Banach proved a contraction principal which states that for a complete metric space (X,d) , a mapping $T : X \to X$ satisfying a contraction condition $d(Tx,Ty) \le \alpha d(x,y)$ for all $x,y \in X$, where $0 < \alpha < 1$ has a unique fixed point in X. Banach contraction principal plays a fundamental role in the emergence of modern fixed point theory and it gains more attention because it is based on iterations so it can be easily applied by using computer. In 1969, Kannan[11] a premier Indian Mathematician proved that a mapping $T : X \to X$ satisfying a contraction condition $d(Tx,Ty) \le \alpha[d(x,Tx)+(y,Ty)]$ for all $x,y \in X$, where $0 < \alpha < \frac{1}{2}$, has a unique fixed point in X. The Kannan[11] contraction mappings need not to be continuous. Since continuity is big requirement, this makes Kannan contraction mappings to be important with respect to application point of view. Azam et.al.[2] published a fixed point theorem for fuzzy locally contractive Chatterjea mappings on a closed ball. The study of fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity, and it has a wide range of applications in different areas such as nonlinear and adoptive control systems, parameterize estimation problems, fractal image decoding, computing magnetostatic fields in a nonlinear medium, and convergence of recurrent networks, (see[13, 14, 20, 23]). In his paper Rhoeds[17] compared different contraction conditions, which become the foundation of development of present fixed point theory for nonlinear contractive operators. The notion of fixed points for fuzzy mappings was introduced by Weiss[24] and Butnariu[7]. Fixed point theorems for fuzzy set-valued mappings have been studied by Heilpern[10] who introduced the concept of fuzzy contraction mappings and established Banach contraction principle for fuzzy mappings in complete metric linear spaces, which is a fuzzy extension of Banach fixed point theorem and Nadler‘s[15] theorem for multivalued mappings. Subsequently several other authors [1, 3, 6, 8, 12, 16, 18, 19, 21, 22] studied the existence of fixed points and common fixed points of

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fuzzy mappings satisfying a contractive type condition. Frigon and O'Regan[9] proved some fuzzy fixed point theorems on closed balls. In this paper, we prove some common fixed point theorems for a pair of fuzzy mappings satisfying Kannan type[11] contractive condition. Let

$$\begin{aligned} 2^X &= \{A : A \text{ is a subet of } X\}, \\ CL(2^X) &= \{A \in 2^X : A \text{ is nonempty closed}\}, \\ C(2^X) &= \{A \in 2^X : A \text{ is nonempty and compact}\}, \\ CB(2^X) &= \{A \in 2^X : A \text{ is nonempty closed and bounded}\}, \end{aligned}$$

For $A,B \in CB(2^X)$,

$$\begin{aligned} d(x,A) &= \inf_{y\in A} d(x,y), \\ d(A,B) &= \inf_{x\in A, y\in B} d(x,y). \end{aligned}$$

Then the Hausdorff metric d_H on $CB(2^X)$ induced by d is defined as:

$$d_H(A,B) = \{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}.$$

A fuzzy set in X is a function with domain X and values in [0,1] and I^X is the collection of all fuzzy sets in X. If A is a fuzzy set and $x \in X$. Then the function values $A(x)$ is called the grade of membership of x in A. The $\alpha - level$ set of a fuzzy set A, is denoted by $[A]_\alpha$, and is defined as:

$$\begin{aligned} [A]_\alpha &= \{x : A(x) \geq \alpha if \alpha \in (0,1]\}, \\ [A]_0 &= \overline{\{x : A(x) \geq 0\}}. \end{aligned}$$

For $x \in X$, we denote the fuzzy set $\chi_{\{x\}}$ by $\{x\}$ unless and until it is stated, where χ_A is the characteristic function of the crisp set A. Now we define a sub-collection of I^X as follows:

$$\Im(X) = \{A \in I^X : [A]_1 \text{ is nonempty and closed}\},$$

For $A,B \in I^X, A \subset B$ means $A(x) \leq B(x)$ for each $x,y \in X$. For $A,B \in \Im(X)$ then define

$$D_1\{A,B\} = d_H([A]_1,[B]_1).$$

A point $x^* \in X$ is called a fixed point of a fuzzy mapping $T : X \to I^X$ if $x^* \subset Tx^*$ (see[15]).

Lemma 1 (see [15]) Let A and B be nonempty closed and bounded subsets of a metric space (X,d). If $a \in A$, then $d(a,B) \leq d_H(A,B)$.

Lemma 2 (see [15]) Let A and B be nonempty closed and bounded subsets of a metric space (X,d) and $0 < \xi \in \Re$ Then for $a \in A$, there exists $b \in B$ such that $d(a,b) \leq d_H(A,B) + \xi$.

Lemma 3 (see [15]) The completeness of (X,d) implies that $(CB(2^X),d_H)$ is complete.

Theorem 4 (see [11]) Let (X,d) be a metric space and a mapping $T : X \to X$. Suppose there exists a constant $\alpha \in (0,\frac{1}{2})$ such that

$$d(Tx,Ty) \leq \alpha[d(x,Tx) + d(y,Ty)],$$

holds. Then T has a unique fixed point in X.

2 Fixed Points of Kannan Type Fuzzy Mappings on closed balls

The mapping satisfies the contractive condition in Theorem 4 is called Kannan mapping. It is mentioned that Kannan contractive condition does not implies that the mapping T is continues, which differentiates it from Banach contractive condition. For $c \in X$ and $0 < r \in \Re$ let

$$S_r(c) = \{x \in X | d(c,x) < r\}$$

be the ball of radius r centered at c. Also, the closure of $S_r(c)$ is denoted by $\overline{S_r(c)}$. We present a result regarding the existence of common fixed point for fuzzy mappings satisfying Kannan type[11] contractive condition on closed balls. The theorem is as follows:

Theorem 5 Let (X,d) be a complete metric space, $x_0 \in X$ and mappings $F,T : \overline{S_r(x_0)} \to \Im(X)$. Suppose there exists a constant $k \in (0,\frac{1}{2})$ with

$$D_1(Fx,Ty) \leq k[d(x,[Fx]_1), d(y,[Ty]_1)], \tag{1}$$

for all $x,y \in \overline{S_r(x_0)}$ and

$$d(x_0,[Fx_0]_1) < \frac{(1-2k)}{(1-k)}r, \tag{2}$$

holds. Then F and T has a common fuzzy fixed point in $\overline{S_r(x_0)}$. That is, there exists $x^* \in \overline{S_r(x_0)}$ with $\{x^*\} \subseteq Fx^* \cap Tx^*$.

Proof. Choose $x_1 \in X$, such that $x_1 \subseteq Fx_0$ and

$$d(x_0,x_1) < \frac{(1-2k)}{(1-k)}r, \tag{3}$$

since $[Fx_0]_1 \neq \phi$ and $d(x_0,[Fx_0]_1) < \frac{(1-2k)}{(1-k)}r$. For the sake of simplicity, choose $\lambda = \frac{k}{1-k}$. This gives us $d(x_0,x_1) < (1-\lambda)r$, which implies that $x_1 \in \overline{S_r(x_0)}$. Now choose $\varepsilon > 0$, such that

$$\lambda d(x_0,x_1) + \frac{\varepsilon}{(1-k)} < \lambda(1-\lambda)r \tag{4}$$

Then choose $x_2 \in X$ such that $\{x_2\} \subseteq Tx_1$ and by using inequality (1) and Lemma 2

$$\begin{aligned} d(x_1,x_2) &\leq D_1(Fx_0,Tx_1) + \varepsilon \\ &\leq k[d(x_0,[Fx_0]) + d(x_1,[Tx_1])] + \varepsilon \\ &\leq k[d(x_0,x_1) + d(x_1,x_2)] + \varepsilon. \end{aligned}$$

Which implies

$$\begin{aligned} (1-k)d(x_1,x_2) &\leq k(x_0,x_1) + \varepsilon \\ d(x_1,x_2) &\leq (\frac{k}{1-k}))d(x_0,x_1) + (\frac{\varepsilon}{1-k}) \\ &= \lambda d(x_0,x_1) + (\frac{\varepsilon}{1-k}). \end{aligned}$$

By using inequality (4) we get

$$d(x_1,x_2) < \lambda(1-\lambda)r. \tag{5}$$

Note that $x_2 \in \overline{S_r(x_0)})$, since

$$d(x_0,x_2) \leq d(x_0,x_1) + d(x_1,x_2),$$

Then by using inequalities (3) and (5) we have

$$\begin{aligned} d(x_0,x_2) &\leq (1-\lambda)r + \lambda(1-\lambda)r \\ &\leq (1-\lambda)r[1+\lambda] \\ &\leq (1-\lambda)r[1+\lambda+\lambda^2+\ldots] = r. \end{aligned}$$

Continue this process and having chosen (x_n) in X such that

$$\{x_{2k+1}\} \subseteq Fx_{2k},$$

and

$$x_{2k+2} \subseteq Tx_{2k+1},$$

with

$$d(x_{2k+1},x_{2k+2}) < \lambda^{2k+1}(1-\lambda)r \text{ for } k = 2,3,\ldots.$$

Notice that (x_n) is Cauchy sequence in $\overline{S_r(x_0)}$, which is complete. Therefore a point $x^* \in \overline{S_r(x_0)}$ exists with $\lim_{n\to\infty} x_n = x^*$. It remains to show that $\{x^*\} \subseteq Tx^*$ and $\{x^*\} \subseteq Fx^*$. Now by Lemma 1 and inequality (1), we get

$$\begin{aligned} d(x^*,[Tx^*]_1) &\leq d(x^*,x_{2n+1}) + d(x_{2n+1},[Tx^*]_1) \\ &\leq d(x^*,x_{2n+1}) + D_1(Fx_{2n+2},Tx^*) \\ &\leq d(x^*,x_{2n+1}) \\ &\quad + k[d(x_{2n+2},[Fx_{2n+2}]_1) + d(x^*,[Tx^*]_1)] \end{aligned}$$

which implies

$$\begin{aligned} (1-k)d(x^*,[Tx^*]_1) &\leq d(x^*,x_{2n+1}) + kd(x_{2n+2},[Fx_{2n+2}]_1) \\ &\leq d(x^*,x_{2n+1}) + kd(x_{2n+2},x_{2n+3}) \to 0 \text{ as } n \to \infty. \end{aligned}$$

This implies

$$d(x^*,[Tx^*]_1) = 0,$$

which implies that $\{x^*\} \subseteq Tx^*$. Similarly consider that

$$d(x^*,[Fx^*]_1) \leq d(x^*,x_{2n+2}) + d(x_{2n+2},[Fx^*]_1),$$

to show that $\{x^*\} \subseteq Fx^*$, which implies that mappings F and T have a common fixed point in $\overline{S_r(x_0)}$, i.e. $\{x^*\} \subseteq Fx^* \cap Tx^*$. □

Corollary 6 Let (X,d) be a complete metric space, $x_0 \in X$ and mapping $F : \overline{S_r(x_0)} \to \Im(X)$. Suppose there exists a constant $k \in (0,\frac{1}{2})$ with

$$D_1(Fx,Fy) \leq k[d(x,[Fx]_1),d(y,[Fy]_1)], \tag{6}$$

for all $x,y \in \overline{S_r(x_0)}$ and

$$d(x_0,[Fx_0]_1) < \frac{(1-2k)}{(1-k)}r, \tag{7}$$

holds. Then F has a fuzzy fixed point in $\overline{S_r(x_0)}$. That is, there exists $x^* \in \overline{S_r(x_0)}$ with $\{x^*\} \subseteq Fx^*$.

Proof. By taking $T = F$ in Theorem 5, we get x^* in $\overline{S_r(x_0)}$ such that $\{x^*\} \subseteq Fx^*$. □

Theorem 7 Let (X,d) be a complete metric space, $x_0 \in X$ and mappings $F,T : X \to \Im(X)$. Suppose there exists a constant $k \in (0,\frac{1}{2})$ with

$$D_1(Fx,Ty) \leq k[d(x,[Fx]_1),d(y,[Ty]_1)], \tag{8}$$

for all $x,y \in X$. Then F and T has a common fuzzy fixed point in X. That is, there exists $x^* \in X$ such that $\{x^*\} \subseteq Fx^* \cap Tx^*$ (i.e. $Fx^*(x^*) = 1$ and $Tx^*(x^*) = 1$).

Proof. Fix $x_0 \in X$ and choose $r > 0$ so that

$$d(x_0,[Fx_0]_1) < \frac{(1-2k)}{(1-k)}r.$$

Now Theorem 5 guarantees that there exists $x^* \in X$ with $\{x^*\} \subseteq Fx^* \cap Tx^*$ □

Corollary 8 Let (X,d) be a complete metric space, $x_0 \in X$ and mapping $F : X \to \Im(X)$. Suppose there exists a constant $k \in (0,\frac{1}{2})$ with

$$D_1(Fx,Fy) \leq k[d(x,[Fx]_1),d(y,[Fy]_1)], \tag{9}$$

for all $x,y \in X$. Then F has a fuzzy fixed point in X. That is, there exists $x^* \in X$ such that $\{x^*\} \subseteq Fx^*$.

Proof. In Theorem 7, take $T = F$. □

An example of a fuzzy mapping which is contractive on the subset of a space but not on the whole space is as follows:

Example 9 Let $X = \Re$ and $d : X \times X \to \Re$ is defined by $d(x,y) = |x-y|$, where $x,y \in X$. Consider the mapping $F : X \to \Im(X)$ defined by

$$F(X) = \begin{cases} \chi_{\{1-x\}} & \text{if x is irrational} \\ \chi_{\{\frac{1+x}{3}\}} & \text{if x is rational} \end{cases}.$$

Then F is Kannan fuzzy mapping on the closed ball $\overline{S_{(\frac{1}{2})}(\frac{1}{2})} = [0,1]$, but not on the whole space X.

3 Conclusion

As far as the application of contraction mapping is concerned, the situation is not yet fully exploited. It is quite possible that a contraction T is defined on the whole space X but it is contractive on the subset Y of the subset of the space rather on the whole space X. Moreover the contraction mapping under consideration may not be continues. If Y is closed, then it is complete, so that the mapping T has a fixed point x in Y, and $x_n \to x$ as in the case of the whole space X, provided we impose a simple restriction on the choice of x_0, so that x_n's remains in Y. In this paper, we have discussed this concept for fuzzy Kannan[11] mappings on a complete metric space X which generalizes/improves several classical fixed point results. We feel that the paper in tendon with [2] will become the foundation of fuzzy theory on closed balls.

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