Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/160511

Exponentiated Flexible Lomax Distribution: Theory and Applications

M. I. Khan

Department of Mathematics, Faculty of Science, Islamic University of Madinah, Saudi Arabia

Received: 1 May 2022, Revised: 12 Jul. 2022, Accepted: 20 Jul. 2022
Published online: 1 Sep. 2022

Abstract: A continuous probability distribution called, exponentiated flexible Lomax distribution is proposed. The proposed probability distribution extends the flexible Lomax distribution. Its basic statistical properties are derived. Technique of maximum likelihood is implemented to estimate the parameters. In the end, real and simulated data are presented and analyzed using this distribution along with some existing distributions for illustrative purposes.

Keywords: Lomax distribution. Exponentiated flexible Lomax distribution. Maximum likelihood estimation.

1 Introduction

The concept of exponentiated distribution is to form a more flexible distribution. To obtain the exponentiated distribution, add an extra parameter to the existing distribution. This approach was coined by [1]. It is extensively applied in different disciplines to analyze data such as medical sciences, engineering, economics, biology, and finance. Several exponentiated distributions were introduced by many authors some of them are notable for example see [1]-[11], among others.

A random variable (rv) X is said to have flexible Lomax distribution if its cumulative density function (cdf) and probability density function (pdf) are given respectively as

$$F(x) = 1 - \left[\left(\frac{x}{\beta}\right)^{\gamma+1}\right]^{-\alpha}, \quad x > 0, \tag{1}$$

$$f(x) = \frac{\alpha\gamma}{\beta}\left[\left(\frac{x}{\beta}\right)^{\gamma+1}\right]^{-(1+\alpha)}\left(\frac{x}{\beta}\right)^{\gamma-1}, \quad x > 0, \tag{2}$$

where $\beta, \alpha, \gamma > 0$ are scale and shape parameters respectively.

The Lomax distribution plays an indispensable role in modeling the lifetime data sets such as business, computer science, biological sciences, and medical, engineering, income, economics, and reliability modeling and wealth inequality due to its heavily skewed property [12].

The present article is summarized as follows. In Section 2, an exponentiated flexible Lomax distribution is introduced. In Section 3, we demonstrate some statistical properties. The maximum-likelihood estimation is performed in Section 4. An empirical application, as well as a simulation study, are presented and discussed in Section 5. Finally, Section 6 ends with concluding remarks.

2 Exponentiated Flexible Lomax Distribution

The cdf of exponentiated flexible Lomax distribution (EFLD) can be obtained by using the following formula

$$G(x) = [F(x)]^{\theta}, \theta > 0. \tag{3}$$

Substituting from (1) into (3), we have the cdf of the EFLD as follows

$$G(x) = \left[1 - \left(\left(\frac{x}{\beta}\right)^{\gamma} + 1\right)^{-\alpha}\right]^{\theta}, x > 0, \tag{4}$$

where $\alpha, \beta, \gamma, \theta > 0$.

* Corresponding author e-mail: izhar.stats@gmail.com

The corresponding pdf is

$$g(x) = \frac{\alpha\gamma\theta}{\beta}\left(\frac{x}{\beta}\right)^{\gamma-1}\left(\left(\frac{x}{\beta}\right)^{\gamma}+1\right)^{-(1+\alpha)} \times \left[1-\left(\left(\frac{x}{\beta}\right)^{\gamma}+1\right)^{-\alpha}\right]^{\theta-1}. \quad (5)$$

The survival function (sf) of EFLD is

$$\bar{G}(x) = 1-\left[1-\left(\left(\frac{x}{\beta}\right)^{\gamma}+1\right)^{-\alpha}\right]^{\theta}. \quad (6)$$

The hazard rate function (hrf) is

$$h(x) = g(x)\left[\bar{G}(x)\right]^{-1} = \frac{\alpha\gamma\theta}{\beta}\frac{\left(\frac{x}{\beta}\right)^{\gamma-1}\left(\left(\frac{x}{\beta}\right)^{\gamma}+1\right)^{-(1+\alpha)}}{1-\left[1-\left(\left(\frac{x}{\beta}\right)^{\gamma}+1\right)^{-\alpha}\right]^{\theta}} \times \left[1-\left(\left(\frac{x}{\beta}\right)^{\gamma}+1\right)^{-\alpha}\right]^{\theta-1}. \quad (7)$$

The Figures 1 – 2, display the pdf, and hrf for specific values of α, β,γ and θ.

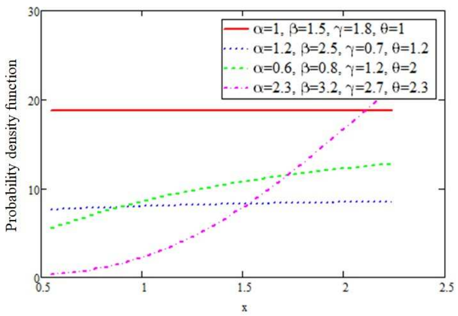

Fig. 1: The pdf's of various EFL distribution.

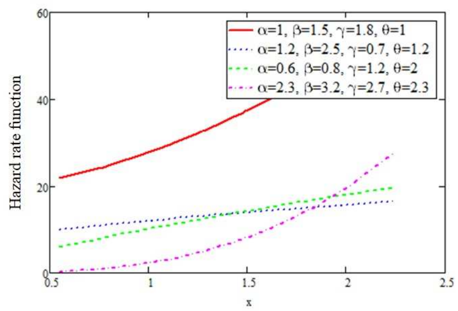

Fig. 2: The hrf of EFL distribution.

From Figures 1 and 2, the EFLD doesn't have a mode (nonmodal), and it has increasing failure rate distribution.

2.1 Sub models of EFL distribution

The exponentiated flexible Lomax distribution is a very adaptable model that generates several distributions. If X is a r.v. with cdf (4) or pdf (5), then we have following sub models.

(i)Setting $\gamma = 1$ and $\beta = \frac{1}{\lambda}$, we obtain exponentiated Lomax distribution, [13] with cdf:

$$G(x) = \left[1-(1+\lambda x)^{-\alpha}\right]^{\theta}.$$

(ii)Setting $\beta = \gamma = 1$, we obtain exponentiated Pareto distribution, [1], with cdf:

$$G(x) = \left[1-(1+x)^{-\alpha}\right]^{\theta}.$$

(iii)Setting $\theta = 1$, we can obtain the flexible Lomax distribution, [12], with cdf:

$$G(x) = 1-\left[1+\left(\frac{x}{\beta}\right)^{\gamma}\right]^{-\alpha}.$$

(iv)Setting $\beta = 1$, we obtain the generalized inverted Kumaraswamy distribution, [14] with cdf:

$$G(x) = \left[1-(1+x^{\gamma})^{-\alpha}\right]^{\theta}.$$

3 Some Statistical Measures

Some statistical measures for the EFL distribution are derived. The moment generating function (mgf) is discussed in the Theorem 1.

Theorem 1. If X is a r.v. having pdf (5), the mgf is

$$M_X(t) = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\theta-1}(-1)^k\binom{\theta-1}{k}\theta\alpha\beta^{\ell} \times B\left(\frac{\ell}{\gamma}+1, \alpha(k+1)-\frac{\ell}{\gamma}\right)\frac{t^{\ell}}{\ell!}, \quad (8)$$

where B is the Beta function defined as

$$B(m,n) = \int_0^{\infty}\frac{u^{n-1}}{(1+u)^{m+n}}du.$$

Proof The mgf is given by

$$M_X(t) = E[e^{tX}] = \int_0^{\infty} e^{tx}g(x)dx \quad (9)$$

From Eq. (5) into Eq. (9), we have

$$M_X(t) = \int_0^\infty e^{tx}\frac{\alpha\gamma\theta}{\beta}\left(\frac{x}{\beta}\right)^{\gamma-1}\left[1+\left(\frac{x}{\beta}\right)^{\gamma}\right]^{-(\alpha+1)} \times \left\{1-\left[1+\left(\frac{x}{\beta}\right)^{\gamma}\right]^{-\alpha}\right\}^{\theta-1} dx$$

$$= \sum_{k=0}^{\theta-1}(-1)^k\binom{\theta-1}{k}\frac{\alpha\gamma\theta}{\beta}\times \int_0^\infty e^{tx}\left(\frac{x}{\beta}\right)^{\gamma-1}\left[1+\left(\frac{x}{\beta}\right)^{\gamma}\right]^{-\alpha)k+1)-1} dx$$

$$= \sum_{k=0}^{\theta-1}(-1)^k\binom{\theta-1}{k}\frac{\alpha\gamma\theta}{\beta}\times \int_0^\infty \sum_{\ell=0}^{\infty}\frac{(tx)^\ell}{\ell!}\left(\frac{x}{\beta}\right)^{\gamma-1}\left[1+\left(\frac{x}{\beta}\right)^{\gamma}\right]^{-\alpha)k+1)-1} dx$$

$$= \frac{\alpha\gamma\theta}{\beta}\sum_{k=0}^{\theta-1}\sum_{\ell=0}^{\infty}(-1)^k\binom{\theta-1}{k}\frac{(t\beta)^\ell}{\ell!}\times \int_0^\infty \left(\frac{x}{\beta}\right)^{\gamma+\ell-1}\left[1+\left(\frac{x}{\beta}\right)^{\gamma}\right]^{-\alpha(k+1)-1} dx.$$

Let $(x/\beta)^\gamma = u$, then $x = \beta u^{1/\gamma}$ and $dx = \frac{\beta}{\gamma}u^{\frac{1}{\gamma}-1}du$, therefore

$$M_X(t) = \alpha\theta\sum_{k=0}^{\theta-1}\sum_{\ell=0}^{\infty}(-1)^k\binom{\theta-1}{k}\frac{(t\beta)^\ell}{\ell!}\times \int_0^\infty u^{\frac{\gamma+\ell-1}{\gamma}}(1+u)^{-\alpha(k+1)-1}u^{\frac{1}{\gamma}-1}du$$

$$= \alpha\theta\sum_{k=0}^{\theta-1}\sum_{\ell=0}^{\infty}(-1)^k\binom{\theta-1}{k}\frac{(t\beta)^\ell}{\ell!}\int_0^\infty \frac{u^{\frac{\ell}{\gamma}}}{(1+u)^{\alpha(k+1)+1}}du.$$

Using Beta function in above integral, we have

$$M_X(t) = \alpha\theta\sum_{k=0}^{\theta-1}\sum_{\ell=0}^{\infty}\left[(-1)^k\binom{\theta-1}{k}\frac{(t\beta)^\ell}{\ell!}\times B\left(\frac{\ell}{\gamma}+1,\alpha(k+1)-\frac{\ell}{\gamma}\right)\right]$$

$$= \alpha\theta\sum_{\ell=0}^{\infty}\sum_{k=0}^{\theta-1}\left[(-1)^k\binom{\theta-1}{k}\beta^\ell\times B\left(\frac{\ell}{\gamma}+1,\alpha(k+1)-\frac{\ell}{\gamma}\right)\frac{t^\ell}{\ell!}\right].$$

This completes the proof.

Corollary 1. The rth moment of EFLD about origin is,

$$\mu_r' = \alpha\theta\sum_{k=0}^{\theta-1}(-1)^k\binom{\theta-1}{k}\beta^r B\left(\frac{r}{\gamma}+1,\alpha(k+1)-\frac{r}{\gamma}\right). \tag{10}$$

Proof The rth moments can be obtained by (10) as the coefficient of $\frac{t^r}{r!}$ in (8).

The mean and variance of EFLD can be obtained as follows

$$\mu = E(X) = \mu_1' = \sum_{k=0}^{\theta-1}(-1)^k\binom{\theta-1}{k}\alpha\theta\beta B\left(\frac{1}{\gamma}+1,\alpha(k+1)-\frac{1}{\gamma}\right).$$

The variance

$$Var(X) = \mu_2' - \mu_1'^2 = \sum_{k=0}^{\theta-1}(-1)^k\binom{\theta-1}{k}\alpha\theta\beta^2\times B\left(\frac{2}{\gamma}+1,\alpha(k+1)-\frac{2}{\gamma}\right) - \left[\sum_{k=0}^{\theta-1}(-1)^k \binom{\theta-1}{k}\alpha\theta\beta B\left(\frac{1}{\gamma}+1,\alpha(k+1)-\frac{1}{\gamma}\right)\right]^2.$$

The skewness and kurtosis can also be obtained as

$$S_k = \frac{\mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3}{(\mu_2' - \mu_1'^2)^{3/2}}, \tag{11}$$

$$K_u = \frac{\mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_2' - 3\mu_1'^4}{(\mu_2' - \mu_1'^2)^2}. \tag{12}$$

Substituting from (10) into (11) and (12), the skewness and kurtosis can be determined.

The Table 1 shows some numeric outcomes for mean, variance, skewness, and kurtosis.

Table 1: At $\alpha = 3, \beta = 2$ and $\gamma = 2$.

θ	E(X)	Var(X)	S_k	K_u
1	1.178	0.612	3.125	12.463
2	1.583	0.694	9.961	13.547
3	1.832	0.744	17.953	14.430
4	2.013	0.784	25.889	15.096
5	2.156	0.818	33.473	15.617
6	2.275	0.848	40.642	16.039
7	2.377	0.875	47.405	16.391

We have the following notes from Table 1:

1. The mean, variance, skewness, and kurtosis are increasing as θ increases.
2. The EFLD is positive skewness.
3. The EFLD is Leptokurtic or heavy-tailed, $K_u > 3$.

The pth percentile can be obtained by using the following formula

$$G(x_p) = p. \tag{13}$$

Substituting from (4) into (13), we get

$$\left\{1-\left[1+\left(\frac{x}{\beta}\right)^{\gamma}\right]^{-\alpha}\right\}^{\theta} = p.$$

Then

$$x_p = \beta\left[\left(1-p^{\frac{1}{\theta}}\right)^{-\frac{1}{\alpha}}-1\right]^{\frac{1}{\gamma}}. \quad (14)$$

The median can be obtained from (14), when $p = 0.5$.
The mode of EFLD can be derived as follows. Take natural log to $g(x)$ in (5), we have

$$\ln g(x) = \ln\left(\frac{\alpha\gamma\theta}{\beta}\right)+(\gamma-1)\ln\left(\frac{x}{\beta}\right)-(\alpha+1)\times$$
$$\ln\left[1+\left(\frac{x}{\beta}\right)^{\gamma}\right]+(\theta-1)\ln\left\{1-\left[1+\left(\frac{x}{\beta}\right)^{\gamma}\right]^{-\alpha}\right\}.$$

Now equating $\frac{d\ln g(x)}{dx} = 0$, as follows

$$\frac{\gamma-1}{x}-\frac{(\alpha+1)\frac{\gamma}{\beta}\left(\frac{x}{\beta}\right)^{\gamma-1}}{1+\left(\frac{x}{\beta}\right)^{\gamma}}+$$
$$\frac{(\theta-1)\alpha\left[1+\left(\frac{x}{\beta}\right)^{\gamma}\right]^{-\alpha-1}\frac{\gamma}{\beta}\left(\frac{x}{\beta}\right)^{\gamma-1}}{1-\left[1+\left(\frac{x}{\beta}\right)^{\gamma}\right]^{-\alpha}} = 0$$

or

$$\frac{\gamma-1}{x}-\frac{\frac{\gamma}{\beta}\left(\frac{x}{\beta}\right)^{\gamma-1}}{1+\left(\frac{x}{\beta}\right)^{\gamma}}\left\{(\alpha+1)-\frac{(\theta-1)\alpha}{\left[1+\left(\frac{x}{\beta}\right)^{\gamma}\right]^{\alpha}-1}\right\} = 0 \quad (15)$$

This equation can't solve theoretically, so we can use some numerical program to find the mode such as Mathematica and Mathcad.

3.1 Order statistics

The pdf of rth order statistic is given as

$$g_{r:n}(x) = \frac{1}{B(r,n-r+1)}[G(x)]^{r-1}[1-G(x)]^{n-r}g(x). \quad (16)$$

If the population has EFL distribution, then pdf of rth order statistic is

$$g_{r,n}(x) = \frac{\alpha\gamma\theta}{\beta B(r,n-r+1)}\left(\frac{x}{\beta}\right)^{\gamma-1}\times$$
$$\left[1+\left(\frac{x}{\beta}\right)^{\gamma}\right]^{-(\alpha+1)}\left\{1-\left[1+\left(\frac{x}{\beta}\right)^{\gamma}\right]^{-\alpha}\right\}^{\theta r-1}\times$$
$$\left(1-\left\{1-\left[1+\left(\frac{x}{\beta}\right)^{\gamma}\right]^{-\alpha}\right\}^{\theta}\right)^{n-r}. \quad (17)$$

If $r = 1$, the pdf for minimum order statistic can be obtained. If $r = n$, the pdf for maximum order statistic can also be obtained.

4 Maximum Likelihood Estimation

Estimation of exponentiated flexible Lomax distribution's parameters is derived in this section.

4.1 Likelihood function

The likelihood function of the EFL distribution based on a sample $(x_1, x_2, \cdots, x_n)$ size n is given by

$$L(\alpha,\beta,\gamma,\theta|x) = \prod_{i=1}^{n} g(x_i;\alpha,\beta,\gamma,\theta)$$
$$= \prod_{i=1}^{n}\frac{\alpha\gamma\theta}{\beta}\left(\frac{x_i}{\beta}\right)^{\gamma-1}\left[1+\left(\frac{x_i}{\beta}\right)^{\gamma}\right]^{-(\alpha+1)}\times$$
$$\left\{1-\left[1+\left(\frac{x_i}{\beta}\right)^{\gamma}\right]^{-\alpha}\right\}^{\theta-1}.$$

The log-likelihood function is

$$\mathscr{L} = n\ln(\alpha\gamma\theta)-n\ln(\beta)+(\gamma-1)\sum_{i=1}^{n}\ln\left(\frac{x_i}{\beta}\right)$$
$$-(\alpha+1)\sum_{i=1}^{n}\left[1+\left(\frac{x}{\beta}\right)^{\gamma}\right]+(\theta-1)\times$$
$$\sum_{i=1}^{n}\ln\left\{1-\left[1+\left(\frac{x}{\beta}\right)^{\gamma}\right]^{-\alpha}\right\}. \quad (18)$$

Differentiating (18) w.r.t. α, β , γ and θ), we have the following

$$\frac{\partial\mathscr{L}}{\partial\alpha} = \frac{n}{\alpha}-\sum_{i=1}^{n}\ln(u_i)+(\theta-1)\sum_{i=1}^{n}\ln(u_i)(u_i^{\alpha}-1)^{-1}, \quad (19)$$

$$\frac{\partial\mathscr{L}}{\partial\beta} = -\frac{n\gamma}{\beta}+\frac{(1+\alpha)\gamma}{\beta}\sum_{i=1}^{n}\xi(x_i,\beta,\gamma)$$
$$-\frac{(\theta-1)}{\beta}\sum_{i=1}^{n}\alpha\gamma\psi(x_i,\alpha,\beta,\gamma) \quad (20)$$

$$\frac{\partial\mathscr{L}}{\partial\gamma} = \frac{n}{\gamma}-n\ln(\beta)+\sum_{i=1}^{n}\ln(x_i)$$
$$-(\alpha+1)\sum_{i=1}^{n}\xi(x_i,\beta,\gamma)\ln\left(\frac{x_i}{\beta}\right)$$
$$+\alpha(\theta-1)\sum_{i=1}^{n}\psi(x_i,\alpha,\beta,\gamma)\ln\left(\frac{x_i}{\beta}\right) \quad (21)$$

$$\frac{\partial\mathscr{L}}{\partial\theta} = \frac{n}{\theta}+\sum_{i=1}^{n}\ln\left(1-u_i^{-\alpha}\right), \quad (22)$$

where $\xi(x_i,\beta,\gamma) = (u_i-1)u_i^{-1}$, $\psi(x_i,\alpha,\beta,\gamma) = \xi(x_i,\beta,\gamma)(u_i^{\alpha}-1)^{-1}$ and $u_i = 1+\left(\frac{x_i}{\beta}\right)^{\gamma}$.

The MLEs can be obtained by equating the derivatives (19)–(22) with zero and solve it, numerically for α, β,γ and θ.

4.2 Asymptotic confidence bounds

As the MLEs do not possess the closed form. It is a very tedious task to derive their exact distributions. Therefore, asymptotic confidence intervals based on normal approximation are adopted. This technique requires applying a variance-covariance matrix see, [15].

$$V=\begin{pmatrix} -\frac{\partial^2\mathscr{L}}{\partial\alpha^2} & -\frac{\partial^2\mathscr{L}}{\partial\alpha\partial\beta} & -\frac{\partial^2\mathscr{L}}{\partial\alpha\partial\gamma} & -\frac{\partial^2\mathscr{L}}{\partial\alpha\partial\theta}\\ -\frac{\partial^2\mathscr{L}}{\partial\alpha\partial\beta} & -\frac{\partial^2\mathscr{L}}{\partial\beta^2} & -\frac{\partial^2\mathscr{L}}{\partial\beta\partial\gamma} & -\frac{\partial^2\mathscr{L}}{\partial\beta\partial\theta}\\ -\frac{\partial^2\mathscr{L}}{\partial\alpha\partial\gamma} & -\frac{\partial^2\mathscr{L}}{\partial\gamma\partial\beta} & -\frac{\partial^2\mathscr{L}}{\partial\gamma^2} & -\frac{\partial^2\mathscr{L}}{\partial\gamma\partial\theta}\\ -\frac{\partial^2\mathscr{L}}{\partial\alpha\partial\theta} & -\frac{\partial^2\mathscr{L}}{\partial\theta\partial\beta} & -\frac{\partial^2\mathscr{L}}{\partial\theta\partial\gamma} & -\frac{\partial^2\mathscr{L}}{\partial\theta^2} \end{pmatrix}^{-1}. \quad (23)$$

where

$$\frac{\partial^2\mathscr{L}}{\partial\alpha^2}=-\frac{n}{\alpha^2}-(\theta-1)\sum_{i=1}^{n}u_i^{\alpha}\left(\frac{\ln(u_i)}{u_i^{\alpha}-1}\right)^2 \quad (24)$$

$$\frac{\partial^2\mathscr{L}}{\partial\alpha\partial\beta}=\frac{\gamma}{\beta}\sum_{i=1}^{n}\xi(u_i)+(\theta-1)\sum_{i=1}^{n}\frac{\psi(u_i)u_i^{\alpha}}{u_i^{\alpha}-1}\times \left[\alpha\ln(u_i)-1-u_i^{-\alpha}\right] \quad (25)$$

$$\frac{\partial^2\mathscr{L}}{\partial\alpha\partial\gamma}=-\sum_{i=1}^{n}\xi(x_i,\beta,\gamma)\ln\left(\frac{x_i}{\beta}\right)-(\theta-1)\frac{\gamma}{\beta}\times \sum_{i=1}^{n}\frac{\psi(u_i)u_i^{\alpha}}{u_i^{\alpha}-1}\ln\left(\frac{x_i}{\beta}\right)\left[\alpha\ln(u_i)-1-u_i^{-\alpha}\right] \quad (26)$$

$$\frac{\partial^2\mathscr{L}}{\partial\alpha\partial\theta}=\sum_{i=1}^{n}\frac{\ln(u_i)}{u_i^{\alpha}-1} \quad (27)$$

$$\frac{\partial^2\mathscr{L}}{\partial\beta^2}=\frac{n\gamma}{\beta^2}+\frac{\gamma(\alpha+1)}{\beta^2}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^2(-\gamma-u_i)}{u_i^2}+ \frac{\alpha\gamma(\theta-1)}{\beta^2}\sum_{i=1}^{n}\frac{\psi(u_i)u_i^{\alpha}}{u_i^{\alpha}-1}\Big\{-\alpha\gamma u_i^{-(\alpha+1)} +(1-u_i^{-\alpha})\left[\gamma+1-(\alpha+1)u_i^{-1}(u_i-1)\right]\Big\} \quad (28)$$

$$\frac{\partial^2\mathscr{L}}{\partial\beta\partial\gamma}=-\frac{n}{\beta}+\left(\frac{\alpha+1}{\beta}\right)\sum_{i=1}^{n}\frac{\xi(u_i)}{u_i}\left[\gamma\ln\left(\frac{x_i}{\beta}\right)+u_i\right]- \frac{(\theta-1)\alpha}{\beta}\sum_{i=1}^{n}\psi(u_i)\left[1+\gamma\ln\left(\frac{x_i}{\beta}\right)\right]+ \frac{(\theta-1)\alpha\gamma}{\beta}\sum_{i=1}^{n}\psi^2(u_i)\left[(\alpha+1)u_i^{\alpha}-1\right]\ln\left(\frac{x_i}{\beta}\right) \quad (29)$$

$$\frac{\partial^2\mathscr{L}}{\partial\beta\partial\theta}=-\frac{\alpha\gamma}{\beta}\sum_{i=1}^{n}\psi(u_i) \quad (30)$$

$$\frac{\partial^2\mathscr{L}}{\partial\gamma^2}=-\frac{n}{\gamma^2}-(\alpha+1)\sum_{i=1}^{n}\frac{\xi(u_i)}{u_i}\ln^2\left(\frac{x_i}{\beta}\right)+ \alpha(\theta-1)\sum_{i=1}^{n}\frac{\psi(u_i)}{u_i^{\alpha}(u_i-1)}\ln^2\left(\frac{x_i}{\beta}\right)\times \left[1-u_i^{-\alpha}-\xi(u_i)(\alpha+1-u_i^{-\alpha})\right] \quad (31)$$

$$\frac{\partial^2\mathscr{L}}{\partial\theta^2}=\alpha\sum_{i=1}^{n}\frac{u_i-1}{u_i^{\alpha}-1}\ln\left(\frac{x_i}{\beta}\right). \quad (32)$$

5 Application

We present an analysis based on real data set and simulation to show that the exponentiated flexible Lomax (EFL) distribution could be a preferred model than exponentiated Lomax (EL) distribution, exponentiated Pareto (EP) distribution, and flexible Lomax (FL) distribution. The model selection criteria are based on (negative Log-likelihood, Akaike Information Criterion, Akaike Information Criterion Corrected, Bayesian Information Criterion and, Hannan-Quinn Information Criterion).

5.1 Real data

The real observing data for the strengths of glass fibers consisting of 1.5 cm thickness, taken from the research findings executed by [16] are reported as:

0.55	1.24	1.48	1.54	1.61	1.68	1.78
0.74	1.25	1.48	1.55	1.62	1.68	1.81
0.77	1.27	1.49	1.55	1.62	1.69	1.82
0.81	1.28	1.49	1.58	1.63	1.70	1.84
0.84	1.29	1.50	1.59	1.64	1.70	1.84
0.93	1.30	1.50	1.60	1.66	1.73	1.89
1.04	1.36	1.51	1.61	1.66	1.76	2.00
1.11	1.39	1.52	1.61	1.66	1.76	2.01
1.13	1.42	1.53	1.61	1.67	1.77	2.24

Table 2: Estimates of parameters.

Models	α	β	γ	θ	λ
EFL	1.201	1.793	16.296	0.289	–
EL	21.763	–	–	36.630	0.135
EP	5.541	–	–	99.305	–
FL	3.496	1.911	6.752	–	–

Table 3: The log- likelihood and measures of distributions.

Models	$\mathscr{L}$	AIC	AICC	BIC	HQIC
EFL	-12.33	32.65	32.99	41.23	36.02
EL	-33.33	72.66	72.87	79.09	75.19
EP	-38.98	81.96	82.06	86.25	83.65
FL	-16.63	39.26	39.46	45.69	41.79

As the value of $\mathscr{L}$, AIC, AICC, BIC and HQIC are smallest for EFL distribution as compared to EL, EP and

FL distribution. Therefore, for the given data set, EFL distribution fits better.
The variance - covariance matrix is calculated as

$$V = \begin{pmatrix} 1.1135 & 0.0892 & -6.9665 & 0.1309 \\ 0.0892 & 9.5503\times 10^{-3} & -0.4845 & 7.7976\times 10^{-3} \\ -6.9665 & -0.4845 & 55.9084 & -1.0719 \\ 0.1309 & 7.7976\times 10^{-3} & -1.0719 & 0.0226 \end{pmatrix}$$

Then the 95% confidence interval for α,β,γ and θ for EFL distribution are (0, 3.2696), (1.601, 1.98408), (1.64076, 30.95136) and (0, 0.58443), respectively.

We depict the $\mathscr{L}$ profile for α,β,γ and θ, in Figure 3. It shows that likelihood equations consist of a unique solution

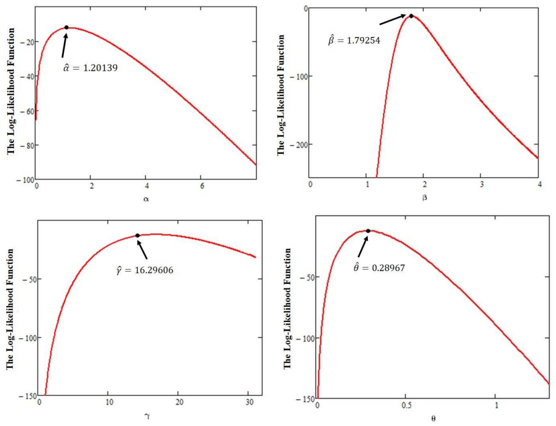

Fig. 3: The $\mathscr{L}$ profile for α, β,γ and θ.

5.2 Simulation Study

Here, we performed a simulation to estimate the parameters for the exponentiated flexible Lomax distribution. The simulation was conducted below mention steps:

(i)For reported values, the sample size $n = 100$.
(ii)For given values, the parameters $(\alpha,\beta,\gamma,\theta) = (1.3, 1.8, 7.4, 0.2)$.
(iii)Generate the sample from the Exponentiated flexible Lomax model according to the method proposed by [17], using $X = \left[(1-U^{1/\theta})^{-1/\alpha}-1\right]^{1/\gamma}$, where U is uniform (0, 1).
(iv)Estimate the parameters α,β,γ and θ, calculate the variance covariance matrix, AIC, AICC, BIC and HQIC.
(v)Repeat the above steps 1000 times. The average of all 1000 estimated values from Step 4 are respectively calculated and summarized.
(vi)The computational results are displayed in Tables 4–5.

Table 4: Estimates of parameters.

Models	α	β	γ	θ	λ
EFL	1.494	2.091	3.342	0.509	–
EL	5.663	–	–	2.871	0.248
EP	2.408	–	–	4.187	–
FL	2.415	2.195	2.141	–	–

Table 5: The log-likelihood, and measures of distributions.

Models	$\mathscr{L}$	AIC	AICC	BIC	HQIC
EFL	-119.53	247.05	247.26	257.47	251.27
EL	-129.37	264.74	264.86	272.55	267.90
EP	-138.10	280.20	280.26	285.41	282.31
FL	-122.24	250.47	250.60	258.29	253.64

As the value of Log-likelihood and measures of distributions are smallest for EFL distribution as compared to EL, EP and FL distribution. Therefore, for the given data set, EFL distribution fits better.
The variance -covariance matrix is obtained as

$$V = \begin{pmatrix} 2.3659 & 0.8494 & -2.7408 & 0.4595 \\ 0.8494 & 0.3395 & -0.9402 & 0.1489 \\ -2.7408 & -0.9402 & 3.4306 & -0.5808 \\ 0.4595 & 0.1489 & -0.5808 & 0.1033 \end{pmatrix}$$

Then the 95% confidence interval for α,β,γ and θ for EFL distribution are (0, 4.50887), (0.94888, 3.23291), (0, 6.97257) and (0, 1.13865), respectively.
The likelihood equations acquire a unique solution as shown in Figure 4.

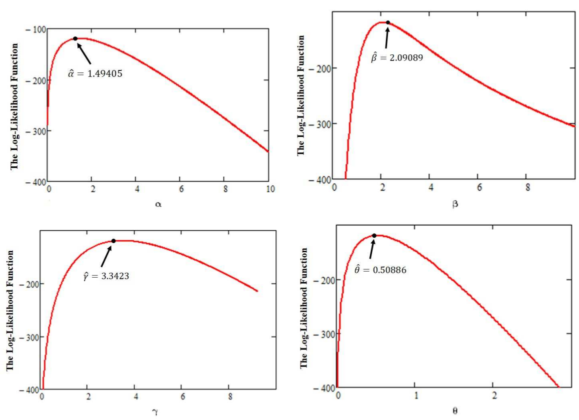

Fig. 4: The $\mathscr{L}$ profile for α, β, γ and θ.

6 Concluding remarks

A new model is proposed, called an exponentiated flexible Lomax distribution. Derive several properties. For the comparison, a real life-data and simulation study is carried out. From Table 3 and Table 5, it may be confirmed that the proposed model suffices better than others.

Acknowledgement

The Post-Publishing Program 1, supported by the Deanship of Scientific Research at the Islamic University of Madinah, is acknowledged.

Conflict of Interest The authors declare that they have no conflict of interest.

References

[1] R. C. Gupta, P. L. Gupta, and R. D. Gupta, Modeling failure time data by Lehmann alternatives, Communications in Statistics - Theory and Methods, **27**, 887-904 (1998).
[2] S. Nadarajah and S. Kotz, The exponentiated type distributions, Acta Applicandae Mathematicae, **92**, 97-111 (2006).
[3] A. Flaih, H. Elsalloukh, E. Mendi, and M. Milanova, The exponentiated inverted Weibull distribution, Applied Mathematics & Information Sciences, **6(2)**, 167-171 (2012).
[4] G. M. Cordeiro, Edwin M. M. Ortega and Daniel C. C. da Cunha, The exponentiated generalized class of distributions, Journal of Data Science, **11**, 1-27. (2013).
[5] A. M. Sarhan and J. Apaloo, Exponentiated modi?ed Weibull extension distribution, Reliability Engineering and System Safety, **112**, 137-144. (2013).
[6] Abu-Zinadah, H. H. and Anhar S. Aloufi, Some characterizations of the exponentiated Gompertz distribution, International Mathematical Forum, **9 (30)**, 1427-1439 (2014).
[7] S. Dey, D. Kumar, P. L. Ramos, and F. Louzada, Exponentiated Chen distribution: Properties and estimation, Communications in Statistics - Simulation and Computation, **46(10)**, 8118-8139 (2017).
[8] A. A. Rather and C. Subramanian, Exponentiated Mukhrejee-Islam distribution, Journal of Statistics Applications & Probability, **7 (2)**, 357-361 (2018).
[9] M. Elgarhy, I. Elbatal, G. G. Hamedani, and A. S. Hassan, On the exponentiated Weibull Rayleigh distribution, Gazi Journal of Science, **32(3)**, 1060-1081 (2019).
[10] A. M. Almarashi, F. Jamal, C. Chesneau, and M. Elgarhy, The Exponentiated truncated inverse Weibull-generated family of distributions with applications, Symmetry, **12(4)**, 1-21 (2020).
[11] M. Badr and M. Ijaz, The Exponentiated Exponential Burr XII distribution: Theory and application to lifetime and simulated data, PLoS One, **16(3)**, p. e0248873 (2021).
[12] M. Ijaz, M. Asim, and A. Khalil, Flexible Lomax distribution, Songklanakarin Journal of Science and Technology, **42 (5)**, 1125-1134 (2020).
[13] I. B. Abdul-Moniem and H. F. Abdel-Hameed, Exponentiated Lomax distribution, International Journal of Mathematical Archive, **3 (5)**, 1-7 (2012).
[14] Z. Iqbal, M. M. Tahir N. Riaz and S.A. Ali and M. Ahmad, Generalized inverted Kumaraswamy distribution: Properties and applications, Open Journal of Statistics, **7**, 645-662 (2017).
[15] J. F. Lawless, Statistical Models and Methods for Lifetime Data John Wiley and Sons, New York, **20**, 1108-1113 (2003).
[16] R. L. Smith and J. C. Naylor, A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution, Journal of the Royal Statistical Society. (Series C), **36(3)**, 358-369 (1987).
[17] C. D. Kemp and W. Kemp, Rapid generation of frequency tables, Journal of Applied Statistics, **36**, 277-282 (1987).