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Bayesian and E-Bayesian Estimation for Rayleigh Distribution Using Unified Progressive Hybrid Censored Samples

Magdy Nagy[1,2]

[1]Department of Statistics and Operation Research, College of Sciences, King Saud University, Riyadh, Kingdom of Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Fayoum University, Fayoum, Egypt

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Abstract: It is believed that the E-Bayesian estimation method will be able to solve the problems caused by the process of choosing the values of hyper-parameters for prior distributions in Bayesian estimates, which has attracted the attention of many authors. In this paper, these approaches are used based on the latest method of censored sample called unified progressive hybrid censored sample from the Rayleigh distribution are discussed. The Bayesian and E-Bayesian estimators are produced using four different loss functions. E-Bayesian estimation also made use of three more hyper-parameter distributions. Finally, in order to assess the applicability of the suggested model and different estimating techniques, the proposed model has been applied to real data.

Keywords: E-Bayesian estimation; Bayesian estimation; Maximum likelihood estimation; unified hybrid progressive censoring; Rayleigh distribution.

1 Introduction

In statistical inference Bayesian approaches are based on the loss function and the a prior distribution which is proposed as appropriate for the likelihood function, but the hyperparameter values may influence the a prior distribution and so we often use the hierarchical Bayesian estimation method in this case. The concept of hierarchical prior distribution was first proposed by Lindley and Smith [1]. The hierarchical Bayesian technique requires two steps to complete the prior distribution setting, making it more resilient than the Bayesian method. The approach for constructing hierarchical prior distribution was developed by Han [2]. Hierarchical Bayesian approaches have recently been applied to data analysis; see Ando and Zellner [3], Osei et al. [4], Han [5] and Fernandez [6] for more information.

Rayleigh distribution was first proposed by Rayleigh [7], and various researchers have employed it in a variety of disciplines of wisdom and technology since then. The Rayleigh distribution is widely used in statistical models, survival analysis, and reliability at the moment. The cornerstone of significant of the treatment of meteorological radar signals statistics is the Rayleigh distribution. It's also commonly used in actuarial science and engineering to simulate population lifetimes with a linearly increasing failure rate.

The Rayleigh distribution's probability density function (pdf) and cumulative distribution function (cdf) are described by

$$g(y|\alpha) = 2\alpha y \exp\left[-\alpha y^2\right], \alpha, y > 0, \tag{1}$$

$$G(y|\alpha) = 1 - \exp\left[-\alpha y^2\right], \alpha, y > 0. \tag{2}$$

Inferences for α in the Rayleigh distribution have been discussed by several authors. Harter and Moore [8] derived an explicit form for the maximum likelihood estimator (MLE) of α based on type II censored data. Moreover, Bayesian estimation and prediction problems for the α based on doubly censored sample have been considered by Raqab and Madi

* Corresponding author e-mail: mn112@fayoum.edu.eg and mnaji@ksu.edu.sa

[9] and Fernandez [10]. Wu et al. [11] have derived the Bayesian estimator and prediction intervals based on progressively type II censored samples. A recent account on progressive censoring schemes can be obtained in the monograph by Balakrishnan [12] or in the excellent reviewarticle by Balakrishnan et al. [13].

Recent Górny and Cramer in [14] are suggested a general type of generalized progressive hybrid Censored scheme (PHCS), called unified PHCS (UPHCS) to address some of the shortcomings of all PHCS. The UPHCS may be described as follows. Consider n of identical items are put on life time test. Let $T_1, T_2 \in (0,\infty)$ are times and integer k, m are pre-fixed integer numbers such that $T_1 < T_2$ and $k < m$ with $R = (R_1, R_2, ..., R_m)$ is also pre-fixed integers satisfying $n = m + R_1 + \ldots + R_m$. At the time of first failure, R_1 of the remaining units are randomly removed. Similarly, at the time of the second failure R_2, of the remaining units are removed and so on. If the k^{th} failure occurs before time T_1, the experiment is terminated at $\min\{\max(Y_{m:m:n}, T_1), T_2\}$. If the k^{th} failure occurs between T_1 and T_2, the experiment is terminated at $\min(Y_{rm:m:n}, T_2)$ and if the k^{th} failure occurs after time T_2, the experiment is terminated at $Y_{k:n}$. Under this censoring scheme, we can guarantee that the experiment would be completed at most in time T_2 with at least k failure and if not, we can guarantee exactly k failures. For recent work on the unified PHCS, see, for example, Kim and Lee in [15], Górny and Cramer in [16] and Nagy and Alrashide in [17].

The primary goal of this study is to present a statistical comparison of Bayesian and E-Bayesian (EB) approaches for estimating the parameter of the Rayleigh distribution under the UPHC data. The resulting estimators are obtained based on four different loss functions. by using the square error loss function (SELF), the Degroot loss function (DLF), the quadratic loss function (QLF) and the Linex loss function (LLF). This article is organized as follows: Section 2 provides an overview of likelihood function depending on the UPHCS. In Section 3, determines the maximum likelihood estimates (ML) of Rayleigh distribution's parameter, In Section 4, Bayesian estimation of unknown parameter under different prior distributions and different loss functions are computed. In Section 5, the formulas of EB are discussed. Comparison between Bayes and EB estimates have been made using simulation study in Section 6. A real data set is analyzed in Section 7. Finally, the paper is concluded in Section 8.

2 unified PHCS and Likelihood Function

Let $\underline{\mathbf{Y}}$ be the unified progressive hybrid censored sample from distribution with the probability density function (PDF) $g(y)$, and the cumulative distribution function (CDF) $G(y)$, then, based on the unified PHCS, the likelihood function is given by

$$L(\alpha|\underline{\mathbf{Y}}) = \left[\prod_{i=1}^{D^*}\sum_{j=i}^{m}\left(\widetilde{R}_j+1\right)\right]\prod_{i=1}^{D^*} g\left(y_{i:m:n}\right)\left[\overline{G}(y_{i:m:n})\right]^{\widetilde{R}_i}\left[\overline{G}(T_1)\right]^{\widetilde{R}_{t_1}}\left[\overline{G}(T_2)\right]^{\widetilde{R}_{t_2}}, \tag{3}$$

where $\overline{G} = 1 - G$ and

$$\underline{\mathbf{Y}} = \begin{cases} (y_{1:m:n}, \ldots, \min\{\max(Y_{m:m:n}, T_1), T_2\}) & \text{if } y_{k:m:n} < T_1, \\ (y_{1:m:n}, \ldots, \min\{\max(y_{k:m:n}, T_2), Y_{m:m:n}\}) & \text{if } T_1 < y_{k:m:n}, \end{cases}$$

$$D^* = \begin{cases} d_1 & y_{m:m:n} < T_1, \\ m & T_1 < y_{m:m:n} < T_2, \\ d_2 & y_{k:m:n} < T_2 < y_{m:m:n}, \\ k & T_2 < y_{k:m:n}, \end{cases}$$

and $\widetilde{R}$, $\widetilde{R}_{k^*} = n - k - \sum_{j=1}^{k-1}\widetilde{R}_j$, $\widetilde{R}_{t_1}$ the number of units which removed at T_1, and $\widetilde{R}_{t_2}$ the number of units which removed at T_2. For more details, see Nagy and Alrashide in [17].

3 The ML Estimation

In this section, we derive the ML inference of the unknown parameter α for Rayleigh distribution based on the UPHCS. From (1), (2), and (3), the likelihood function of α under the UPHCS can be derived as

$$L(\alpha|\underline{\mathbf{Y}}) = \left[\prod_{i=1}^{D^*}\sum_{j=i}^{m}\left(\widetilde{R}_j+1\right)\right](2\alpha)^{D^*}\left(\prod_{i=1}^{D^*} y_i\right)\exp\left\{-\alpha\left[\psi\left(\underline{\mathbf{y}}, T_1, T_2\right)\right]\right\}, \tag{4}$$

where $\psi\left(\underline{\mathbf{y}},T_1,T_2\right)=\sum_{i=1}^{D^*}\left(\widetilde{R}_i+1\right)y_i^2+\widetilde{R}_{t_1}\ln T_1+\widetilde{R}_{t_2}\ln T_2$, and $y_i=y_{i:D^*:n}$ for simplicity of notation. From (4), The log-likelihood function of α is given by

$$\ln\left[L\left(\alpha|\underline{\mathbf{Y}}\right)\right]\propto D^*\ln\left(\alpha\right)-\alpha\left[\psi\left(\underline{\mathbf{y}},T_1,T_2\right)\right]. \tag{5}$$

By differentiate (5) with respect to α and solve the equation

$$\frac{\partial\ln\left[L\left(\alpha|\underline{\mathbf{Y}}\right)\right]}{\partial\alpha}=0,$$

so the ML estimator $\widehat{\alpha}_{ML}$ of α is obtained as

$$\widehat{\alpha}_{ML}=\frac{D^*}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)}. \tag{6}$$

4 Bayesian and EB Estimation

In this Section, the Bayesian and EB etimators for the parameter α using SELF, DLF, QLF and LLF are derived. For creating the Bayesian estimation, we suppose that the parameter α has gamma prior distribution with scale parameter b and shape parameter a can be expressed as follows

$$P\left(\alpha\right)=\frac{b^a}{\Gamma\left(a\right)}\alpha^{a-1}\exp\left(-b\alpha\right),b,a>0. \tag{7}$$

The posterior PDF of α is given from (4), (7), as follows.

$$P^*\left(\alpha|\underline{\mathbf{x}}\right)=\frac{\left[\psi\left(\underline{\mathbf{y}},T_1,T_2\right)+b\right]^{\left(D^*+a\right)}}{\Gamma\left(D^*+a\right)}\alpha^{D^*+a-1}\exp\left\{-\alpha\left[\psi\left(\underline{\mathbf{y}},T_1,T_2\right)\right]\right\}, \tag{8}$$

Here, Three different prior distributions of hyper-parameters are investigated in this section to see how they affect the EB estimates of α. We select the hyper-parameters a and b to prove that $P\left(\alpha\right)$ is a decreasing function of α. The first derivative of $P\left(\alpha\right)$ regarding α is as follows:

$$\frac{\partial P(\alpha)}{\partial\alpha}\propto\alpha e^{-b\alpha}\left[(\alpha-1)-b\alpha\right]. \tag{9}$$

Thus, for $0<a<1$ and $b>0$, the prior PDF $P(\alpha)$ is a decreasing function of α. Suppose that a and b, are independent with bivariate PDF given by

$$p(a,b)=p(a)p(b), \tag{10}$$

the EB estimates of the parameter α are expectation of the Bayesian estimate of α can be obtained as follows:

$$\widehat{\alpha}_{EB}=E\left[\widehat{\alpha}_B|\underline{\mathbf{x}}\right]=\int_A\widehat{\alpha}_B\left(a,b\right)p\left(a,b\right)dadb, \tag{11}$$

According to three various prior PDF of the hyper-parameters a and b, the EB estimates of the parameter α can be derived. As a result, prior distributions chosen to show how different prior distributions affect the estimation of the EB of α. We suggest the following prior PDFs

$$\begin{aligned} p_1\left(a,b\right)&=\tfrac{1}{c}, && 0<a<1,0<b<c,\\ p_2\left(a,b\right)&=\tfrac{2b}{c^2}, && 0<a<1,0<b<c,\\ p_3\left(a,b\right)&=\tfrac{2(c-b)}{c^2}, && 0<a<1,0<b<c, \end{aligned} \tag{12}$$

4.1 Bayesian and EB under SELF

Based on the SELF, the Bayes estimators of α is given by,

$$\begin{aligned}\widehat{\alpha}_{BS} &= E_{P^*}\left[\alpha\right] = \int_0^\infty \alpha\pi^*\left(\alpha|\underline{\mathbf{x}}\right)d\alpha\\ &= \frac{D^*+a}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)+b}.\end{aligned} \tag{13}$$

The EB estimate of α under the SEL based on $p_1\left(a,b\right)$, $p_2\left(a,b\right)$, and $p_3\left(a,b\right)$ are computed from (11), (12) and (13), respectively, as follows:

$$\begin{aligned}\widehat{\alpha}_{EBS1} &= \int_0^1\int_0^c \frac{1}{c}\left[\frac{D^*+a}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)+b}\right]dbda\\ &= \frac{2D^*+1}{2c}\ln\left(1+\frac{c}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)}\right),\end{aligned} \tag{14}$$

$$\begin{aligned}\widehat{\alpha}_{EBS2} &= \int_0^1\int_0^c \frac{2b}{c}\left[\frac{D^*+a}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)+b}\right]dbda\\ &= \frac{2D^*+1}{c}\left[1-\frac{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)}{c}\ln\left(1+\frac{c}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)}\right)\right],\end{aligned} \tag{15}$$

and

$$\begin{aligned}\widehat{\alpha}_{EBS3} &= \int_0^1\int_0^c \frac{2\left(c-b\right)}{c}\left[\frac{D^*+a}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)+b}\right]dbda\\ &= \frac{2D^*+1}{c}\left[\left(1+\frac{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)}{c}\right)\ln\left(1+\frac{c}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)}\right)-1\right].\end{aligned} \tag{16}$$

4.2 Bayesian and EB under DLF

The Bayesian estimate of α under DLF loss function is given by

$$\begin{aligned}\widehat{\alpha}_{BD} &= \frac{E_{P^*}\left[\alpha^2\right]}{E_{P^*}\left[\alpha\right]} = \frac{\int_0^\infty \alpha^2 P^*\left(\alpha|\underline{\mathbf{x}}\right)d\alpha}{\int_0^\infty \alpha\pi^*\left(\alpha|\underline{\mathbf{x}}\right)d\alpha}\\ &= \frac{D^*+a+1}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)+b}.\end{aligned} \tag{17}$$

Based on $p_1\left(a,b\right)$, $p_2\left(a,b\right)$, and $p_3\left(a,b\right)$, under the DLF, the EB estimate of α, can be derived from (11), (12) and (17), respectively as follows:

$$\begin{aligned}\widehat{\alpha}_{EBD1} &= \int_0^1\int_0^c \frac{1}{c}\left[\frac{D^*+a+1}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)+b}\right]dbda\\ &= \frac{2D^*+3}{2c}\ln\left(1+\frac{c}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)}\right),\end{aligned} \tag{18}$$

$$\begin{aligned}\widehat{\alpha}_{EBD2} &= \int_0^1 \int_0^c \frac{2b}{c}\left[\frac{D^*+a+1}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)+b}\right] dbda \\ &= \frac{2D^*+3}{c}\left[1-\frac{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)}{c}\ln\left(1+\frac{c}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)}\right)\right], \end{aligned} \tag{19}$$

and

$$\begin{aligned}\widehat{\alpha}_{EBD3} &= \int_0^1 \int_0^c \frac{2(c-b)}{c}\left[\frac{D^*+a+1}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)+b}\right] dbda \\ &= \frac{2D^*+3}{c}\left[\left(1+\frac{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)}{c}\right)\ln\left(1+\frac{c}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)}\right)-1\right]. \end{aligned} \tag{20}$$

4.3 Bayesian and EB under QLF

The Bayesian estimate of α under QLF is given by

$$\begin{aligned}\widehat{\alpha}_{BQ} &= \frac{E_{P^*}\left[\alpha^{-1}\right]}{E_{P^*}\left[\alpha^{-2}\right]} = \frac{\int_0^\infty \alpha^{-1}P^*\left(\alpha|\underline{\mathbf{x}}\right)d\alpha}{\int_0^\infty \alpha^{-2}P^*\left(\alpha|\underline{\mathbf{x}}\right)d\alpha} \\ &= \frac{D^*+a-2}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)+b}. \end{aligned} \tag{21}$$

The EB estimate of α under the QLF based on $p_1(a,b)$, $p_2(a,b)$, and $p_3(a,b)$ are computed from (11), (12) and (21), respectively, as follows:

$$\begin{aligned}\widehat{\alpha}_{EBQ1} &= \int_0^1 \int_0^c \frac{1}{c}\left[\frac{D^*+a-2}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)+b}\right] dbda \\ &= \frac{2D^*-3}{2c}\ln\left(1+\frac{c}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)}\right), \end{aligned} \tag{22}$$

$$\begin{aligned}\widehat{\alpha}_{EBQ2} &= \int_0^1 \int_0^c \frac{2b}{c}\left[\frac{D^*+a-2}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)+b}\right] dbda \\ &= \frac{2D^*-3}{c}\left[1-\frac{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)}{c}\ln\left(1+\frac{c}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)}\right)\right], \end{aligned} \tag{23}$$

and

$$\begin{aligned}\widehat{\alpha}_{EBQ3} &= \int_0^1 \int_0^c \frac{2(c-b)}{c}\left[\frac{D^*+a-2}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)+b}\right] dbda \\ &= \frac{2D^*-3}{c}\left[\left(1+\frac{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)}{c}\right)\ln\left(1+\frac{c}{\psi\left(\underline{\mathbf{y}},T_1,T_2\right)}\right)-1\right]. \end{aligned} \tag{24}$$

4.4 Bayesian and EB under LLF

The Bayesian estimate of α under LLF is given by

$$
\begin{aligned}
\widehat{\alpha}_{BL} &= \frac{-1}{\upsilon}\ln\left\{E_{P^*}\left[\exp(-\upsilon\alpha)\right]\right\} = \frac{-1}{\upsilon}\ln\left\{\int_0^\infty \exp(-\upsilon\alpha)P^*(\alpha|\underline{\mathbf{x}})\,d\alpha\right\} \\
&= \frac{-1}{\upsilon}\ln\left\{\left[\frac{\psi(\underline{\mathbf{y}},T_1,T_2)+b}{\psi(\underline{\mathbf{y}},T_1,T_2)+b+\upsilon}\right]^{(a+D^*)}\right\}. \qquad (25)
\end{aligned}
$$

Also, based on $p_1(a,b)$, $p_2(a,b)$, and $p_3(a,b)$, under the LINEX loss function, the EB estimate of α, can be derived from (11), (12) and (25), respectively as follows:

$$
\begin{aligned}
\widehat{\alpha}_{EBL1} &= \frac{-1}{\upsilon}\int_0^1\int_0^c \frac{1}{c}\ln\left\{\left[\frac{\psi(\underline{\mathbf{y}},T_1,T_2)+b}{\psi(\underline{\mathbf{y}},T_1,T_2)+b+\upsilon}\right]^{(a+D^*)}\right\}dbda \\
&= \frac{2D^*+1}{2\upsilon c}\left\{\left[c+\psi(\underline{\mathbf{y}},T_1,T_2)\right]\ln\left(1+\frac{\upsilon}{\psi(\underline{\mathbf{y}},T_1,T_2)+c}\right)+\upsilon\right. \\
&\quad \left.\times\ln\left(1+\frac{c}{\psi(\underline{\mathbf{y}},T_1,T_2)+\upsilon}\right)-\psi(\underline{\mathbf{y}},T_1,T_2)\ln\left(1+\frac{\upsilon}{\psi(\underline{\mathbf{y}},T_1,T_2)}\right)\right\}, \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\alpha}_{EBL2} &= \frac{-1}{\upsilon}\int_0^1\int_0^c \frac{2b}{c}\left\{\left[\frac{\psi(\underline{\mathbf{y}},T_1,T_2)+b}{\psi(\underline{\mathbf{y}},T_1,T_2)+b+\upsilon}\right]^{(a+D^*)}\right\}dbda \\
&= \frac{2D^*+1}{2\upsilon c^2}\left\{\upsilon c_i-\upsilon\left(2\psi(\underline{\mathbf{y}},T_1,T_2)+\upsilon\right)\ln\left(1+\frac{c}{\psi(\underline{\mathbf{y}},T_1,T_2)+\upsilon}\right)+\left[\psi(\underline{\mathbf{y}},T_1,T_2)\right]^2\right. \\
&\quad \left.\times\ln\left(1+\frac{\upsilon}{\psi(\underline{\mathbf{y}},T_1,T_2)}\right)-\left(c-\left[\psi(\underline{\mathbf{y}},T_1,T_2)\right]^2\right)\ln\left(1+\frac{\upsilon}{\psi(\underline{\mathbf{y}},T_1,T_2)+c}\right)\right\} \qquad (27)
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{\alpha}_{EBL3} &= \frac{-1}{\upsilon}\int_0^1\int_0^c \frac{2(c-b)}{c}\ln\left\{\left[\frac{\psi(\underline{\mathbf{y}},T_1,T_2)+b}{\psi(\underline{\mathbf{y}},T_1,T_2)+b+\upsilon}\right]^{(a+D^*)}\right\}dbda \\
&\quad \frac{2D^*+1}{2c^2\upsilon}\left\{-c\upsilon+\left(\psi(\underline{\mathbf{y}},T_1,T_2)+2c\right)\ln\left(1+\frac{\upsilon}{\psi(\underline{\mathbf{y}},T_1,T_2)}\right)\right. \\
&\quad +\upsilon\left(2\left(\psi(\underline{\mathbf{y}},T_1,T_2)+c\right)+\upsilon\right)\ln\left(1+\frac{c}{\psi(\underline{\mathbf{y}},T_1,T_2)+\upsilon}\right) \\
&\quad \left.+\left(c+\left[\psi(\underline{\mathbf{y}},T_1,T_2)\right]^2\right)\ln\left(1+\frac{\upsilon}{\psi(\underline{\mathbf{y}},T_1,T_2)+c}\right)\right\} \qquad (28)
\end{aligned}
$$

The properties of E-Bayesian estimates under the proposed priors of the hyperparameters a and b have been discussed by many authors, see for example Nagy et al. in [18]

5 Simulation Study

In this section, simulation research compared the effectiveness of the Bayesian and E-Bayesian estimate algorithms under different unified PHCS. To generate unified PHC data from Rayleigh distribution, we use the algorithms which used by Nagy and Alrashide in [17].

We simulate unified PHCS by different selections of (n,m,k,T_1,T_2) and $R=(R_1,\ldots,R_m)$, where, $n=40$ and $k=\frac{1}{2}m$, $m=10,20,30$ with different values of $T_1=\frac{1}{2}T_2$, $T_2=0.5,1.0$ and 2. We consider the true values of the Rayleigh parameter $\alpha=2$. We computed the ML estimate, Bayesian and E-Bayesian estimates of α, under SELF, DLF, QLF, and LLF (with $\varepsilon=0.5$). The values of (a,b) are chosen to be $(\alpha b, \frac{\alpha}{0.005})$ for informative prior (IP), $(0,0)$ for noninformative prior (NIP) and the E-Bayesian estimates are computed with $c=0.75$. To determine the efficiency of the estimation, the mean square error (MSE) and the estimated bias (EB) are calculated for each estimate. We take the different censoring schemes as follows:

1. Scheme 1: $R_{\frac{m}{2}}=R_m=\frac{n-m}{2}$, and $R_i=0$ for all $i\neq k,m$.
2. Scheme 2: $R_1=R_k=\frac{n-m}{2}-2, R_m=2$, and $R_i=0$ for all $i\neq 1,k$, and m.

The values MSE and EB for ML, Bayesian and E-Bayesian estimates under SELF have been reported in Table 1 and Table 2, also, the values MSE and EB for Bayesian and E-Bayesian estimates under DLF have been reported in Table 3 and Table 4, while Tables 5 and Table 6 are present the values MSE and EB for Bayesian and E-Bayesian estimates under QLF. Finally, Tables 7 and Table 8 are present the values MSE and EB for Bayesian and E-Bayesian estimates under LLF, respectively.

Table 1: The values of MSE of ML and Bayesian with E-Bayesian estimates for α under SELF based on the different unified UHCSs.

				$\widehat{\alpha}_B$				
				$\widehat{\alpha}_{BS}$				
(n,m)	Sch.	T_2	$\widehat{\alpha}_{ML}$	IP	NIP	$\widehat{\alpha}_{EBS1}$	$\widehat{\alpha}_{EBS2}$	$\widehat{\alpha}_{EBS3}$
(40,10)	1	1	0.1319	0.0527	0.0580	0.0448	0.0475	0.0464
	2		0.1412	0.0565	0.0621	0.0480	0.0508	0.0497
(40,20,)	1		0.1096	0.0438	0.0482	0.0373	0.0394	0.0386
	2		0.1096	0.0439	0.0482	0.0373	0.0395	0.0386
(40,30)	1		0.1005	0.0402	0.0442	0.0342	0.0362	0.0354
	2		0.0998	0.0399	0.0439	0.0339	0.0359	0.0351
(40,10)	1	2	0.1191	0.0476	0.0524	0.0405	0.0429	0.0419
	2		0.0974	0.0390	0.0429	0.0331	0.0351	0.0343
(40,20,)	1		0.0751	0.0301	0.0331	0.0255	0.0271	0.0264
	2		0.0836	0.0334	0.0368	0.0284	0.0301	0.0294
(40,30)	1		0.0836	0.0334	0.0368	0.0284	0.0301	0.0294
	2		0.0938	0.0375	0.0413	0.0319	0.0338	0.0330
(40,10)	1	3	0.1893	0.0757	0.0833	0.0644	0.0682	0.0666
	2		0.1745	0.0698	0.0768	0.0593	0.0628	0.0614
(40,20,)	1		0.0604	0.0242	0.0266	0.0205	0.0217	0.0213
	2		0.0549	0.0220	0.0242	0.0187	0.0198	0.0193
(40,30)	1		0.0897	0.0359	0.0395	0.0305	0.0323	0.0316
	2		0.0885	0.0354	0.0389	0.0301	0.0319	0.0312

Table 2: The values of EB of ML and Bayesian with E-Bayesian estimates for α under SELF based on the different unified UHCSs.

				$\widehat{\alpha}_B$				
				$\widehat{\alpha}_{BS}$				
(n,m)	$Sch.$	T_2	$\widehat{\alpha}_{ML}$	IP	NIP	$\widehat{\alpha}_{EBS1}$	$\widehat{\alpha}_{EBS2}$	$\widehat{\alpha}_{EBS3}$
(40,10)	1	1	0.0551	0.0220	0.0242	0.0187	0.0198	0.0194
	2		0.0589	0.0236	0.0259	0.0200	0.0212	0.0207
(40,20,)	1		0.0335	0.0134	0.0147	0.0114	0.0120	0.0118
	2		0.0321	0.0128	0.0141	0.0109	0.0116	0.0113
(40,30)	1		0.0257	0.0103	0.0113	0.0088	0.0093	0.0091
	2		0.0263	0.0105	0.0116	0.0090	0.0095	0.0093
(40,10)	1	2	0.1151	0.0460	0.0506	0.0391	0.0414	0.0405
	2		0.0930	0.0372	0.0409	0.0316	0.0335	0.0327
(40,20,)	1		0.0160	0.0064	0.0070	0.0054	0.0057	0.0056
	2		0.0240	0.0096	0.0106	0.0082	0.0086	0.0084
(40,30)	1		0.0272	0.0109	0.0120	0.0092	0.0098	0.0096
	2		0.0272	0.0109	0.0120	0.0092	0.0098	0.0096
(40,10)	1	3	0.1883	0.0753	0.0828	0.0640	0.0678	0.0663
	2		0.1736	0.0694	0.0764	0.0590	0.0625	0.0611
(40,20,)	1		0.0369	0.0147	0.0162	0.0125	0.0133	0.0130
	2		0.0235	0.0094	0.0103	0.0080	0.0085	0.0083
(40,30)	1		0.0260	0.0104	0.0114	0.0088	0.0094	0.0091
	2		0.0255	0.0102	0.0112	0.0087	0.0092	0.0090

Table 3: The values of MSE of ML and Bayesian with E-Bayesian estimates for α under DLF based on the different unified UHCSs.

			$\widehat{\alpha}_B$				
			$\widehat{\alpha}_{BD}$				
(n,m)	$Sch.$	T_2	IP	NIP	$\widehat{\alpha}_{EBD1}$	$\widehat{\alpha}_{EBD2}$	$\widehat{\alpha}_{EBD3}$
(40,10)	1	1	0.0580	0.0638	0.0493	0.0522	0.0511
	2		0.0621	0.0683	0.0528	0.0559	0.0547
(40,20,)	1		0.0482	0.0530	0.0410	0.0434	0.0424
	2		0.0482	0.0531	0.0410	0.0434	0.0425
(40,30)	1		0.0442	0.0486	0.0376	0.0398	0.0389
	2		0.0439	0.0483	0.0373	0.0395	0.0386
(40,10)	1	2	0.0524	0.0577	0.0446	0.0472	0.0461
	2		0.0429	0.0472	0.0364	0.0386	0.0377
(40,20,)	1		0.0331	0.0364	0.0281	0.0298	0.0291
	2		0.0368	0.0405	0.0313	0.0331	0.0324
(40,30)	1		0.0368	0.0405	0.0313	0.0331	0.0324
	2		0.0413	0.0454	0.0351	0.0372	0.0363
(40,10)	1	3	0.0833	0.0916	0.0708	0.0750	0.0733
	2		0.0768	0.0845	0.0653	0.0691	0.0676
(40,20,)	1		0.0266	0.0292	0.0226	0.0239	0.0234
	2		0.0242	0.0266	0.0205	0.0217	0.0213
(40,30)	1		0.0395	0.0434	0.0335	0.0355	0.0347
	2		0.0389	0.0428	0.0331	0.0350	0.0343

Table 4: The values of EB of ML and Bayesian with E-Bayesian estimates for α under DLF based on the different unified UHCSs.

			$\widehat{\alpha}_B$				
			$\widehat{\alpha}_{BD}$				
(n,m)	$Sch.$	T_2	IP	NIP	$\widehat{\alpha}_{EBD1}$	$\widehat{\alpha}_{EBD2}$	$\widehat{\alpha}_{EBD3}$
(40,10)	1	1	0.0242	0.0267	0.0206	0.0218	0.0213
	2		0.0259	0.0285	0.0220	0.0233	0.0228
(40,20,)	1		0.0147	0.0162	0.0125	0.0133	0.0130
	2		0.0141	0.0155	0.0120	0.0127	0.0124
(40,30)	1		0.0113	0.0125	0.0096	0.0102	0.0100
	2		0.0116	0.0127	0.0098	0.0104	0.0102
(40,10)	1	2	0.0506	0.0557	0.0430	0.0456	0.0445
	2		0.0409	0.0450	0.0348	0.0368	0.0360
(40,20,)	1		0.0070	0.0077	0.0060	0.0063	0.0062
	2		0.0106	0.0116	0.0090	0.0095	0.0093
(40,30)	1		0.0120	0.0132	0.0102	0.0108	0.0105
	2		0.0120	0.0132	0.0102	0.0108	0.0105
(40,10)	1	3	0.0828	0.0911	0.0704	0.0746	0.0729
	2		0.0764	0.0840	0.0649	0.0687	0.0672
(40,20,)	1		0.0162	0.0178	0.0138	0.0146	0.0143
	2		0.0103	0.0114	0.0088	0.0093	0.0091
(40,30)	1		0.0114	0.0126	0.0097	0.0103	0.0101
	2		0.0112	0.0123	0.0095	0.0101	0.0099

Table 5: The values of MSE of ML and Bayesian with E-Bayesian estimates for α under QLF based on the different unified UHCSs.

			$\widehat{\alpha}_B$				
			$\widehat{\alpha}_{BQ}$				
(n,m)	$Sch.$	T_2	IP	NIP	$\widehat{\alpha}_{EBQ1}$	$\widehat{\alpha}_{EBQ2}$	$\widehat{\alpha}_{EBQ3}$
(40,10)	1	1	0.0591	0.0650	0.0502	0.0532	0.0520
	2		0.0632	0.0696	0.0538	0.0569	0.0557
(40,20,)	1		0.0491	0.0540	0.0417	0.0442	0.0432
	2		0.0491	0.0540	0.0417	0.0442	0.0432
(40,30)	1		0.0450	0.0495	0.0383	0.0405	0.0396
	2		0.0447	0.0492	0.0380	0.0402	0.0393
(40,10)	1	2	0.0534	0.0587	0.0454	0.0480	0.0470
	2		0.0436	0.0480	0.0371	0.0393	0.0384
(40,20,)	1		0.0337	0.0370	0.0286	0.0303	0.0296
	2		0.0375	0.0412	0.0318	0.0337	0.0330
(40,30)	1		0.0375	0.0412	0.0318	0.0337	0.0330
	2		0.0420	0.0462	0.0357	0.0378	0.0370
(40,10)	1	3	0.0848	0.0933	0.0721	0.0763	0.0746
	2		0.0782	0.0860	0.0664	0.0704	0.0688
(40,20,)	1		0.0271	0.0298	0.0230	0.0244	0.0238
	2		0.0246	0.0271	0.0209	0.0221	0.0216
(40,30)	1		0.0402	0.0442	0.0341	0.0362	0.0354
	2		0.0397	0.0436	0.0337	0.0357	0.0349

Table 6: The values of *EB* of ML and Bayesian with E-Bayesian estimates for α under QLF based on the different unified UHCSs.

			$\widehat{\alpha}_B$				
			$\widehat{\alpha}_{BQ}$				
(n,m)	*Sch.*	T_2	*IP*	*NIP*	$\widehat{\alpha}_{EBQ1}$	$\widehat{\alpha}_{EBQ2}$	$\widehat{\alpha}_{EBQ3}$
(40,10)	1	1	0.0247	0.0272	0.0210	0.0222	0.0217
	2		0.0264	0.0290	0.0224	0.0237	0.0232
(40,20,)	1		0.0150	0.0165	0.0127	0.0135	0.0132
	2		0.0144	0.0158	0.0122	0.0129	0.0127
(40,30)	1		0.0115	0.0127	0.0098	0.0104	0.0102
	2		0.0118	0.0130	0.0100	0.0106	0.0104
(40,10)	1	2	0.0515	0.0567	0.0438	0.0464	0.0454
	2		0.0417	0.0458	0.0354	0.0375	0.0367
(40,20,)	1		0.0072	0.0079	0.0061	0.0064	0.0063
	2		0.0107	0.0118	0.0091	0.0097	0.0095
(40,30)	1		0.0122	0.0134	0.0104	0.0110	0.0107
	2		0.0122	0.0134	0.0104	0.0110	0.0107
(40,10)	1	3	0.0843	0.0928	0.0717	0.0759	0.0742
	2		0.0778	0.0855	0.0661	0.0700	0.0684
(40,20,)	1		0.0165	0.0182	0.0140	0.0149	0.0145
	2		0.0105	0.0116	0.0089	0.0095	0.0093
(40,30)	1		0.0116	0.0128	0.0099	0.0105	0.0102
	2		0.0114	0.0126	0.0097	0.0103	0.0100

Table 7: The values of *MSE* of ML and Bayesian with E-Bayesian estimates for α under LLF based on the different unified UHCSs.

			$\widehat{\alpha}_B$				
			$\widehat{\alpha}_{BL}$				
(n,m)	*Sch.*	T_2	*IP*	*NIP*	$\widehat{\alpha}_{EBL1}$	$\widehat{\alpha}_{EBL2}$	$\widehat{\alpha}_{EBL3}$
(40,10)	1	1	0.0570	0.0627	0.0484	0.0513	0.0501
	2		0.0610	0.0671	0.0518	0.0549	0.0537
(40,20,)	1		0.0473	0.0521	0.0402	0.0426	0.0417
	2		0.0474	0.0521	0.0403	0.0426	0.0417
(40,30)	1		0.0434	0.0477	0.0369	0.0391	0.0382
	2		0.0431	0.0474	0.0366	0.0388	0.0379
(40,10)	1	2	0.0515	0.0566	0.0437	0.0463	0.0453
	2		0.0421	0.0463	0.0358	0.0379	0.0370
(40,20,)	1		0.0325	0.0357	0.0276	0.0292	0.0286
	2		0.0361	0.0397	0.0307	0.0325	0.0318
(40,30)	1		0.0361	0.0397	0.0307	0.0325	0.0318
	2		0.0405	0.0446	0.0345	0.0365	0.0357
(40,10)	1	3	0.0818	0.0900	0.0695	0.0736	0.0720
	2		0.0754	0.0829	0.0641	0.0678	0.0663
(40,20,)	1		0.0261	0.0287	0.0222	0.0235	0.0230
	2		0.0237	0.0261	0.0202	0.0213	0.0209
(40,30)	1		0.0387	0.0426	0.0329	0.0349	0.0341
	2		0.0382	0.0421	0.0325	0.0344	0.0336

Table 8: The values of EB of ML and Bayesian with E-Bayesian estimates for α under LLF based on the different unified UHCSs.

			$\widehat{\alpha}_B$				
			$\widehat{\alpha}_{BL}$				
(n,m)	$Sch.$	T_2	IP	NIP	$\widehat{\alpha}_{EBL1}$	$\widehat{\alpha}_{EBL2}$	$\widehat{\alpha}_{EBL3}$
(40,10)	1	1	0.0238	0.0262	0.0202	0.0214	0.0210
	2		0.0254	0.0280	0.0216	0.0229	0.0224
(40,20,)	1		0.0145	0.0159	0.0123	0.0130	0.0127
	2		0.0139	0.0153	0.0118	0.0125	0.0122
(40,30)	1		0.0111	0.0122	0.0095	0.0100	0.0098
	2		0.0114	0.0125	0.0097	0.0102	0.0100
(40,10)	1	2	0.0497	0.0547	0.0422	0.0447	0.0437
	2		0.0402	0.0442	0.0341	0.0361	0.0353
(40,20,)	1		0.0069	0.0076	0.0059	0.0062	0.0061
	2		0.0104	0.0114	0.0088	0.0093	0.0091
(40,30)	1		0.0118	0.0129	0.0100	0.0106	0.0103
	2		0.0117	0.0129	0.0100	0.0106	0.0103
(40,10)	1	3	0.0813	0.0895	0.0691	0.0732	0.0716
	2		0.0750	0.0825	0.0637	0.0675	0.0660
(40,20,)	1		0.0159	0.0175	0.0135	0.0143	0.0140
	2		0.0101	0.0112	0.0086	0.0091	0.0089
(40,30)	1		0.0112	0.0124	0.0095	0.0101	0.0099
	2		0.0110	0.0121	0.0094	0.0099	0.0097

6 Real Data Application

In this Section, we showcase an actual example of real data to show how the offered options perform in the application. We give a real data example. These data, is used by Bhaumik et al. [19] to represent vinyl chloride data from clean upgradient monitoring wells. This set of data has 34 observations, as 5.1 , 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8, 0.8, 0.4, 0.6 , 0.1, 1.8, 0.9, 2, 4 , 0.9, 0.4, 2, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1, 0.2 , 6.8, 1.2, 0.4, 0.2, and 0.1. We will use these data to generate the unified PHCS, suppose $m = 20$, $k = 10$, $R_i = 3$ for $i = 5,10,15$, $R_m = 5$, and $R_i = 0$ otherwise, with different values of $T_2 = 2T_1$ with $T_2 = 3,5$ and 8.

Based on the unified PHCS and two different choices IP and NIP priors, the ML, Bayesian E-Bayesian estimates for the unknown parameter α based on different unified PHCSs., are presented in Table 9.

7 Conclusions

In this Paper a statistical comparison of Bayesian and E-Bayesian approaches for estimating the parameter of the Rayleigh distribution is present under the UPHC data. From the resulting estimators which obtained based on four different loss functions, from the results in the above tables, we can note the following:

1.The values of mean squared error of all Bayesian and E-Bayesian estimatition are less than it in ML case.
2.The values of MSE of all E-Bayesian estimatition are less than it in Bayesian case.
3.The values of MSE of all E-Bayesian estimatition are less than it in Bayesian case.
4.In most cases, the values of MSE of E-Bayesian estimatition under DLF are less than it in all E-Bayesian cases.
5.The values of MSE of all estimatition with IP are less than it in NIP case.

Table 9: The values of MSE of Bayesian and E-Bayesian estimates for α under LLF based on the different unified UHCSs.

Method	$T_2 = 3$	$T_2 = 5$	$T_2 = 7$
$\widehat{\alpha}_{ML}$	0.3756	0.2835	0.1433
$\widehat{\alpha}_{BS}$ with IP	0.3778	0.2857	0.1455
$\widehat{\alpha}_{BS}$ with NIP	0.3752	0.2826	0.1433
$\widehat{\alpha}_{EBS1}$	0.3757	0.2831	0.1437
$\widehat{\alpha}_{EBS2}$	0.3791	0.2865	0.1472
$\widehat{\alpha}_{EBS3}$	0.3686	0.2760	0.1366
$\widehat{\alpha}_{BD}$ with IP	0.3982	0.2996	0.1517
$\widehat{\alpha}_{BD}$ with NIP	0.3961	0.2968	0.1495
$\widehat{\alpha}_{EBD1}$	0.3960	0.2967	0.1494
$\widehat{\alpha}_{EBD2}$	0.3968	0.2975	0.1502
$\widehat{\alpha}_{EBD3}$	0.3897	0.2904	0.1431
$\widehat{\alpha}_{BQ}$ with IP	0.3369	0.2578	0.1331
$\widehat{\alpha}_{BQ}$ with NIP	0.3335	0.2544	0.1308
$\widehat{\alpha}_{EBQ1}$	0.3321	0.2530	0.1294
$\widehat{\alpha}_{EBQ2}$	0.3267	0.2476	0.1240
$\widehat{\alpha}_{EBQ3}$	0.3237	0.2446	0.1210
$\widehat{\alpha}_{BL}$ with IP	0.3759	0.2847	0.1452
$\widehat{\alpha}_{BL}$ with NIP	0.3733	0.2817	0.1430
$\widehat{\alpha}_{EBL1}$	0.3717	0.2801	0.1415
$\widehat{\alpha}_{EBL2}$	0.3661	0.2745	0.1359
$\widehat{\alpha}_{EBL3}$	0.3676	0.2759	0.1373

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