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Coefficient Estimate of A Uniform Lipschitz Mapping Failing Fixed Point Property on A Class in The Köthe-Toeplitz Duals for Generalized Cesàro Difference Sequence Spaces

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Abstract: In 1983, Orhan introduced Cesàro Difference Sequence Spaces. Later, various authors generalized them. In this study, we take a generalized Cesàro difference sequence spaces and especially we consider their Köthe-Toeplitz Duals. In fact, recalling that Dowling et al. proved that Banach spaces containing isomorphic copies of ℓ^1 cannot have the fixed point property for uniform Lipschitz mappings, we work on a well-known invariant mapping defined on a certain class in a Köthe-Toeplitz Dual of a generalized Cesàro difference sequence space so that the right shift mapping can be a uniform Lipschitz mapping. For this aim, we find an upper bound estimate of the Lipschitz coefficient. Next, we investigate the second power of the mapping we care so that it can be uniformly Lipschitz while it is supposed to fail the fixed point property on the class we study in those spaces.

Keywords: Right shift, affine mapping, fixed point property, uniformly Lipschitz mapping, Cesáro difference sequences, Köthe-Toeplitz dual

1 Introduction and Preliminaries

The Cesàro sequence spaces

$$\mathrm{ces}_p = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\}$$

and

$$\mathrm{ces}_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}$$

were introduced by Shiue [1] in 1970, where $1 \leq p < \infty$. It has been shown that $\ell^p \subset \mathrm{ces}_p$ for $1 < p \leq \infty$. Moreover, it has been shown that Cesàro sequence spaces ces_p for $1 < p < \infty$ are separable reflexive Banach spaces.

Later, in 1981, Kızmaz [2] introduced difference sequence spaces for ℓ^∞, c and c_0 where they are the Banach spaces of bounded, convergent and null sequences $x = (x_n)_n$, respectively. As it is seen below, his definitions for these spaces were given using difference operator applied to the sequence x, $\triangle x = (x_k - x_{k+1})_k$.

$$\ell^\infty(\triangle) = \{ x = (x_n)_n \subset \mathbb{R} \,|\, \triangle x \in \ell^\infty \},$$

$$\mathrm{c}(\triangle) = \{ x = (x_n)_n \subset \mathbb{R} \,|\, \triangle x \in \mathrm{c} \},$$

$$\mathrm{c}_0(\triangle) = \{ x = (x_n)_n \subset \mathbb{R} \,|\, \triangle x \in \mathrm{c}_0 \}.$$

Kızmaz investigated Köthe-Toeplitz Duals and some properties of these spaces.

Furthermore, Cesàro sequence spaces X^p of non-absolute type were defined by Ng and Lee [3] in 1977 as follows:

$$X^p = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p \right)^{1/p} < \infty \right\}$$

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and

$$X^{\infty}=\left\{x=(x_n)_n\subset\mathbb{R}\;\middle|\;\sup_n\left|\frac{1}{n}\sum_{k=1}^{n}x_k\right|<\infty\right\},$$

where $1\leq p<\infty$.

Later, in 1983, Orhan [4] introduced Cesàro Difference Sequence Spaces by the following definitions:

$$C_p=\left\{x=(x_n)_n\subset\mathbb{R}\;\middle|\;\left(\sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{k=1}^{n}\triangle x_k\right|^p\right)^{1/p}<\infty\right\}$$

and

$$C_\infty=\left\{x=(x_n)_n\subset\mathbb{R}\;\middle|\;\sup_n\left|\frac{1}{n}\sum_{k=1}^{n}\triangle x_k\right|<\infty\right\},$$

where $1\leq p<\infty$ and $\triangle x=(\triangle x_k)=(x_k-x_{k+1})_k$. He noted that their norms are given as below for any $x=(x_n)_n$:

$$\|x\|_p^*=|x_1|+\left(\sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{k=1}^{n}\triangle x_k\right|^p\right)^{1/p}$$

and

$$\|x\|_\infty^*=|x_1|+\sup_n\left|\frac{1}{n}\sum_{k=1}^{n}\triangle x_k\right|$$

respectively.

Orhan [4] showed that there exists a linear bounded operator $S:C_p\to C_p$ for $1\leq p\leq\infty$ such that Köthe-Toeplitz $\beta-$Duals of these spaces are given respectively as follows: Let $1<p<\infty$ and $q=\frac{p}{p-1}$; then,

$$S(C_p)^\beta=\{a=(a_n)_n\subset\mathbb{R}\mid(na_n)_n\in\ell^q\},$$

$$S(C_1)^\beta=\{a=(a_n)_n\subset\mathbb{R}\mid(na_n)_n\in\ell^\infty\}\text{ and}$$

$$S(C_\infty)^\beta=\left\{a=(a_n)_n\subset\mathbb{R}\;\middle|\;(na_n)_n\in\ell^1\right\}.$$

It might be better to use the notation $X^p(\triangle)$ instead of C_p for $1\leq p\leq\infty$ since we also recalled the difference sequence spaces and used similar type of notation.

Note also that Köthe-Toeplitz Dual for $p=\infty$ case in Orhan's study and ℓ^∞ case in Kızmaz study coincides.

Furthermore, Et and Çolak [5] generalized the spaces introduced in Kızmaz's work [2] in the following way for $m\in\mathbb{N}$.

$$\ell^\infty(\triangle^m)=\{x=(x_n)_n\subset\mathbb{R}\mid\triangle^m x\in\ell^\infty\},$$

$$\mathrm{c}(\triangle^m)=\{x=(x_n)_n\subset\mathbb{R}\mid\triangle^m x\in\mathrm{c}\},$$

$$\mathrm{c}_0(\triangle^m)=\{x=(x_n)_n\subset\mathbb{R}\mid\triangle^m x\in\mathrm{c}_0\}$$

where $\triangle x=(\triangle x_k)=(x_k-x_{k+1})_k$, $\triangle^0 x=(x_k)_k$, $\triangle^m x=(\triangle^m x_k)=\left(\triangle^{m-1}x_k-\triangle^{m-1}x_{k+1}\right)_k$ and $\triangle^m x_k=\sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k+i}$.

Also, Et [6] and Tripathy et. al. [7] generalized the space introduced by Orhan in the following way for $m\in\mathbb{N}$.

$$X^p(\triangle^m)=\left\{(x_n)_n\subset\mathbb{R}\;\middle|\;\left(\sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{k=1}^{n}\triangle^m x_k\right|^p\right)^{1/p}<\infty\right\}$$

and

$$X^\infty(\triangle^m)=\left\{x=(x_n)_n\subset\mathbb{R}\;\middle|\;\sup_n\left|\frac{1}{n}\sum_{k=1}^{n}\triangle^m x_k\right|<\infty\right\}.$$

Then, it is seen that that Köthe-Toeplitz Dual for $p=\infty$ case in Et's study [6] and ℓ^∞ case in Et and Çolak study [5] coincides such that Köthe-Toeplitz Dual was given as below for any $m\in\mathbb{N}$.

$$D_m:=\left\{a=(a_n)_n\subset\mathbb{R}\;\middle|\;(n^m a_n)_n\in\ell^1\right\}$$

$$=\left\{a=(a_k)_k\subset\mathbb{R}:\;\|a\|=\sum_{k=1}^{\infty}k^m|a_k|<\infty\right\}.$$

Note that $D_m\subset\ell^1$.

One can see that corresponding function space for these duals can be given as below:

$$U_m:=\left\{\begin{matrix}f:[0,1]\to\mathbb{R}\\ \text{measurable}\end{matrix}:\;\|f\|=\int_0^1 t^m|f(t)|\,dt<\infty\right\}.$$

Note that $L_1[0,1]\subset U_m$ and D_m is the space when counting measure is used for U_m.

Now recall that we say that a Banach space $(\mathrm{X},\|.\|)$ has the fixed point property for non-expansive mappings (fpp-n.e.) if every invariant nonexpansive mapping defined on every nonempty closed, bounded and convex subset (c.b.c.) in X has a fixed point. If the above expression holds for every uniform Lipshcitz mapping, then $(\mathrm{X},\|.\|)$ is said to have the fixed point property for uniformly Lipschitz mappings (fpp-u.L.). Therefore, once we find a nonempty closed, bounded and convex subset E and a uniformly Lipschitz mapping $\mathrm{T}:\mathrm{E}\to\mathrm{E}$ without any fixed point, then X is said to fail the fpp-u.L..

There have been works to research fixed point property for Cesàro sequence spaces. For example, it was proved by Cui and Hudzik [8], Cui, Hudzik and Li [9] and Cui, Meng and Pluciennik [10] that Cesàro sequence spaces ces_p for $1<p<\infty$ have fpp-ne. They prove this result using different methods. One method is to calculate Garcia-Falset coefficient. It is known that if Garcia-Falset coefficient is less than 2 for a Banach space, then it has the fixed point property for nonexpansive mappings [11]. Using this fact, since they calculate this coefficient for ces_p as $2^{1/p}$ similarly to what it is for ℓ^p, they point the result for the Cesàro sequence spaces. Another fact is that the space has normal structure for $1 < p < \infty$. By the fact via Kirk [12] that reflexive Banach spaces with normal structure has the fixed point property, it is easily deduced that the space has the fixed point property for $1 < p < \infty$. These results on Cesàro sequence spaces as a survey can be also seen in [13]. There are various other interesting summable spaces as well as interesting facts and useful features in the domain of summability theory (see, for example, [14], [15], [16]) which may be taken up for further study in the framework of fixed point property.

Furthermore, like Cesàro sequence spaces introduced by Shiue [1], Ng and Lee [3] proved that Cesàro sequence spaces X^p of non-absolute type are linearly isomorphic and isometric to ℓ^p for $1 \leq p \leq \infty$. Thus, one would easily deduce that these spaces too have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

Considering the fixed point theory oriented questions for Cesàro Difference Sequence Spaces, we note that Orhan [4] proved that $X^p \subset X^p(\triangle)$ for $1 \leq p \leq \infty$ strictly. Also, one can clearly see that $X^p(\triangle)$ is linearly isomorphic and isometric to ℓ^p for $1 \leq p \leq \infty$. Therefore, it easy to deduce that Cesàro Difference Sequence Spaces also have similar properties to ℓ^p spaces for $1 \leq p \leq \infty$ in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

In this study, we take a generalized Cesàro difference sequence spaces and especially we consider their Köthe-Toeplitz Duals. There are, of course, other generalized difference operators for constructing other generalized difference sequence spaces (see, for example, [17], [18], [19]). In fact, recalling that Dowling et al. [20] proved that Banach spaces containing isomorphic copies of ℓ^1 cannot have the fixed point property for uniform Lipschitz mappings, we work on a well-known invariant mapping defined on a certain class in a Köthe-Toeplitz Dual of a generalized Cesàro difference sequence space so that the right shift mapping can be a uniform Lipschitz mapping. For this aim, we find an upper bound estimate of the Lipschitz coefficient. Next, we investigate the second power of the mapping we care so that it can be uniformly Lipschitz while it is supposed to fail the fixed point property on the class we study in those spaces.

The followings are needed as preliminaries.

Definition 1.*Let* $(\mathrm{X},\|\cdot\|)$ *be a Banach space ,* E *be a nonempty c.b.c. subset and* $T : E \to E$ *be a mapping.*

1. T *is called an affine mapping if for every* $\lambda \in [0,1]$ *and* $\mathrm{x},\mathrm{y} \in \mathrm{E}$, $\mathrm{T}((1-\lambda)\mathrm{x}+\lambda\ \mathrm{y}) = (1-\lambda)\mathrm{T}(\mathrm{x})+\lambda\ \mathrm{T}(\mathrm{y})$.

2. T *is called a* $\|\cdot\|-$*nonexpansive mapping if for every* $\mathrm{x},\mathrm{y} \in \mathrm{E}$, $\|\mathrm{T}(\mathrm{x})-\mathrm{T}(\mathrm{y})\| \leq \|\mathrm{x}-\mathrm{y}\|$.

Furthermore, if every $\|\cdot\|-$*nonexpansive mapping* $\mathrm{T} : \mathrm{E} \to \mathrm{E}$ *has a fixed point; i.e., if there exists a* $\mathrm{u} \in \mathrm{E}$ *such that* $\mathrm{T}(\mathrm{u}) = \mathrm{u}$, *then we say that* E *has the fpp(n.e.).*

3. T *is called a uniform Lipshcitz mapping if there exists a scalar* $\mathrm{L} \in [1,\infty)$ *such that for every* $\mathrm{x},\mathrm{y} \in \mathrm{E}$, $\|\mathrm{T}^{\mathrm{n}}(\mathrm{x})-\mathrm{T}^{\mathrm{n}}(\mathrm{y})\| \leq \mathrm{L}\|\mathrm{x}-\mathrm{y}\|$ *for every* $n \in \mathbb{N}$. *Moreover, here L is called a uniform Lipshcitz constant.*

Furthermore, if every uniformly Lipschitz mapping $\mathrm{T} : \mathrm{E} \to \mathrm{E}$ *has a fixed point; i.e., if there exists a* $\mathrm{u} \in \mathrm{E}$ *such that* $\mathrm{T}(\mathrm{u}) = \mathrm{u}$, *then we say that* E *has the fpp-u.L..*

We also note that through the study, the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1, where i^{th} term is 1 and others 0 for the e_i.

2 Main Result

In this section, we consider a large class of c.b.c. subsets in Köthe-Toeplitz Dual for $X^\infty(\triangle^m)$, the space D_m given above for $m \in \mathbb{N}$, and investigate a well-known mapping mostly used in fixed point theory researches. The mapping we care is the right shift mapping. Since we know by Dowling et al. [20], isomorphic copies or spaces containing them cannot have the fpp(u.L.), we check how our mapping becomes a uniformly Lipschitz invariant mapping on the class we study. In the case the mapping is a uniformly Lipschitz, we find an upper bound estimate of the Lipschitz coefficient. In fact, we find the minimum constant as the Lipschitz coefficient. Now, firstly we consider the following class of c.b.c. subsets. Note that here we use the similar ideas to those in [21, section 3.2], written under supervision of Chris Lennard. We note that case $m = 1$ has recently been done by Nezir and Güven and submitted to a refereed international journal. As we stated, here we present the general case for any $m \in \mathbb{N}$.

*Example 1.*Fix $m \in \mathbb{N}$ and $b \in (0,1)$. Define a sequenc $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\ e_1$, and $f_n := \frac{1}{n^m} e_n$, for every integer $n \geq 2$. Next, define the c.b.c. subset $E^{(m)} = E_b{}^{(m)}$ of D_m by

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} \beta_n\ f_n :\ \ \forall n \in \mathbb{N},\ \beta_n \geq 0 \ \text{ and } \ \sum_{n=1}^{\infty} \beta_n = 1 \right\} .$$

Consider the right shift mapping $T : E^{(m)} \to E^{(m)}$ defined by

$$T(x) = T\left(\sum_{n=1}^{\infty} \beta_n\ f_n\right) = \sum_{n=1}^{\infty} \beta_n\ f_{n+1}.$$

Then, for any $x = \sum_{k=1}^{\infty} \beta_k f_k$ and $y = \sum_{k=1}^{\infty} \gamma_k f_k$ in $E^{(m)}$. It is easy to see that T is affine and fixed point free. Moreover, for every $q \in \mathbb{N}$,

$$\begin{aligned}\|T^q y - T^q x\| &= \left\|\sum_{k=1}^{\infty} \beta_k f_{k+q} - \sum_{k=1}^{\infty} \gamma_k f_{k+q}\right\| \\ &= |\beta_1 - \gamma_1| + \sum_{k=2}^{\infty} |\beta_k - \gamma_k| \\ &= \left|1 - \sum_{k=2}^{\infty} \beta_k - 1 + \sum_{k=2}^{\infty} \gamma_k\right| + \sum_{k=2}^{\infty} |\beta_k - \gamma_k| \\ &= \left|\sum_{k=2}^{\infty} \beta_k - \gamma_k\right| + \sum_{k=2}^{\infty} |\beta_k - \gamma_k| \\ &\le 2 \sum_{k=2}^{\infty} |\beta_k - \gamma_k|.\end{aligned}$$

Hence,

$$\|T^q y - T^q x\| \le 2\left(b|\beta_1 - \gamma_1| + \sum_{k=2}^{\infty} |\beta_k - \gamma_k|\right).$$

But since

$$\begin{aligned}\|y - x\| &= \left\|\sum_{k=1}^{\infty} \beta_k f_k - \sum_{k=1}^{\infty} \gamma_k f_k\right\| \\ &= b|\beta_1 - \gamma_1| + \sum_{k=2}^{\infty} |\beta_k - \gamma_k|, \qquad (1)\end{aligned}$$

we get that for any $q \in \mathbb{N}$,

$$\|T^q y - T^q x\| \le 2\|y - x\|.$$

Also, for any $q \in \mathbb{N}$,

$$\begin{aligned}\|T^q y - T^q x\| &= |\beta_1 - \gamma_1| + \sum_{k=2}^{\infty} |\beta_k - \gamma_k| \\ &= \frac{1}{b}\left(b|\beta_1 - \gamma_1| + b\sum_{k=2}^{\infty} |\beta_k - \gamma_k|\right) \\ &\le \frac{1}{b}\left(b|\beta_1 - \gamma_1| + \sum_{k=2}^{\infty} |\beta_k - \gamma_k|\right) \\ &= \frac{1}{b}\|y - x\| \quad \text{by the equality (1)}\end{aligned}$$

Hence, for any $q \in \mathbb{N}$, $\|T^q y - T^q x\| \le \min\left\{2, \frac{1}{b}\right\} \|y - x\|$. Therefore, the right shift T is uniformly Lipschitz with Lipschitz coefficient $M_b = \min\left\{2, \frac{1}{b}\right\}$.

Now, we find the minimum Lipschitz coefficient in the following theorem.

In fact, as we see that we have exactly same findings and computations as those in section 3.2 of [21], the next theorem is also obtained similarly to the proof method of Everest.

Theorem 1. *Fix $m \in \mathbb{N}$ and $b \in (0,1)$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\ e_1$, and $f_n := \frac{1}{n^m} e_n$, for every integer $n \ge 2$. Next, define the c.b.c. subset $E^{(m)} = E_b{}^{(m)}$ of D_m by*

$$E^{(m)} := \left\{\sum_{n=1}^{\infty} \beta_n f_n : \ \forall n \in \mathbb{N},\ \beta_n \ge 0 \ \text{and} \ \sum_{n=1}^{\infty} \beta_n = 1\right\}.$$

Consider the right shift mapping $T : E^{(m)} \to E^{(m)}$ defined by

$$T(x) = T\left(\sum_{n=1}^{\infty} \beta_n f_n\right) = \sum_{n=1}^{\infty} \beta_n f_{n+1}.$$

Then, for any $x = \sum_{k=1}^{\infty} \beta_k f_k$ and $y = \sum_{k=1}^{\infty} \gamma_k f_k$ in $E^{(m)}$ and $\forall q \in \mathbb{N}$,

$$\|T^q y - T^q x\| \le \frac{2}{1+b}\|y - x\|$$

such that $\frac{2}{1+b}$ is the minimum uniform Lipschitz constant that satisfies the condition above.

Proof. Fix $m \in \mathbb{N}$ and $q \in \mathbb{N}$. Let $= \sum_{k=1}^{\infty} \beta_k f_k$ and $y = \sum_{k=1}^{\infty} \gamma_k f_k$ in $E^{(m)}$. Let $\tau \in [0,1]$. Then we have

$$\begin{aligned}\|T^q y - T^q x\| &= \left\|\sum_{k=1}^{\infty} \beta_k f_{k+q} - \sum_{k=1}^{\infty} \gamma_k f_{k+q}\right\| \\ &= |\beta_1 - \gamma_1| + \sum_{k=2}^{\infty} |\beta_k - \gamma_k| \\ &= \tau|\beta_1 - \gamma_1| + (1-\tau)|\beta_1 - \gamma_1| + \sum_{k=2}^{\infty} |\beta_k - \gamma_k| \\ &= \tau|\beta_1 - \gamma_1| + (1-\tau)\left|1 - \sum_{k=2}^{\infty} \beta_k - 1 + \sum_{k=2}^{\infty} \gamma_k\right| \\ &\quad + \sum_{k=2}^{\infty} |\beta_k - \gamma_k| \\ &= \tau|\beta_1 - \gamma_1| + (1-\tau)\left|\sum_{k=2}^{\infty} \beta_k - \gamma_k\right| + \sum_{k=2}^{\infty} |\beta_k - \gamma_k| \\ &\le \tau|\beta_1 - \gamma_1| + (2-\tau)\sum_{k=2}^{\infty} |\beta_k - \gamma_k| \qquad (2) \\ &= \frac{\tau}{b}\left(b|\beta_1 - \gamma_1| + \frac{(2-\tau)b}{\tau}\sum_{k=2}^{\infty} |\beta_k - \gamma_k|\right) \qquad (3) \\ &= (2-\tau)\left(\frac{\tau}{(2-\tau)}|\beta_1 - \gamma_1| + \sum_{k=2}^{\infty} |\beta_k - \gamma_k|\right). \qquad (4)\end{aligned}$$

Here, to use (3) so that we can get uniform Lipschitz estimate, we need

$$\frac{(2-\tau)b}{\tau} \le 1 \Leftrightarrow 2b - \tau b \le \tau \Leftrightarrow \frac{2b}{b+1} \le \tau.$$

Then, to minimize coefficient in (3), which is $\frac{\tau}{b}$, we minimize τ, so by the above fact, minimum value for τ

satifying (3) would be $\frac{2b}{b+1}$. Thus, minimum coefficient $\frac{\tau}{b}$ is $\frac{2}{b+1}$ satisfying (3).

But to use (4) so that we can get uniform Lipschitz estimate, we need $\frac{\tau}{(2-\tau)} \leq b$ and we minimize $2-\tau$. Here we note that

$$\frac{\tau}{(2-\tau)} \leq b \Leftrightarrow \tau \leq 2b - \tau b \Leftrightarrow \tau \leq \frac{2b}{b+1}.$$

Then, to minimize $2-\tau$ we would maximize τ and that maximum value of τ would be $\frac{2b}{b+1}$ by the above fact. So minimum value for $2-\tau$ in (4) is $\frac{2}{b+1}$. That is, minimum coefficient $2-\tau$ in (4) is $\frac{2}{b+1}$.

Therefore, from both results, we can say that for any $q \in \mathbb{N}$,

$$\|T^q y - T^q x\| \leq M_b \left(b|\beta_1 - \gamma_1| + \sum_{k=2}^{\infty} |\beta_k - \gamma_k| \right) = M_b \|y - x\|$$

and M_b might be $\frac{2}{b+1}$. In fact, the following fact tells us that minimum coefficient M_b is $\frac{2}{b+1}$.

Indeed, consider $x := f_1$ and $y := f_2$. Then,

$$\|y - x\| = \left\| \frac{1}{2} e_2 - b e_1 \right\| = \left\| (-b, \frac{1}{2}, 0, 0, 0, \ldots) \right\| = b + 1.$$

Then,

$$\begin{aligned}
\|T^q y - T^q x\| &= \|T^q f_2 - T^q f_1\| = \|f_{q+2} - f_{q+1}\| \\
&= \left\| \frac{1}{q+2} e_{q+2} - \frac{1}{q+1} e_{q+1} \right\| \\
&= \frac{q+2}{q+2} + \frac{q+1}{q+1} = 2 = \frac{2}{b+1}(b+1) \\
&= \frac{2}{b+1} \|y - x\| = M_b \|y - x\|.
\end{aligned}$$

So this shows where the equality occurs and this verifies that $\frac{2}{b+1}$ is the smallest Lispchitz coefficient.

Theorem 2. *Fix $m \in \mathbb{N}$. Define a sequence $(f_n)_{n \in \mathbb{N}}$ by setting $f_1 := \frac{1}{2} e_1$, $f_2 := \frac{1}{2^{m+1}} e_2$, and $f_n := \frac{1}{n^m} e_n$, for every integer $n \geq 3$. Next, define the c.b.c. subset $S^{(m)}$ of D_m by*

$$S^{(m)} := \left\{ \sum_{n=1}^{\infty} \beta_n f_n : \ \forall n \in \mathbb{N},\ \beta_n \geq 0 \ \text{and} \ \sum_{n=1}^{\infty} \beta_n = 1 \right\}.$$

Consider the second power of the right shift mapping $T^2 : S^{(m)} \to S^{(m)}$. Then, the second power of the right shift is fixed point free affine Uniformly Lipschitz mapping on $S^{(m)}$ such that its Lipschitz coefficient is 2.

Proof. We can easily see that for any $x \in S^{(m)}$,

$$T^2(x) = T^2 \left(\sum_{n=1}^{\infty} \beta_n f_n \right) = \sum_{n=1}^{\infty} \beta_n f_{n+2}.$$

Then, for any $x = \sum_{k=1}^{\infty} \beta_k f_k$ and $y = \sum_{k=1}^{\infty} \gamma_k f_k$ in $S^{(m)}$. It is easy to see that T^2 is affine and fixed point free. Moreover, for every $q \in \mathbb{N}$,

$$\begin{aligned}
\|T^{q+1} y - T^{q+1} x\| &= \left\| \sum_{k=1}^{\infty} \beta_k f_{k+q+1} - \sum_{k=1}^{\infty} \gamma_k f_{k+q+1} \right\| \\
&= |\beta_1 - \gamma_1| + |\beta_2 - \gamma_2| + \sum_{k=3}^{\infty} |\beta_k - \gamma_k|
\end{aligned}$$

Also,

$$\begin{aligned}
\|y - x\| &= \left\| \sum_{k=1}^{\infty} \beta_k f_k - \sum_{k=1}^{\infty} \gamma_k f_k \right\| \\
&= \frac{1}{2} |\beta_1 - \gamma_1| + \frac{1}{2} |\beta_2 - \gamma_2| + \sum_{k=3}^{\infty} |\beta_k - \gamma_k|. \qquad (5)
\end{aligned}$$

Case 1: Assume $|\beta_1 - \gamma_1| \leq |\beta_2 - \gamma_2|$

Then, from (5),

$$2\|y - x\| \geq 2|\beta_1 - \gamma_1| + 2\sum_{k=3}^{\infty} |\beta_k - \gamma_k|$$

On the other hand,

$$\begin{aligned}
\|T^{q+1} y - T^{q+1} x\| &= |\beta_1 - \gamma_1| + \left| 1 - \beta_1 - \sum_{k=3}^{\infty} \beta_k - 1 + \gamma_1 + \sum_{k=3}^{\infty} \gamma_k \right| \\
&\quad + \sum_{k=3}^{\infty} |\beta_k - \gamma_k| \\
&\leq 2|\beta_1 - \gamma_1| + \left| \sum_{k=3}^{\infty} \beta_k - \gamma_k \right| + \sum_{k=3}^{\infty} |\beta_k - \gamma_k| \\
&\leq 2|\beta_1 - \gamma_1| + 2\sum_{k=2}^{\infty} |\beta_k - \gamma_k| \\
&\leq 2\|y - x\|.
\end{aligned}$$

Case 2: Assume $|\beta_2 - \gamma_2| \leq |\beta_1 - \gamma_1|$

Then, from (5),

$$2\|y - x\| \geq 2|\beta_2 - \gamma_2| + 2\sum_{k=3}^{\infty} |\beta_k - \gamma_k|$$

On the other hand,

$$\begin{aligned}
\|T^{q+1} y - T^{q+1} x\| &= \left| 1 - \beta_2 - \sum_{k=3}^{\infty} \beta_k - 1 + \gamma_2 + \sum_{k=3}^{\infty} \gamma_k \right| \\
&\quad + |\beta_2 - \gamma_2| + \sum_{k=3}^{\infty} |\beta_k - \gamma_k| \\
&\leq 2|\beta_2 - \gamma_2| + \left| \sum_{k=3}^{\infty} \beta_k - \gamma_k \right| + \sum_{k=3}^{\infty} |\beta_k - \gamma_k| \\
&\leq 2|\beta_2 - \gamma_2| + 2\sum_{k=2}^{\infty} |\beta_k - \gamma_k| \\
&\leq 2\|y - x\|.
\end{aligned}$$

So from two cases we get that for any $q \in \mathbb{N}$,

$$\|T^{q+1} y - T^{q+1} x\| \leq 2\|y - x\|.$$

Also, for any $q \in \mathbb{N}$,

$$\begin{aligned}
\left\|T^{q+1}y - T^{q+1}x\right\| &= |\beta_1 - \gamma_1| + |\beta_2 - \gamma_2| + \sum_{k=3}^{\infty} |\beta_k - \gamma_k| \\
&= 2\left(\frac{1}{2}|\beta_1 - \gamma_1| + \frac{1}{2}|\beta_2 - \gamma_2| + \frac{1}{2}\sum_{k=3}^{\infty} |\beta_k - \gamma_k|\right) \\
&\leq 2\left(\frac{1}{2}|\beta_1 - \gamma_1| + \frac{1}{2}|\beta_2 - \gamma_2| + \sum_{k=3}^{\infty} |\beta_k - \gamma_k|\right) \\
&= 2\left\|y - x\right\|. \qquad \text{(by the equality (5))}
\end{aligned}$$

Hence, for any $q \in \mathbb{N}$, $\left\|T^{q+1}y - T^{q+1}x\right\| \leq 2\left\|y - x\right\|$. Therefore, the second power of the right shift T is uniformly Lipschitz with Lipschitz coefficient $M = 2$.

Conflict of Interest The authors declare that they have no conflict of interest

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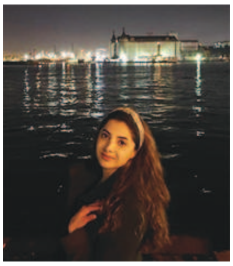

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