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Factorization of Rotation and Lorentz Matrices via Euler– Olinde Rodrigues Parameters

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Abstract: In this paper, the factorization of 3-rotation matrices in terms of the Euler-Olinde Rodrigues parameters from the generation of Lorentz transformations through quaternions is obtained.

Keywords: Dirac equation, Euler-Olinde Rodrigues parameters, Lorentz transformations, quaternions, 3-rotations

1 Introduction

The Lorentz matrix $L = (L^{\nu}{}_{\mu})$ between the frames of reference $(x^{\mu}) = (ct, x, y, z)$ and $(\tilde{x}^{\nu})$ has six degrees of freedom:

$$\tilde{x}^{\nu} = L^{\nu}{}_{\mu}\, x^{\mu}, \tag{1}$$

and it can be generated using quaternions [1,2,3,4] via the expression [5,6,7,8,9,10]:

$$\tilde{\mathbf{X}} = \mathbf{A}\mathbf{X}\bar{\mathbf{A}}^{*}, \qquad \mathbf{X} = \frac{1}{\sqrt{2}}\left(ct - ix\mathbf{I} - iy\mathbf{J} - iz\mathbf{K}\right),$$

$$\mathbf{A} = a_0 + a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K},$$

$$\bar{\mathbf{A}}^{*} = a_0^{*} - a_1^{*}\mathbf{I} - a_2^{*}\mathbf{J} - a_3^{*}\mathbf{K}, \quad a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1. \tag{2}$$

The Lorentz matrix $(L^{\mu}{}_{\nu})$ accepts an interesting factorization [11] in terms of the Cayley-Klein parameters [12]. Matrix factorization is a mathematical procedure used in various situations and contexts; for example, in relation to the Helmholtz operator in mathematical physics and in the theory of differential equations [13,14,15]. Here we are interested in the factorization of an arbitrary 3-rotation matrix [16,17,18] using the Euler-Olinde Rodrigues parameters [19,20,21,22] a_{μ}, $\mu = 0, \ldots, 3$. In fact, if all a_{ν} are real, then (2) implies $\tilde{t} = t$ and:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad R\,R^{T} = I, \tag{3}$$

where the matrix R represents a passive rotation (in the terminology of Ryder [23]:

$$R = \begin{pmatrix} 1-2(a_2^2+a_3^2) & 2(a_1a_2 - a_0a_3) & 2(a_1a_3 + a_0a_2) \\ 2(a_1a_2 + a_0a_3) & 1-2(a_1^2+a_3^2) & 2(a_2a_3 - a_0a_1) \\ 2(a_1a_3 - a_0a_2) & 2(a_0a_1 + a_2a_3) & 1-2(a_1^2+a_2^2) \end{pmatrix}. \tag{4}$$

2 3-Rotations

With any quaternion we can associate a 4×4 matrix, for example [24]:

$$\mathbf{X} \quad \longleftrightarrow \quad \frac{1}{\sqrt{2}}\begin{pmatrix} ct & -ix & -iy & -iz \\ ix & ct & iz & -iy \\ iy & -iz & ct & ix \\ iz & iy & -ix & ct \end{pmatrix},$$

$$\mathbf{A} \quad \longleftrightarrow \quad \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ -a_1 & a_0 & -a_3 & a_2 \\ -a_2 & a_3 & a_0 & -a_1 \\ -a_3 & -a_2 & a_1 & a_0 \end{pmatrix}, \tag{5}$$

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then (2), in the form $\bar{\mathbf{A}}\tilde{\mathbf{X}} = \mathbf{X}\bar{\mathbf{A}}$, and (5) imply:

$$\begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{pmatrix} \begin{pmatrix} ct & -i\tilde{x} & -i\tilde{y} & -i\tilde{z} \\ i\tilde{x} & ct & i\tilde{z} & -i\tilde{y} \\ i\tilde{y} & -i\tilde{z} & ct & i\tilde{x} \\ i\tilde{z} & i\tilde{y} & -i\tilde{x} & ct \end{pmatrix} =$$

$$= \begin{pmatrix} ct & -ix & -iy & -iz \\ ix & ct & iz & -iy \\ iy & -iz & ct & ix \\ iz & iy & -ix & ct \end{pmatrix} \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{pmatrix}, \quad (6)$$

that is:

$$a_1\tilde{x} + a_2\tilde{y} + a_3\tilde{z} = a_1x + a_2y + a_3z, \quad (7)$$

$$a_0\tilde{x} + a_3\tilde{y} - a_2\tilde{z} = a_0x - a_3y + a_2z, \quad (8)$$

$$a_3\tilde{x} - a_0\tilde{y} - a_1\tilde{z} = -a_3x - a_0y + a_1z, \quad (9)$$

$$a_2\tilde{x} - a_1\tilde{y} + a_0\tilde{z} = -a_2x + a_1y + a_0z, \quad (10)$$

If $a_0 \neq 0$, then (8), (9), (10) imply (7), thus the solution of the linear system (8, 9, 10) has the structure (3) such that:

$$R = \frac{1}{a_0} \begin{pmatrix} a_0a_2 + a_1a_3 & a_0^2 + a_1^2 & a_0a_3 - a_1a_2 \\ -a_0a_1 + a_2a_3 & a_0a_3 + a_1a_2 & -a_0^2 - a_2^2 \\ a_0^2 + a_3^2 & -a_0a_2 + a_1a_3 & -a_0a_1 - a_2a_3 \end{pmatrix}$$

$$\times \begin{pmatrix} -a_2 & a_1 & a_0 \\ a_0 & -a_3 & a_2 \\ -a_3 & -a_0 & a_1 \end{pmatrix}, \quad (11)$$

which is a factorization of the matrix of rotation (4) in terms of the Euler-Olinde Rodrigues parameters.

Similarly, if $a_1 \neq 0$, then (7), (9), (10) generate to (8), therefore:

$$R = \frac{1}{a_1} \begin{pmatrix} a_0^2 + a_1^2 & -a_0a_3 + a_1a_2 & a_0a_2 + a_1a_3 \\ a_0a_3 + a_1a_2 & -a_1^2 - a_3^2 & -a_0a_1 + a_2a_3 \\ -a_0a_2 + a_1a_3 & a_0a_1 + a_2a_3 & -a_1^2 - a_2^2 \end{pmatrix}$$

$$\times \begin{pmatrix} a_1 & a_2 & a_3 \\ -a_2 & a_1 & a_0 \\ -a_3 & -a_0 & a_1 \end{pmatrix}, \quad (12)$$

for the case $a_2 \neq 0$ the relations (7), (8), (10) imply (9), thus:

$$R = \frac{1}{a_2} \begin{pmatrix} -a_0a_3 + a_1a_2 & a_2^2 + a_3^2 & a_0a_2 + a_1a_3 \\ a_0^2 + a_2^2 & -a_0a_3 - a_1a_2 & -a_0a_1 + a_2a_3 \\ a_0a_1 + a_2a_3 & a_0a_2 - a_1a_3 & -a_1^2 - a_2^2 \end{pmatrix}$$

$$\times \begin{pmatrix} a_1 & a_2 & a_3 \\ -a_2 & a_1 & a_0 \\ a_0 & -a_3 & a_2 \end{pmatrix}, \quad (13)$$

and finally, if $a_3 \neq 0$ the expressions (7), (8), (9) generate to (10), hence:

$$R = \frac{1}{a_3} \begin{pmatrix} a_0a_2 + a_1a_3 & a_0a_3 - a_1a_2 & a_2^2 + a_3^2 \\ -a_0a_1 + a_2a_3 & a_1^2 + a_3^2 & -a_0a_3 - a_1a_2 \\ a_0^2 + a_3^2 & -a_0a_1 - a_2a_3 & a_0a_2 - a_1a_3 \end{pmatrix} \times$$

$$\times \begin{pmatrix} a_1 & a_2 & a_3 \\ a_0 & -a_3 & a_2 \\ -a_3 & -a_0 & a_1 \end{pmatrix}, \quad (14)$$

The relations (11), (12), (13) and (14) are factorizations of the matrix of rotation (4), and their possible geometrical meaning, is an open problem.

3 Lorentz transformation

The quaternionic relation (2) gives the following components for an arbitrary Lorentz matrix:

$$L^0_{\ 0} = a_0^*a_0 + a_1^*a_1 + a_2^*a_2 + a_3^*a_3,\ L^0_{\ 1} = i\left(a_0^*a_1 + a_3^*a_2\right) + cc,$$

$$L^0_{\ 2} = i\left(a_0^*a_2 + a_1^*a_3\right) + cc,\ L^0_{\ 3} = i\left(a_0^*a_3 + a_2^*a_1\right) + cc,$$

$$L^1_{\ 0} = i\left(a_0^*a_1 + a_2^*a_3\right) + cc,\ L^1_{\ 1} = a_0^*a_0 + a_1^*a_1 - a_2^*a_2 - a_3^*a_3,$$

$$L^1_{\ 2} = -a_0^*a_3 + a_1^*a_2 + cc,\ L^1_{\ 3} = a_0^*a_2 + a_1^*a_3 + cc,$$

$$L^2_{\ 0} = i\left(a_0^*a_2 + a_3^*a_1\right) + cc,\ L^2_{\ 1} = a_1^*a_2 + a_3^*a_2 + cc,$$

$$L^2_{\ 2} = a_0^*a_0 - a_1^*a_1 + a_2^*a_2 - a_3^*a_3,\ L^2_{\ 3} = -a_0^*a_1 + a_2^*a_3 + cc,$$

$$L^3_{\ 0} = i\left(a_0^*a_3 + a_1^*a_2\right) + cc,\ L^3_{\ 1} = -a_0^*a_2 + a_1^*a_3 + cc,$$

$$L^3_{\ 2} = a_0^*a_1 + a_2^*a_3 + cc,\ L^3_{\ 3} = a_0^*a_0 - a_1^*a_1 - a_2^*a_2 + a_3^*a_3, \quad (15)$$

where cc means the complex conjugate of all the previous terms; therefore, any Lorentz transformations accepts the splitting:

$$(L^\mu_{\ \nu}) = \begin{pmatrix} -ia_0^* & -ia_1^* & -ia_2^* & -ia_3^* \\ -a_1^* & a_0^* & -a_3^* & a_2^* \\ -a_2^* & a_3^* & a_0^* & -a_1^* \\ -a_3^* & -a_2^* & a_1^* & a_0^* \end{pmatrix} \begin{pmatrix} ia_0 & -a_1 & -a_2 & -a_3 \\ ia_1 & a_0 & -a_3 & a_2 \\ ia_2 & a_3 & a_0 & -a_1 \\ ia_3 & -a_2 & a_1 & a_0 \end{pmatrix}, \quad (16)$$

in terms of the Euler−Olinde Rodrigues parameters. For example, for a boost with arbitrary direction [25]:

$$a_0 = \cosh\left(\frac{\phi}{2}\right), \quad a_1 = i\frac{v_x}{v}\sinh\left(\frac{\phi}{2}\right),$$

$$a_2 = i\frac{v_y}{v}\sinh\left(\frac{\phi}{2}\right), \quad a_3 = i\frac{v_z}{v}\sinh\left(\frac{\phi}{2}\right), \tag{17}$$

where ϕ is determined by the relative speed between the frames of reference:

$$\cosh\phi = \tilde{\gamma} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \sinh\phi = \frac{v}{c}\tilde{\gamma}, \quad v^2 = v_x^2+v_y^2+v_z^2,$$

$$\mathbf{p} = m\mathbf{v} = m_0\tilde{\gamma}\,\mathbf{v}, \quad |\mathbf{p}| = p = mv,$$

$$E^2 = p^2c^2 + m_0^2c^4, \quad \cosh\left(\frac{\phi}{2}\right) = \sqrt{\frac{E+m_0c^2}{2\,m_0\,c^2}},$$

$$\sinh\left(\frac{\phi}{2}\right) = \sqrt{\frac{E-m_0c^2}{2\,m_0\,c^2}}, \quad \tanh\phi = \frac{v}{c},$$

$$\tanh\left(\frac{\phi}{2}\right) = \frac{\sinh\phi}{1+\cosh\phi} = \frac{pc}{E+m_0c^2},$$

$$2\sinh^2\left(\frac{\phi}{2}\right) = \cosh\phi - 1 = \tilde{\gamma} - 1, \tag{18}$$

then the parameters (17) generate a proper Lorentz transformation future preserving $\tilde{x}^\mu = L^\mu_{\ \lambda}x^\lambda$ via the expressions (15). Thus, we obtain the following Lorentz symmetric matrix representing a boost in an arbitrary direction [25,26]:

$$(L^\mu_{\ \lambda}) = $$

$$\begin{pmatrix} \tilde{\gamma} & -\frac{v_x}{c}\tilde{\gamma} & -\frac{v_y}{c}\tilde{\gamma} & -\frac{v_z}{c}\tilde{\gamma} \\ -\frac{v_x}{c}\tilde{\gamma} & 1+\frac{v_x^2}{v^2}(\tilde{\gamma}-1) & \frac{v_xv_y}{v^2}(\tilde{\gamma}-1) & \frac{v_xv_z}{v^2}(\tilde{\gamma}-1) \\ -\frac{v_y}{c}\tilde{\gamma} & \frac{v_xv_y}{v^2}(\tilde{\gamma}-1) & 1+\frac{v_y^2}{v^2}(\tilde{\gamma}-1) & \frac{v_yv_z}{v^2}(\tilde{\gamma}-1) \\ -\frac{v_z}{c}\tilde{\gamma} & \frac{v_xv_z}{v^2}(\tilde{\gamma}-1) & \frac{v_yv_z}{v^2}(\tilde{\gamma}-1) & 1+\frac{v_z^2}{v^2}(\tilde{\gamma}-1) \end{pmatrix}, \tag{19}$$

that is [26]:

$$\tilde{t} = \tilde{\gamma}\left(t - \frac{1}{c^2}\mathbf{v}\cdot\mathbf{x}\right), \quad \tilde{\mathbf{x}} = \mathbf{x} + \left(\frac{\tilde{\gamma}-1}{v^2}\mathbf{v}\cdot\mathbf{x} - \tilde{\gamma}t\right)\mathbf{v}. \tag{20}$$

Then the factorization (16) applied to (19) implies the splitting:

$$(L^\mu_{\ \nu}) = \frac{\tilde{\gamma}-1}{2v^2}\begin{pmatrix} -iQv & -v_x & -v_y & -v_z \\ iv_x & Qv & iv_z & -iv_y \\ iv_y & -iv_z & Qv & iv_x \\ iv_z & iv_y & -iv_x & Qv \end{pmatrix}$$

$$\times \begin{pmatrix} iQv & -iv_x & -iv_y & -iv_z \\ -v_x & Qv & -iv_z & iv_y \\ -v_y & iv_z & Qv & -iv_x \\ -v_z & -iv_y & iv_x & Qv \end{pmatrix} \tag{21}$$

such that:

$$Q = \coth\left(\frac{\phi}{2}\right) = \sqrt{\frac{E+m_0c^2}{E-m_0c^2}} = \frac{E+m_0c^2}{pc}$$

$$= \frac{v\tilde{\gamma}}{c(\tilde{\gamma}-1)} = \frac{c(\tilde{\gamma}+1)}{v\tilde{\gamma}},$$

$$\frac{\tilde{\gamma}-1}{2v^2} = \frac{E+m_0c^2}{2m_0c^2v^2}. \tag{22}$$

4 Dirac equation with zero mass

The quaternionic form of Dirac equation for spin 1/2 without the mass term is given by [27,28,29,30]:

$$\nabla\mathbf{D} = \mathbf{0}\,, \qquad \nabla = \frac{i}{c}\frac{\partial}{\partial t} + \mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\,, \tag{23}$$

$$\mathbf{D} = i(\eta^1+\xi^2) + (\eta^2+\xi^1)\mathbf{I} + i(\eta^2-\xi^1)\mathbf{J} + (\eta^1-\xi^2)\mathbf{K}\,, \tag{24}$$

where:

$$\xi^1 = \psi_2 - \psi_4, \qquad \xi^2 = \psi_3 - \psi_1,$$

$$\eta^1 = \psi_1^* + \psi_3^*, \qquad \eta^2 = \psi_2^* + \psi_4^*, \tag{25}$$

in terms of the components of Dirac spinor, in the standard representation of the gamma matrices [23,26,31,32]. Thus (23-24), under the association (5), is equivalent to the matrix relation:

$$\begin{pmatrix} \frac{1}{c}\partial_t & \partial_x & \partial_y & \partial_z \\ -\partial_x & \frac{1}{c}\partial_t & -\partial_z & \partial_y \\ -\partial_y & \partial_z & \frac{1}{c}\partial_t & -\partial_x \\ -\partial_z & -\partial_y & \partial_x & \frac{1}{c}\partial_t \end{pmatrix} \times$$

$$\begin{pmatrix} i(\eta^1+\xi^2) & (\eta^2+\xi^1) & i(\eta^2-\xi^1) & (\eta^1-\xi^2) \\ -(\eta^2+\xi^1) & i(\eta^1+\xi^2) & -(\eta^1-\xi^2) & i(\eta^2-\xi^1) \\ i(-\eta^2+\xi^1) & (\eta^1-\xi^2) & i(\eta^1+\xi^2) & -(\eta^2+\xi^1) \\ (-\eta^1+\xi^2) & i(-\eta^2+\xi^1) & (\eta^2+\xi^1) & i(\eta^1+\xi^2) \end{pmatrix}$$

$$= 0, \tag{26}$$

which implies four independent equations:

$$\eta^1_{\ ,1\dot{1}} + \eta^2_{\ ,2\dot{1}} + \xi^1_{\ ,1\dot{2}} + \xi^2_{\ ,2\dot{2}} = 0, \ \eta^1_{\ ,1\dot{1}} + \eta^2_{\ ,2\dot{1}} - \xi^1_{\ ,1\dot{2}} - \xi^2_{\ ,2\dot{2}} = 0,$$

$$\eta^1_{\ ,1\dot{2}} + \eta^2_{\ ,2\dot{2}} + \xi^1_{\ ,1\dot{1}} + \xi^2_{\ ,2\dot{1}} = 0, \ \eta^1_{\ ,1\dot{2}} + \eta^2_{\ ,2\dot{2}} - \xi^1_{\ ,1\dot{1}} - \xi^2_{\ ,2\dot{1}} = 0, \tag{27}$$

such that:

$$\partial_{1\dot{1}} = \frac{1}{c}\partial_t + \partial_z, \quad \partial_{1\dot{2}} = \partial_x - i\partial_y,$$

$$\partial_{2\dot{1}} = \partial_x + i\partial_y, \quad \partial_{2\dot{2}} = \frac{1}{c}\partial_t - \partial_z, \tag{28}$$

then from (27) are immediate the Weyl spinor equations [23,33,34]:

$$\partial_{A\dot{B}}\,\eta^{\dot{B}} = 0, \qquad \partial_{B\dot{A}}\,\xi^B = 0. \tag{29}$$

5 Conclusion

Lorentz transformations can arise through quaternions; which here has given rise to the factorization of an arbitrary 3-rotation matrix, as well as to the factorization of Lorentz matrices, in terms of the Euler-Olinde Rodrigues parameters. The quaternionic form of the Dirac equation, without the mass term, together with the mathematical fact that a 4×4 matrix can be associated with any quaternion, led to a set of independent relations from which the Weyl spinor equations arose.

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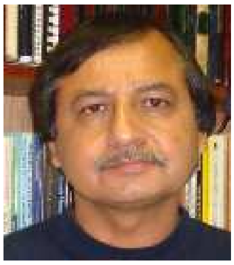

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