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Statistical Inference under Copula Approach of Accelerated Dependent Generalized Inverted Exponential Failure Time with Progressive Hybrid Censoring Scheme

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Abstract: The problem of statistical inference in reliability theory for the competing risks model under accelerated life testing (ALT) have a great significance. In practice, independent variables are assumed for convenience, which do not agree with the nature of the problem at hand. In this paper, we consider the constant stress accelerated life testing (CS-ALT) of dependent competing risks model for generalized inverted exponential distribution (GIED). The dependence structure is described by the copula approach between variable. Under consideration that units is failing by only two dependent causes of failure under constant stress ALTs and type-I progressive hybrid censoring scheme (PHCS), the model parameters are estimated with maximum likelihood method by using the bivariate Pareto copula function. The asymptotic confidence intervals with approximate Bootstrap confidence intervals are constructed. Under consideration two stress levels the set of real data are analyzed for illustrative purposes. For different measures of Kendall's tau and censoring schemes Monto Carlo simulation study is constructed.

Keywords: Generalized inverted exponential distribution; Competing risks model; Copula function; Accelerated life testing; Progressively hybrid censoring; Maximum likelihood estimation; Bootstrap confidence intervals

1 Introduction

More information about the lifetime of product (systems or components) in short period of time can be obtained under ALTs. Hence, the ALTs have attention in a recently year to present a quickly source of data. In an ALT, the units are placed under stress levels higher than use to accelerate the failures of the units. Different types of ALTs are avialable and the key reference of these types are presented by [1]. The first type, known by constant stress ALTs, in which the experimenter run the experiment under constant stress until the final point of the experiment. For more surf of constant stress ALTs see, [2], [3] and [4]. Second type called step stress ALTs, in which the experimenter run the experiment at different stress levels and changing at prefixed time or number of failures. For more surf of step stress ALTs, see [5], [6], [7], [8], [9] and [10] .The final type, which the stress is kept with continuously increasing at all experiment steps is called by progressive stress ALTs. For more details of progressive stress ALTs, see [11] and [12]. Recently, [13] have discussed the accelerated competing risks model from Gompertz lifetime distribution.

The GIED and its properties is presented early as a generalization of inverted exponential distribution by [14]. Also, GIED discussed as a special case of exponentiated Frechet distribution. The random variable X is called GIE random variable if distributed with cumulative distribution function (CDF) given by

$$F(x) = 1-(1-e^{-\lambda/x})^{\alpha},\ \alpha,\ \lambda > 0,\ x > 0, \tag{1}$$

where α and λ are called the shape and scale parameters, respectively. Also, GIED has probability density function (PDF) and survival function, respectively given by

$$f(x) = (\alpha\lambda/x^2)e^{-\lambda/x}(1-e^{-\lambda/x})^{\alpha-1}, \tag{2}$$

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and

$$S(x) = (1 - e^{-\lambda/x})^{\alpha}. \tag{3}$$

Many of the studies examined the characteristics and inferences of the GIED using different types of data. Under a complete sample, the properties and reliability characteristics as well as maximum likelihood estimators (MLEs) of GIED are derived by [14]. Under progressive type-II censored sample, the model parameters are estimate with ML and least squares methods by [15]. The necessary and sufficient conditions for existence and uniqueness of the MLEs of the GIE parameters have been provided by [16]. [17] and [18] discussed the Bayes estimators of the GIE parameters under hybrid censoring scheme. [19], provided the estimated stress-strength parameter $P(Y < X)$ for GIE model under progressive first-failure censoring scheme. Under adaptive type-II progressive censoring scheme, the ML and Bayes estimators of the model parameters are obtained by [20].

Measuring the quality of the life product required to put some units of the product under life testing experiment. The obtaining data may be complete or censoring. The experimenter under some restricted of cost or time determine the suitable method to obtain the data. The word complete data is used when information come from all units under the test. But, censoring data appear when some but not all lifetime of units is observed. The oldest common censoring schemes are called type-I and type-II censoring schemes. If the experimenter reported the test time and number of failure is random then, we mean type-I censoring scheme. But, if reported the number of failure units and the test time is random then, we mean type-II censoring scheme. When, the experimenter report to remove units at any step of experiment which is more suitable for another purposes then, we mean progressive censoring scheme. For more details in this topic, interested readers are referred to [22] and [23]. Hybrid censoring scheme is appear when the experimenter report the suitable number of failure units and the test time before the experiment is running. Different types of hybrid censoring schemes are available in literature, among them type-I progressive hybrid and type-II progressive hybrid censoring schemes are described as follows:

Suppose that, n selected units are putted under the test and the suitable number of failure units needing for statistical inferences and the ideal test time are determined by $(m,\ \tau)$. In type-I progressive hybrid censoring scheme, the failure time is record until the $\min(T_m,\ \tau)$ is observed. In type-II progressive hybrid censoring scheme the failure time is record until the $\max(T_m,\ \tau)$ is observed. In progressive type-I hybrid censoring scheme, suppose censoring scheme $\mathbf{R}=\{R_1, R_2, \ldots, R_m\}$ is prefixed to satisfies $n = m + \sum_{i=1}^{m} R_i$. At each failure time T_i, then R_i survival units are removed from the test with $i = 1, 2, \ldots, m$.

The paper aim to analysis the competing risks model under dependent GIE failure time when the failure times are accelerated with constant stress ALTs. And, the lifetime data is collected under type-I progressive hybrid censoring scheme (type-I PHCS). The copula approach is used to describing the dependence structure between variables. The parameters of the proposed model are estimated with ML method for point and interval estimators. Aslo, the bootstrap technique is used to constructed the approximated confidence intervals. The developed results are used for analysis some data set. Also, estimators are assessed and compared with Monto Carlo simulation study.

The remainder of this paper is planed as the following sections, the dependent structure of competing risks model with copula function and bivariate Pareto copula in Section 2. The model description and its properties in Section 3. The parameters and Reliability of the system are estimated with maximum likelihood method as well as the corresponding asymptotic confidence intervals based on the observed Fisher's information matrix are obtained in Section 4. Bootstrap technique is used to construct the confidence intervals of the model parameters in Section 5. Data analysis are presented in Section 6. Simulation study with numerical comparisons of the estimates are presented in Section 7. The article concludes in Section 8.

2 Copulas

The problem of modeling the dependence structure between variable with copula function is convenient way for dependent competing failure modes. The selected copula determine what is the type of dependence structure. More detail about the properties and definition of copula function are given in [24]. Under consideration that, the variables X_i distributed with marginal distributions F_i and the corresponding survival functions $S_i = \bar{F}_i, i = 1, 2, \ldots, m$, respectively. Then, a unique m dimensional copula C is exists to define the joint distribution function $H(x_1, \ldots, x_m)$ by

$$H(x_1, \ldots, x_m) = C(F_1(x_1), \ldots, F_m(x_m)). \tag{4}$$

In contrast, for given m-dimensional copula function C of the marginal functions $F_i, i = 1, 2, \ldots, m$, then H present m joint distribution function given by (4). Also, the dependence structure of joint distribution function H depend on the choice of copula function C and the corresponding marginal functions $F_i, i = 1, 2, \ldots, m$. The corollary of Sklar's theorem is used to construct a copula function of the continuos marginal functions $F_i, i = 1, 2, \ldots, m$. Also, for m-dimensional invariance distributions marginal functions $F_i^{-1}(u_i)$, where $u_i = F_i(x_i)$ and $i = 1, 2, \ldots, m$ the copula function defined by,

$$C(u_1, \ldots, u_m) = H(F_1^{-1}(u_1), \ldots, F_m^{-1}(u_m)). \tag{5}$$

The multivariate survival function $S(x_1, \ldots, x_m)$ under transformation $X_i \to F_i(X_i) = 1 - S_i(X_i)$ with Sklar's theorem via an appropriate copula $\bar{C}$ called the survival copula of $(X_1,...,X_m)$ can be expressed by

$$S(x_1,...,x_m) = \bar{C}(S_1(x_1),...,S_m(x_m)), \quad (6)$$

where $\bar{C}$ called the appropriate survival copula of X_i, $i = 1,2,...,m$. Then, we can say copula functions C and $\bar{C}$ are relate with marginal distribution functions F_i and marginal survival functions S_i, i = 1, 2, ..., m, with multivariate distribution and survival functions, respectively.

2.1 Archimedean copula

The function ϕ under some situations satisfies

$$\phi(C(u,v)) = \phi(u) + \phi(v), \quad (7)$$

is called Archimedean copula. And, the inverse function $\phi^{[-1]}$ define the solution of C to such that

$$C(u,v) = \phi^{[-1]}(\phi(u) + \phi(v)). \quad (8)$$

When $\phi(t) = t^{-1/\theta} - 1$, $\theta \geq 1$, the Bivariate Pareto Copula (BPC) is defined by

$$C_\theta(u,v) = (u^{-1/\theta} + v^{-1/\theta} - 1)^{-\theta} \quad (9)$$

2.2 Measure of association

Different types of copula are available with different parameters values and so, copula function does not comparable. Therefore, the compare case under copula need to define Kendall's tau as the follows

$$\tau = 4\int_0^1\int_0^1 C(u,v)c(u,v)dudv = 4E[C(U,V)] - 1, \quad (10)$$

Hence, (10) under BPC is reduced to

$$\tau = 4\int_0^1 \frac{\phi(t)}{\phi'(t)}dt + 1 = 1/(2\theta+1). \quad (11)$$

3 The Model Description

Let, n identical independent units are randomly selected from a life product, this sample are randomly distributed to (n_1, n_2 ...,n_k), $n = \sum_{l=1}^{k} n_l$ to test under k different constant stress levels. Suppose, the prefixed, integers (m_1, m_2 ...,m_k), ideal tested times (τ_1, τ_2 ...,τ_k) and censoring schemes ($\mathbf{R}_1$, $\mathbf{R}_2$ ...,$\mathbf{R}_k$) are proposed. Let n_l units are tested under constant stress level $\mathbf{S}_l$, $l = 1,2,...,k$, respectively. Under consideration that unit fails under one of two dependent causes of failure denoted by $\rho = \{1, 2\}$ with the mechanism of type-I PHCS. The failure time and corresponding cause of failure is record (T_{il}, ρ_{il}), $i = 1$, 2, ..., m_l, l = 1, 2, ..., k. At each failure time (T_{il}, ρ_{il}), R_{il}, $i = 1,2$, ...,m_l survival units are removed from the test. The experiment is continual until the min(T_{m_l}, τ_l), l = 1, 2, ..., k is observed. Suppose that, m_l^* denote to number of failure units at the τ_l^* =min(T_{m_l}, τ_l), then the remaining $R^*_{m_l^* l} = n_l - m_l^* - \sum_{i=1}^{m_l^*-1} R_{il}$ survival units are removed from the test. If the min(T_{m_l}, τ_l) is T_{m_l} then $m_l^* = m_l$ and $R^*_{m_l^* l} = R_{m_l l}$. The observed sample define by $\underline{\mathbf{t}}_l$={$(t_{1l}, \rho_{1l}), (t_{2l}, \rho_{2l}), ..., (t_{m_l^* l}, \rho_{m_l^* l})$}, where, ρ_{il} = {1, 2} indicate the failure cause. The joint likelihood function at any stress level l of the observed data is given by

$$L_l = \prod_{i=1}^{m_l^*}\left\{\left[\frac{\partial C(u_1,u_2)}{\partial u_1}|_{u_1=S_{1l}(t_{il})} f_{1l}(t_{il})\right]^{I_1(\rho_{il}=1)} \times \left[\frac{\partial C(u_1,u_2)}{\partial u_2}|_{u_2=S_{2l}(t_{il})} f_{2l}(t_{il})\right]^{I_2(\rho_{il}=2)} S_l(t_{il})^{R_{il}}\right\} \times S_l(\tau_l^*)^{n_l - m_l^* - \sum_{i=1}^{m_l^*-1} R_{il}}. \quad (12)$$

And the joint likelihood function is given by

$$\mathbf{L} = \prod_{l=1}^{k} L_l \quad (13)$$

where

$$I_j(\rho_{il}) = \begin{cases} 1, \text{ if } \rho_{il} = j \\ 0, \text{ if } \rho_{il} \neq j \end{cases}. \quad (14)$$

Let $n_{lj} = \sum_{i=1}^{m_l^*} I_j(\rho_{il})$ denote to number of units fails under caused j and stress level $\mathbf{S}_l$.

Model Assumption

1. Over different stress levels $\mathbf{S}_l$, $l = 1,2,...,k$, there exist only two dependence competing failure modes.
2. The failure time T_{lj} under different stress level $\mathbf{S}_l$, $l = 1,2,...,k$ and cause j is distributed by GIED with shape and scale parameters λ_{lj} and α_{lj}, respectively with CDFs given by

$$F_{lj}(t\,|\alpha_{lj},\lambda_{lj}) = 1 - \left(1 - e^{-\frac{\lambda_{lj}}{t}}\right)^{\alpha_{lj}}, \alpha_{lj},\ \lambda_{lj} > 0,\ t > 0, \quad (15)$$

and the corresponding PDFs

$$f_{lj}(t\,|\alpha_{lj},\lambda_{lj}) = \frac{\alpha_{lj}\lambda_{lj}}{t^2} e^{-\frac{\lambda_{lj}}{t}} \left(1 - e^{-\frac{\lambda_{lj}}{t}}\right)^{\alpha_{lj}-1}. \quad (16)$$

3. The shape parameters under different stress level $\mathbf{S}_l$, $l = 1,2,...,k$ are equal and equal to α_j. ($\alpha_{1j} = \alpha_{2j} =$

$\ldots = \alpha_{kj} = \alpha_j,\ j = 1,2)$. Then, the survival function given by

$$S_l(t) = \left(\left(1-e^{-\frac{\lambda_{l1}}{t}}\right)^{\frac{-\alpha 1}{\theta}} + \left(1-e^{-\frac{\lambda_{l2}}{t}}\right)^{\frac{-\alpha 2}{\theta}} - 1\right)^{-\theta}. \tag{17}$$

4. The stress function $\phi(\mathbf{S}_l)$ of the j−th competing failure mode satisfies the log linear function of the parameter λ

$$\log\lambda_{lj} = a_j + b_j\phi(s_l),\ l = 0,1,\ldots,k;\ j = 1,2, \tag{18}$$

where $a_j, b_j > 0$ are unknown parameters.

4 ML Estimation

4.1 Point estimation

The likelihood function given by (12) with stress level $\mathbf{S}_l$, $l = 1,2,\ldots,k$ is reduced to

$$\begin{aligned} L_l =& \alpha_1^{n_{l1}}\alpha_2^{n_{l2}}\lambda_{l1}^{n_{l1}}\lambda_{l2}^{n_{l2}}\Bigg\{(1-e^{-\frac{\lambda_{l1}}{\tau_l^*}})^{-\frac{\alpha_1}{\theta}} \\ &+ (1-e^{-\frac{\lambda_{l2}}{\tau_l^*}})^{-\frac{\alpha_2}{\theta}} - 1\Bigg\}^{-\theta(n_l - m_l^* - \sum_{i=1}^{m_l^*-1} R_{il})} \\ &\times \prod_{i=1}^{m_l^*}\Bigg\{\left[\frac{e^{-\frac{\lambda_{l1}}{t_{il}}}}{t_{il}^2}(1-e^{-\frac{\lambda_{l1}}{t_{il}}})^{-\frac{\alpha_1}{\theta}-1}\right]^{I_1(\rho_{il})} \\ &\times\left[\frac{e^{-\frac{\lambda_{l2}}{t_{il}}}}{t_{il}^2}(1-e^{-\frac{\lambda_{l2}}{t_{il}}})^{-\frac{\alpha_2}{\theta}-1}\right]^{I_2(\rho_{il})} \\ &\times\Bigg[(1-e^{-\frac{\lambda_{l1}}{t_{il}}})^{-\frac{\alpha_1}{\theta}} \\ &+ (1-e^{-\frac{\lambda_{l2}}{t_{il}}})^{-\frac{\alpha_2}{\theta}} - 1\Bigg]^{-\theta(R_{il}+1)-1}\Bigg\}. \end{aligned} \tag{19}$$

Under the joint likelihood function given by (13) the log-likelihood function is

$$\begin{aligned} \ell =& \sum_{l=1}^{k}\log L_l = n_{l1}\log\alpha_1 + n_{l2}\log\alpha_2 + n_{l1}\log\lambda_{l1} \\ &+ n_{l2}\log\lambda_{l2} - \sum_{l=1}^{k}\Bigg\{\sum_{i=1}^{m_l^*}(\theta(R_{il}+1)+1) \\ &\times\log\left((1-e^{-\frac{\lambda_{l1}}{t_{il}}})^{-\frac{\alpha_1}{\theta}} + (1-e^{-\frac{\lambda_{l2}}{t_{il}}})^{-\frac{\alpha_2}{\theta}} - 1\right) \\ &+ \sum_{i=1}^{m_l^*} I_1(\rho_{il})\log\left[\frac{e^{-\frac{\lambda_{l1}}{t_{il}}}}{t_{il}^2}(1-e^{-\frac{\lambda_{l1}}{t_{il}}})^{-\frac{\alpha_1}{\theta}-1}\right] \\ &+ \sum_{i=1}^{m_l^*} I_2(\rho_{il})\log\left[\frac{e^{-\frac{\lambda_{l2}}{t_{il}}}}{t_{il}^2}(1-e^{-\frac{\lambda_{l2}}{t_{il}}})^{-\frac{\alpha_2}{\theta}-1}\right] \\ &- \theta(n_l - m_l^* - \sum_{i=1}^{m_l^*-1} R_{il})\log\Bigg[(1-e^{-\frac{\lambda_{l1}}{\tau_l^*}})^{-\frac{\alpha_1}{\theta}} \\ &+ (1-e^{-\frac{\lambda_{l2}}{\tau_l^*}})^{-\frac{\alpha_2}{\theta}} - 1\Bigg]\Bigg\}. \end{aligned} \tag{20}$$

After equating the first partially derivative of ℓ with respect to parameters vector $\alpha_1, \alpha_2, \lambda_{l1}, \lambda_{l2}, \theta, l = 1,2,\ldots k$ to zero, we get the likelihood equations as

$$\begin{aligned} \frac{\partial\ell}{\partial\alpha_1} =& \sum_{l=1}^{k}\Bigg\{\frac{n_{l1}}{\alpha_1} + \frac{1}{\theta}\sum_{i=1}^{m_l^*}(\theta(R_{il}+1)+1) \\ &\times\frac{(1-e^{-\frac{\lambda_{l1}}{t_{il}}})^{-\frac{\alpha_1}{\theta}}\log(1-e^{-\frac{\lambda_{l1}}{t_{il}}})}{(1-e^{-\frac{\lambda_{l1}}{t_{il}}})^{-\frac{\alpha_1}{\theta}} + (1-e^{-\frac{\lambda_{l2}}{t_{il}}})^{-\frac{\alpha_2}{\theta}} - 1} \\ &- \sum_{i=1}^{m_l^*}\frac{I_1(\rho_{il})}{\theta}\log(1-e^{-\frac{\lambda_{l1}}{t_{il}}}) \\ &+ \left(n_l - m_l^* - \sum_{i=1}^{m_l^*-1} R_{il}\right) \\ &\times\frac{(1-e^{-\frac{\lambda_{l1}}{\tau_l^*}})^{-\frac{\alpha_1}{\theta}}\log(1-e^{-\frac{\lambda_{l1}}{\tau_l^*}})}{(1-e^{-\frac{\lambda_{l1}}{\tau_l^*}})^{-\frac{\alpha_1}{\theta}} + (1-e^{-\frac{\lambda_{l2}}{\tau_l^*}})^{-\frac{\alpha_2}{\theta}} - 1}\Bigg\} = 0, \end{aligned} \tag{21}$$

$$\begin{aligned} \frac{\partial\ell}{\partial\alpha_2} =& \sum_{l=1}^{k}\Bigg\{\frac{n_{l2}}{\alpha_2} - \frac{1}{\theta}\sum_{i=1}^{m_l^*}(\theta(R_{il}+1)+1) \\ &\times\frac{(1-e^{-\frac{\lambda_{l2}}{t_{il}}})^{-\frac{\alpha_2}{\theta}}\log(1-e^{-\frac{\lambda_{l2}}{t_{il}}})}{(1-e^{-\frac{\lambda_{l1}}{t_{il}}})^{-\frac{\alpha_1}{\theta}} + (1-e^{-\frac{\lambda_{l2}}{t_{il}}})^{-\frac{\alpha_2}{\theta}} - 1} \\ &+ \sum_{i=1}^{m_l^*} -\frac{I_2(\rho_{il})}{\theta}\log(1-e^{-\frac{\lambda_{l2}}{t_{il}}}) \end{aligned}$$

$$+\left(n_l-m_l^*-\sum_{i=1}^{m_l^*-1}R_{il}\right)\times\frac{(1-e^{-\frac{\lambda_{l2}}{\tau_l^*}})^{-\frac{\alpha_2}{\theta}}\log(1-e^{-\frac{\lambda_{l2}}{\tau_l^*}})}{(1-e^{-\frac{\lambda_{l1}}{\tau_l^*}})^{-\frac{\alpha_1}{\theta}}+(1-e^{-\frac{\lambda_{l2}}{\tau_l^*}})^{-\frac{\alpha_2}{\theta}}-1}\Bigg\}=0, \quad (22)$$

$$\frac{\partial\ell}{\partial\lambda_{l1}}=\sum_{l=1}^{k}\Bigg\{\frac{n_{l1}}{\lambda_{l1}}+\sum_{i=1}^{m_l^*}\left(\theta\left(R_{il}+1\right)+1\right)\times\frac{\frac{\alpha_1}{\theta}(1-e^{-\frac{\lambda_{l1}}{t_{il}}})^{-\frac{\alpha_1}{\theta}-1}e^{-\frac{\lambda_{l1}}{t_{il}}}}{t_{il}\left((1-e^{-\frac{\lambda_{l1}}{t_{il}}})^{-\frac{\alpha_1}{\theta}}+(1-e^{-\frac{\lambda_{l2}}{t_{il}}})^{-\frac{\alpha_2}{\theta}}-1\right)}$$
$$-\sum_{i=1}^{m_l^*}\frac{I_1(\rho_{il})}{t_{il}}-\sum_{i=1}^{m_l^*}\frac{I_2(\rho_{il})\left(\frac{\alpha_1}{\theta}+1\right)e^{-\frac{\lambda_{l1}}{t_{il}}}}{t_{il}(1-e^{-\frac{\lambda_{l1}}{t_{il}}})}$$
$$+\left(n_l-m_l^*-\sum_{i=1}^{m_l^*-1}R_{il}\right)\frac{\alpha_1}{\tau_l^*}\times\frac{(1-e^{-\frac{\lambda_{l1}}{\tau_l^*}})^{-\frac{\alpha_1}{\theta}-1}e^{-\frac{\lambda_{l1}}{\tau_l^*}}}{(1-e^{-\frac{\lambda_{l1}}{\tau_l^*}})^{-\frac{\alpha_1}{\theta}}+(1-e^{-\frac{\lambda_{l2}}{\tau_l^*}})^{-\frac{\alpha_1}{\theta}}-1}\Bigg\}=0, \quad (23)$$

and

$$\frac{\partial\ell}{\partial\lambda_{l2}}=\sum_{l=1}^{k}\Bigg\{\frac{n_{l2}}{\lambda_{l2}}+\frac{\alpha_2}{\theta}\sum_{i=1}^{m_l^*}\frac{\theta\left(R_{il}+1\right)+1}{t_{il}}\times\frac{\frac{\alpha_2}{\theta}(1-e^{-\frac{\lambda_{l2}}{t_{il}}})^{-\frac{\alpha_2}{\theta}-1}e^{-\frac{\lambda_{l2}}{t_{il}}}}{(1-e^{-\frac{\lambda_{l1}}{t_{il}}})^{-\frac{\alpha_1}{\theta}}+(1-e^{-\frac{\lambda_{l2}}{t_{il}}})^{-\frac{\alpha_2}{\theta}}-1}$$
$$-\sum_{i=1}^{m_l^*}\frac{I_2(\rho_{il})}{t_{il}}-\sum_{i=1}^{m_l^*}\frac{I_2(\rho_{il})\left(\frac{\alpha_2}{\theta}+1\right)e^{-\frac{\lambda_{l2}}{t_{il}}}}{t_{il}(1-e^{-\frac{\lambda_{l2}}{t_{il}}})}$$
$$+\frac{\alpha_2\left(n_l-m_l^*-\sum_{i=1}^{m_l^*-1}R_{il}\right)}{\tau_i^*}\times\frac{(1-e^{-\frac{\lambda_{l2}}{\tau_l^*}})^{-\frac{\alpha_2}{\theta}-1}e^{-\frac{\lambda_{l2}}{\tau_l^*}}}{(1-e^{-\frac{\lambda_{l1}}{\tau_l^*}})^{-\frac{\alpha_1}{\theta}}+(1-e^{-\frac{\lambda_{l2}}{\tau_l^*}})^{-\frac{\alpha_2}{\theta}}-1}\Bigg\}=0. \quad (24)$$

The equations (21), (22), (23) and (24) cannot be solved analytically for α_1, α_2, $\lambda_{l1},\lambda_{l2}$, $l=1,2,\ldots k$. Iterative techniques such as Newton Raphson can be used to obtain the estimates $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\lambda}_{l1},\hat{\lambda}_{l2}$.

4.2 Approximate confidence intervals (ACIs)

The property of asymptotic normality distribution of the MLEs can be used to constructed confidence bounds for the parameter values. It is known that the asymptotic distribution of the MLE of α_1, α_2, λ_{l1}, and λ_{l2}, $l=1,2,\ldots k$ under some regularity conditions is approximately distributed as bivariate normal: $(\varphi_{ML}-\varphi)\longrightarrow N_2(0,I^{-1}(\varphi))$, where $I^{-1}(\varphi)$ is the inverse form of the observed information matrix of the unknown parameters $\varphi=(\alpha_1,\alpha_2,\lambda_{l1},\lambda_{l2})$, given as

$$I^{-1}(\varphi)=\begin{pmatrix} I_{11}\cdots & I_{1k} & I_{1(k+1)} & I_{1(k+2)}\\ \vdots & \vdots & \vdots & \vdots\\ I_{k1}\cdots & I_{kk} & I_{k(k+1)} & I_{k(k+2)}\\ I_{(k+1)1}\cdots & I_{(k+1)k} & I_{(k+1)(k+1)} & I_{(k+1)(k+2)}\\ I_{(k+2)1}\cdots & I_{(k+2)k} & I_{(k+2)(k+1)} & I_{(k+2)(k+2)}\end{pmatrix}^{-1}, \quad (25)$$

where the elements of the observed information matrix is negative of the second partial derivatives of (20) with respected to parameters vector, where

$$I_{il}=-\left(\frac{\partial^2\ell}{\partial\lambda_{ij}\partial\lambda_{lj}}\right),\ i,\ l=1,\ 2,\ \ldots,\ k,\ j=1,\ 2. \quad (26)$$

$$I_{i(k+1)}=I_{(k+1)i}=-\left(\frac{\partial^2\ell}{\partial\lambda_{ij}\partial\alpha_1}\right),\ i=1,\ 2,\ \ldots,\ k. \quad (27)$$

$$I_{i(k+2)}=I_{(k+2)i}=-\left(\frac{\partial^2\ell}{\partial\lambda_{ij}\partial\alpha_2}\right),i=1,\ 2,\ \ldots,\ k. \quad (28)$$

and

$$I_{(k+1)(k+1)}=-\left(\frac{\partial^2\ell}{\partial\alpha_1\partial\alpha_1}\right),I_{(k+2)(k+2)}=-\left(\frac{\partial^2\ell}{\partial\alpha_2\partial\alpha_2}\right). \quad (29)$$

In different cases more serious obtaining the minus expectation of second derivative of log likelihood. Therefore, we applied the approximate information matrixA $100(1-\gamma)\%$ two-sided approximate confidence intervals for the parameters α_1, α_2, λ_{l1}, and λ_{l2}, $i=1,2,\ldots k$ can be given by

$$\hat{\alpha}_1\mp Z_{\gamma/2}\sqrt{\mathrm{var}(\hat{\alpha}_1)}\ \text{and}\ \hat{\alpha}_2\mp Z_{\gamma/2}\sqrt{\mathrm{var}(\hat{\alpha}_2)}. \quad (30)$$

and

$$\hat{\lambda}_{l1}\mp Z_{\gamma/2}\sqrt{\mathrm{var}(\hat{\lambda}_{l1})}\ \text{and}\ \hat{\lambda}_{l2}\mp Z_{\gamma/2}\sqrt{\mathrm{var}(\hat{\lambda}_{l2})}, \quad (31)$$

$Z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail probability $\gamma/2$.

4.3 Reliability estimation

The estimate of the reliability of the system, at any time t is given by

$$\hat{S}_0(t)=\left(\left(1-e^{-\frac{\hat{\lambda}_{01}}{t}}\right)^{\frac{-\hat{\alpha}_1}{\theta}}+\left(1-e^{-\frac{\hat{\lambda}_{02}}{t}}\right)^{\frac{-\hat{\alpha}_2}{\theta}}-1\right)^{-\theta}. \quad (32)$$

The estimates of the lifetime $\hat{\lambda}_{0j}$ under normal stress S_0 is $log\hat{\lambda}_{0j} = \hat{a}_j + \hat{b}_j\phi(s_0)$.
Putting $\hat{\lambda}_{lj}$, $j = 1,2$ into eq(18), we used the least-square method to obtain the estimates of a_j, b_j

$$\begin{cases} \hat{a}_j = \dfrac{\sum_{l=1}^{k} \ln \hat{\lambda}_{lj} - \hat{b}_j \sum_{l=1}^{k} \phi(s_l)}{k}, \\ \hat{b}_j = \dfrac{k\sum_{l=1}^{k} \ln \hat{\lambda}_{lj}\phi(s_l) - \sum_{l=1}^{k} \ln \hat{\lambda}_{lj} \sum_{l=1}^{k} \phi(s_l)}{k\sum_{l=1}^{k} \phi^2(s_l) - \left(\sum_{l=1}^{k} \phi(s_l)\right)^2}. \end{cases} \tag{33}$$

5 Bootstrap Confidence Interval

The bootstrap technique, not only used in parameters estimation with point and interval estimate, but can be used to estimate the bias and variance of an estimators as well as calibrate hypothesis testing. The bootstrap technique as given in [25] and [26] has the parametric or nonparametric forms. In this section, we applied the parametric bootstrap technique to built confidence intervals, for more detail see [27] and [28]. The bootstrap confidence intervals can be obtained according the following steps

Step 1:For given type-I progressive hybrid censored sample $\underline{\mathbf{t}}_l$={ (t_{1l}, ρ_{1l}), (t_{2l}, ρ_{2l}), ..., $(t_{m_l^* l}, \rho_{m_l^* l})$} with $\mathbf{S}_l$, n_l, m_l, τ_l and $\mathbf{R}_l$, $l = 1, 2, ..., k$, compute the MLEs $\hat{\lambda}_{lj}$, $\hat{\alpha}_j$, $j = 1,2$.

Step 2:Under the values $\hat{\lambda}_{lj}$, $\hat{\alpha}_j$, $j = 1,2$ and censoring parameters n_l, m_l, τ_l and $\mathbf{R}_l$, $l = 1, 2, ..., k$, generate a type-I progressively hybrid censored sample $\underline{\mathbf{t}}_l^*$={ (t_{1l}^*, ρ_{1l}), (t_{2l}^*, ρ_{2l}), ..., $(t_{m_l^* l}^*, \rho_{m_l^* l})$}.

a1. Generate a random sample $w_{1j}, \ldots, w_{m_l j}$, $j = 1,2$ from Uniform distribution U(0, 1). Let $v_{ij} = w_{ij}^{1/(i+R_{im_l}+R_{im_l-1}+\cdots+R_{im_l-i+1})}$
$U_{ij} = 1 - v_{m_l}v_{m_l-1}...v_{m_l-i+1}, i = 1,...,m_l$ be uniform progressive type-II order statistics.

a2. We obtain the failures m_l^* before time τ_l and the terminal time τ_l^*.

If $U_{m_l j} \leq 1 - \left(1 - e^{-\frac{\lambda_{lj}}{\tau_l}}\right)^{\alpha_j}$, $m_l^* = m_l, \tau_l^* = \dfrac{-\lambda_{lj}}{\log\left(1 - \left(1 - U_{m_l j}\right)^{\frac{1}{\alpha_j}}\right)}$;

If $U_{m_l j} > 1 - \left(1 - e^{-\frac{\lambda_{lj}}{\tau_l}}\right)^{\alpha_j}$, $m_l^* = J_l, \tau_l^* = \tau_l$, where J_l is obtained from the inequality

$$U_{J_l j} < 1 - \left(1 - e^{-\frac{\lambda_{lj}}{\tau_l}}\right)^{\alpha_j} < U_{(J_l+1)j},$$

for $1 \leq i \leq m_l^*$, $l = 1, 2, ..., k$ we set $t_{ilj}^* = \dfrac{-\lambda_{ij}}{\log\left(1 - \left(1 - U_{ij}\right)^{\frac{1}{\alpha_j}}\right)}$, where t_{il}^* =min(t_{il1}^*, t_{il2}^*)

Step 3:Based on n_l, m_l^*, τ_l^*, $\mathbf{R}_l$ and $\underline{\mathbf{t}}_l^*$={ (t_{1l}^*, ρ_{1l}), (t_{2l}^*, ρ_{2l}), ..., $(t_{m_l^* l}^*, \rho_{m_l^* l})$}, we compute the MLEs $\hat{\lambda}_{lj}^*$, $\hat{\alpha}_j^*$.

Step 4: Repeat steps 2–3 N times, we obtain N estimates $\left\{\hat{\lambda}_{lj}^{*\eta}, \hat{\alpha}_j^{*\eta}\right\}$ $(\eta = 1,...,N)$. Arrange them in ascending order to obtain the bootstrap sample $\left\{\hat{\lambda}_{lj}^{*[1]},...,\hat{\lambda}_{lj}^{*[N]}; \hat{\alpha}_j^{*[1]},...,\hat{\alpha}_j^{*[N]}\right\}$.
The approximate $100(1-\gamma)\%$ confidence interval for parameters $\hat{\lambda}_{lj}$, $\hat{\alpha}_j$, $l = 1,2,...,k$; $j = 1,2$ are given, respectively, by

$$\left(\hat{\lambda}_{lj}^{*[N\,\gamma/2]}, \hat{\lambda}_{lj}^{*[N\,(1-\gamma/2)]}\right)$$
$$\left(\hat{\alpha}_j^{*[N\,\gamma/2]}, \hat{\alpha}_j^{*[N\,(1-\gamma/2)]}\right).$$

6 Data Analysis

6.1 Example 1:

In this section, a real life data from [29] is used to illustrate the proposed model. Under the laboratory experiment, the observed data present the lifetime of two groups of RFM strain male mice putted under a radiation dose of 300r at an age of 5-6 weeks. The conventional laboratory environment is applied on the first group of mice but, the germ-ree environment is applied on the second group. We consider that, only two major causes of death Thymic Lymphoma and combined all the other causes into a single group. The data are displayed in Table 1. We use transform for this data $\left(\frac{\text{data}}{200}\right)^{\frac{1}{2}}$, the new data by using Kolmogorov–Smirnov (K-S) test, the (K-S) of the case 1 in Data 1 is 0.1845, the (K-S) of the case 2 in Data 1 is 0.1698, the (K-S) of the case 1 in Data 2 is 0.1772 and the (K-S) of the case 2 in Data 2 is 0.1849 see figures (1-4). So, the dependent distributions considered for two risk factors of the device failure may follow generalized inverted exponential distribution at significance level take the value 0.05. The misleading results will be obtained under consideration dependent structure of two competing risks. Hence, we used this structure between capacitor failure and controller failure under Bivariate Pareto Copula (BPC). The Type-I PHCS are generated from the original data in Table 2, with n_1=61, n_2=67, m_1=40, m_2=40, $\tau_1 = 1.8$, $\tau_2 = 2.1$, R_1 = $(10^1, 0^4, 10^1, 0^4, 1^1, 0^{29})$, and R_2 = $(10^1, 0^4, 10^1, 0^4, 7^1, 0^{29},)$. Let the parameter of Bivariate Pareto Copula $\theta = 2$ or equivalently the Kendall's τ association $\tau = 1/5$. The estimates of the unknown parameters, confidence intervals and bootstrap CIs using Bivariate Pareto Copula (BPC) are obtained and reported in Table 3.

Table 1: Autopsy data for 61 RFM conventional male mice which received a radiation dose of 300r at age 5-6 weeks

Data 1:															
case 1	159	189	191	198	200	207	220	235	245	250	256	261	265	266	280
	343	356	383	403	414	428	432								
case 2	40	42	51	62	163	179	206	222	228	252	249	282	324	333	341
	366	385	407	420	431	441	461	462	482	517	517	524	564	567	586
	619	620	621	622	647	651	686	761	763						
Data 2:															
case 1	158	192	193	194	195	202	212	215	229	230	237	240	244	247	259
	300	301	321	337	415	434	444	485	496	529	537	624	707	800	
case 2	136	246	255	376	421	565	616	617	652	655	658	660	662	675	681
	734	736	737	757	769	777	800	807	825	855	857	864	868	870	870
	873	882	895	910	934	942	1015	1019							

Table 2: Observed Type-I PHC data sets, for two levels of stress

$\mathbf{S_1}$	0.4472 2	0.4583 2	0.5050 2	0.5568 2	0.8916 1	0.9028 2	0.9460 2	0.9721 1
	0.9950 1	1 1	1.0149 2	1.0677 2	1.0840 1	1.1158 2	1.1180 1	1.1314 1
	1.1424 1	1.1511 1	1.1832 1	1.1874 2	1.2728 2	1.2904 2	1.3058 2	1.3096 1
	1.3874 2	1.4195 1	1.4265 2	1.4388 1	1.4629 1	1.4680 2	1.4697 1	1.5182 2
	1.6186 2	1.6793 2	1.7593 2	1.7621 2	1.7986 2			
$\mathbf{S_2}$	0.824621 2	0.8888 1	0.9798 1	0.9823 1	0.9849 1	1.005 1	1.0296 1	1.0368 1
	1.07005 1	1.0724 1	1.0886 1	1.0955 1	1.1045 1	1.1091 2	1.1113 1	1.1380 1
	1.2669 1	1.2981 1	1.3711 2	1.4731 1	1.6808 2	1.8138 2	1.8166 2	1.8193 2
	1.8371 2	1.8802 1	1.9157 2	1.9710 2	2 1	2.0087 2	2.0310 2	2.070 2
	2.0857 2	2.0857 2	2.0893 2					

Table 3. MLEs and 95% CIs of the parameters

Par	MLE	ACI	Boot-P	Boot-P CI
α_1	8.485	(1.0903,15.8797)	9.0830	(2.8765,16.3486)
α_2	26.8558	(0.7284,52.9831)	23.7927	(3.5443,50.3531)
λ_{11}	4.1772	(2.9689,5.3855)	5.4214	(2.3467,5.4798)
λ_{12}	5.00501	(3.6840,6.3261)	4.7026	(2.9895,6.4512)
λ_{21}	4.65926	(3.3145,6.0040)	5.1322	(3.5447,6.2312)
λ_{22}	6.98344	(5.2182,8.7487)	6.9669	(4.1456,8.9845)

6.2 Example 2:

The proposed model and the corresponding developed methods are tested through numerical example considered under two-levels of constant stress ALT with two dependent competing failure modes. Suppose that, experiment are done under Type-I PHCS, the stress is taken under temperature: $s_1 = 30°C = 303K$, $s_2 = 60°C = 333K$ and the normal stress level is taken under $s_0 = 5°C = 278K$. The accelerated function is defined by $\varphi(s) = 1/s$. Censoring scheme is presented with, sample size under each stress level s_i, i = 1, 2 as $n_1 = n_2 = 50$, $m_1 = m_2 = 40$, the pre-fixed sampling scheme $R_1 = (1^{10}, 0^{30})$, $R_2 = (1^{10}, 0^{30})$. Let the parameter of Bivariate Pareto Copula $\theta = 2$ or equivalently the Kendall's τ association $\tau = 1/5$, the terminal times are $(\tau_1 = \tau_2 = 130)$. The initial values for the shape parameters are $\alpha_1 = 0.6$, $\alpha_2 = 0.8$, and the the scale parameters take the values for λ_{ij} are given by $\log \lambda_{ij} = a_j + b_j \phi(s_i)$, $i = 1,2$; $j = 1,2$, where $a_1 = -5$, $a_2 = -8$, $b_1 = 1600$, and $b_2 = 2700$. The value of λ_{ij}, $i = 1,2$; $j = 1,2$ is given by: $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) =$

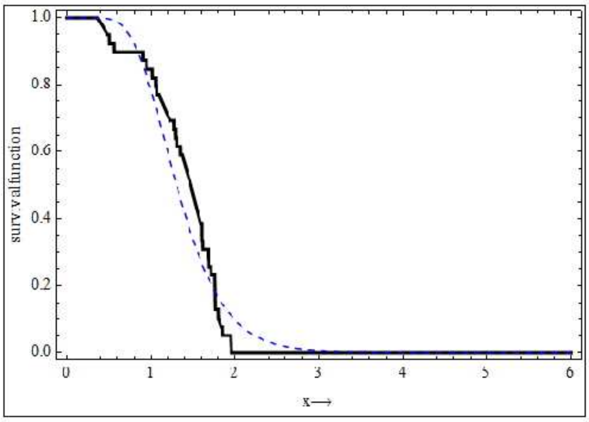

Fig. 1: Empirical and fitted survival functions of the case1 in data1

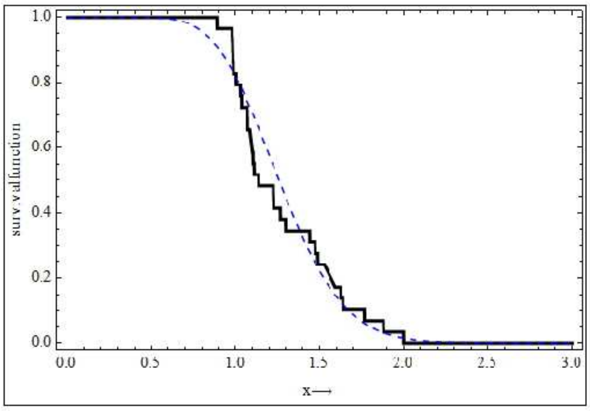

Fig. 3: Empirical and fitted survival functions of the case1 in data2

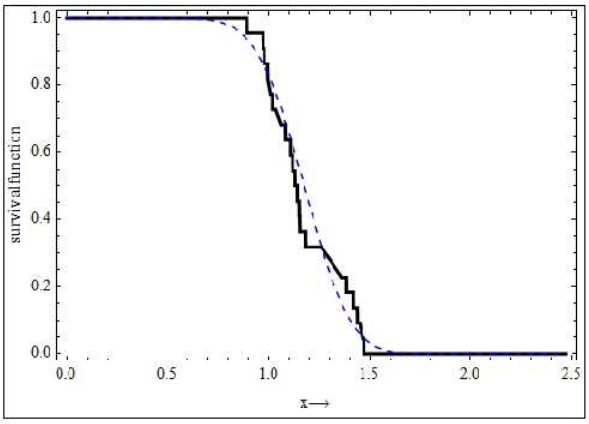

Fig. 2: Empirical and fitted survival functions of the case2 in data1

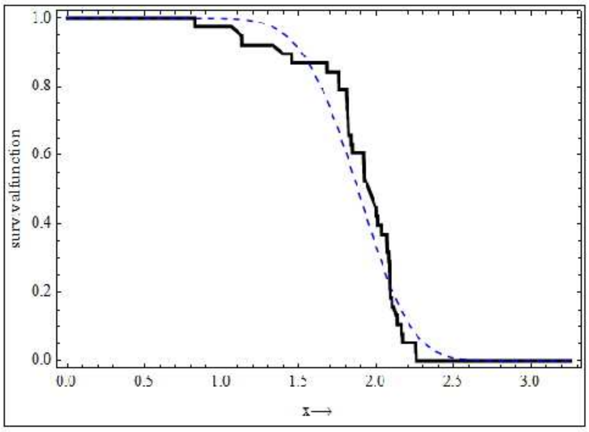

Fig. 4: Empirical and fitted survival functions of the case2 in data2

$(1.3238, 2.4865, 0.82267, 1.114168)$.
The Type-I progressively hybrid censoring sample is generated according to [30], for two competing failure mode from generalized inverted exponential distribution GIE(λ_{ij}, α_{ij}). Table 4 has preseted the simulated data sets and the point estimates and the corresponding confidence intervals of the model parameters are shown in Table 5. Then we can calculate the estimates of the acceleration coefficients a_1, a_2, b_1 and b_2 by using Equation (33), $(\hat{a}_1, \hat{b}_1, \hat{a}_2, \hat{b}_2) = (-7.47, 2347, -7.99, 2686)$ and the reliability estimates under normal stress, by using Equation (32) $\hat{S}_0(t = 2) = 0.8814$

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7 Simulation study

In this section, we adopted Type-I PHCS scheme, two-levels of constant stress ALT and two dependent competing failure modes. Firstly, stress is taken temperature with s_1 = 30°C = 303 K, s_2 = 60°C =333 K. The sample size under each stress level s_i, i = 1, 2 is taken to be $n_1 = n_2$ = 40, 50, 60, 70, m_1 = 30, 40, 50, 60. m_2 =25, 40, 45, 55. The initial values for the shape parameters are $\alpha_1 = 0.6$, $\alpha_2 = 0.8$, and the scale parameters λ_{ij} is given by $log\lambda_{ij} = a_j + bj\phi(si)$, $i = 1,2,3$, $j = 1,2$, where $a_1 = -5$, $a_2 = -8$, $b_1 = 1600$, and $b_2 = 2700$. The value of λ_{ij}, $i = 1,2$; $j = 1,2$ is given by $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) = (1.3238, 2.4865, 0.82267, 1.114168)$.Type-I progressively hybrid censoring sample is generated according to [30], for competing failure mode j, $j = 1,2$ from generalized inverted exponential distribution GIE(λ_{ij}, α_j). We are repeated this process

Table 4: The generated Type-I PHC data sets for two levels of stress

S$_1$	0.4444	0.5760	0.6343	0.8116	0.9179	1.02417	1.0263	1.2658
	1	2	2	2	2	1	1	1
	1.2878	1.3369	1.3588	1.5043	1.5882	1.8571	2.5559	2.5761
	1	1	1	1	1	1	2	2
	2.9775	3.0408	3.04256	3.2826	3.3682	3.7983	6.3784	6.9868
	2	2	2	2	2	2	1	1
	7.1608	7.5791	9.3350	9.5575	13.7002	18.7324	21.5904	23.937
	1	1	1	1	1	2	1	1
	25.305	51.9586	68.0655	91.7614	103.868	120.43		
	1	2	2	2	2	2		
S$_2$	0.1796	0.1938	0.3345	0.3754	0.4133	0.4218	0.5730	0.6611
	1	1	1	1	2	2	1	1
	0.6973	0.7138	0.7604	0.8330	0.9403	1.01419	1.11729	1.4999
	2	2	2	2	2	2	2	2
	1.6666	1.7452	1.9436	2.1160	2.2764	2.43873	2.5243	2.5677
	1	1	2	2	1	2	1	2
	2.7732	2.7954	2.9198	3.0260	3.040	3.99007	5.5222	7.7276
	2	2	2	2	2	2	2	2
	9.8888	10.9556	15.0169	21.6692	30.7255	62.1754	73.2467	
	2	2	2	1	1	2	1	

Table 5. MLEs and 95% CIs of the parameters

Par	MLE	ACI	Boot-P	Boot-P CI
α_1	0.2949	(0.1637,0.6958)	0.3120	(0.2027,0.6425)
α_2	0.5034	(0.2787,0.9054)	0.6585	(0.3147,1.0806)
λ_{11}	1.4419	(0.5534,2.2436)	1.8960	(0.9690,3.6945)
λ_{12}	2.3973	(0.8285,2.4063)	2.9056	(1.6272,6.7192)
λ_{21}	0.7118	(0.3066,1.1135)	0.6845	(0.3783,1.1033)
λ_{22}	1.0786	(0.4825,1.4833)	1.8422	(0.8791,6.5208)

Table 6. Mean estimates and MSEs of the parameters with $\theta = 1$, $\alpha_1 = 0.6$, $\alpha_2 = 0.8$ and $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) = (1.324, 2.4865, 0.8227, 1.1142)$

n_1	n_2	m_1	m_2	scheme	λ_{11}	λ_{12}	λ_{21}	λ_{22}	α_1	α_2
40	40	30	25	$R_1 = 5^1, 0^3, 5^1, 0^{25}$	1.2395	3.2288	0.83	1.9198	0.4611	0.7296
				$R_2 = 5^1, 0^3, 5^1, 0^4, 5^1, 0^{15}$	0.2781	0.9057	0.1269	0.5878	0.0373	0.0381
				$R_1 = 1^{10}, 0^{20}$	1.203	3.4243	0.8468	1.8095	0.446	0.7218
				$R_2 = 1^{15}, 0^{10}$	0.2429	0.6692	0.1852	0.5118	0.0406	0.0374
				$R_1 = 0^{20}, 1^{10}$	1.303	3.4343	0.8568	1.7995	0.486	0.7328
				$R_2 = 0^{10}, 1^{15}$	0.2829	0.7692	0.1975	0.618	0.0506	0.0398
50	50	40	40	$\mathrm{R}_1 = 5^1, 0^3, 5^1, 0^{35}$	1.0927	2.8195	0.7674	2.1459	0.407	0.7389
				$R_2 = 5^1, 0^3, 5^1, 0^{35}$	0.3413	0.8753	0.0497	0.4999	0.0478	0.0291
				$R_1 = 1^{10}, 0^{30}$	1.0362	2.3684	0.7935	2.029	0.3827	0.6919
				$R_2 = 1^{10}, 0^{30}$	0.2220	0.6541	0.1768	0.474	0.0581	0.0326
				$R_1 = 0^{30}, 1^{10}$	1.0498	2.6412	0.7775	1.8387	0.3716	0.6586
				$R_2 = 0^{30}, 1^{10}$	0.3014	0.6929	0.1045	0.4885	0.0614	0.0429
60	60	50	45	$\mathrm{R}_1 = 5^1, 0^3, 5^1, 0^{45}$	1.1324	3.2587	0.8502	1.8964	0.4143	0.7035
				$R_2 = 5^1, 0^3, 5^1, 0^4, 5^1, 0^{35}$	0.18483	0.7816	0.2299	0.3936	0.0440	0.0349
				$R_1 = 1^{10}, 0^{40}$	1.0436	3.262	0.7948	2.1953	0.4201	0.7558
				$R_2 = 1^{15}, 0^{30}$	0.1635	0.5807	0.1067	0.3894	0.0439	0.0373
				$R_1 = 0^{40}, 1^{10}$	0.9773	2.8986	0.7688	1.9993	0.3248	0.5342
				$R_2 = 0^{30}, 1^{15}$	0.2239	0.6827	0.1683	0.3947	0.0804	0.0836
70	70	60	55	$R_1 = 5^1, 0^3, 5^1, 0^{55}$	0.9787	3.6447	0.8794	2.0552	0.4081	0.7562
				$R_2 = 5^1, 0^3, 5^1, 0^4, 5^1, 0^{45}$	0.1796	0.4709	0.1537	0.2957	0.0507	0.0346
				$R_1 = 1^{10}, 0^{50}$	0.9907	3.5417	0.8184	2.0552	0.4081	0.7562
				$R_2 = 1^{15}, 0^{40}$	0.1596	0.3709	0.1337	0.2557	0.0408	0.034
				$R_1 = 0^{50}, 1^{10}$	0.9938	5.2227	0.653	1.8881	0.3126	0.5563
				$R_2 = 0^{40}, 1^{15}$	0.2006	0.5496	0.1729	0.2972	0.0867	0.0830

Table 7. Coverage percentages (95%) and Average Length of the parameters with, $\theta = 1, \alpha_1 = 0.6$, $\alpha_2 = 0.8$ and $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) = (1.324, 2.4865, 0.8227, 1.1142)$

m_1	m_2	scheme	λ_{11}		λ_{12}		λ_{21}		λ_{22}		α_1		α_2	
$n_1 = n_2 = 40$			ACI	Boot	ACI	Boot	ACI	Boot	ACI	Boot	ACI	Boot	ACI	Boot
30	25	$R_1=5^1,0^3,5^1,0^{25}$	0.88	0.99	0.87	0.98	0.89	0.96	0.88	0.95	0.88	0.95	0.90	0.96
		$R_2= 5^1,0^3,5^1,0^4,5^1,0^{15}$	1.4128	1.4358	3.7999	6.5182	0.9309	1.0281	2.3402	5.4659	0.4154	0.3368	0.6925	1.3022
		$R_1=1^{10},0^{20}$	0.90	1	0.89	0.94	0.88	1	0.92	0.95	0.89	0.92	0.88	0.93
		$R_2=1^{15},0^{10}$	1.3665	1.4312	4.4005	6.0036	0.9540	1.0599	2.1232	5.2029	0.4087	0.3305	0.7002	1.3028
		$R_1=0^{20},1^{10}$	0.89	0.97	0.88	0.96	0.89	0.96	0.88	0.96	0.89	0.95	0.90	0.96
		$R_2=0^{10},1^{15}$	1.4328	1.4758	3.8999	6.7182	0.9609	1.3281	2.6402	5.7659	0.4754	0.3868	0.7925	1.4022
$n_1=n_2=50$														
40	40	$R_1=5^1,0^3,5^1,0^{35}$	0.90	0.95	0.89	0.94	0.95	0.96	0.92	0.95	0.89	0.92	0.88	0.93
		$R_2=5^1,0^3,5^1,0^{35}$	1.1639	1.1164	2.745	14.7188	0.7776	0.8132	2.1559	6.1392	0.3081	0.2351	0.5617	1.1506
		$R_1=1^{10},0^{30}$	0.89	0.94	0.90	0.94	0.89	0.96	0.90	0.94	0.89	0.92	0.89	0.93
		$R_2=1^{10},0^{30}$	1.1279	0.9589	2.2399	15.2324	0.8292	0.8491	2.0359	6.1890	0.2976	0.2164	0.5169	1.0313
		$R_1=0^{30},1^{10}$	0.89	0.97	0.87	0.96	0.89	0.96	0.88	0.95	0.88	0.95	0.90	0.96
		$R_2=0^{30},1^{10}$	1.1359	1.0870	2.6095	13.6752	0.8028	0.7387	1.7769	5.7282	0.2876	0.2061	0.5837	1.2637
$n_1=n_2=60$														
50	45	$R_1=5^1,0^3,5^1,0^{45}$	0.88	1	0.89	0.95	0.88	0.90	0.89	0.96	0.88	0.95	0.90	0.97
		$R_2= 5^1,0^3,5^1,0^4,5^1,0^{35}$	1.093	0.9761	3.0850	5.5025	0.8200	0.6982	1.7435	5.5777	0.2869	0.2305	0.4975	0.9795
		$R_1=1^{10},0^{40}$	0.89	0.97	0.87	0.94	0.88	0.97	0.89	0.96	0.88	0.95	0.89	0.95
		$R_2=1^{15},0^{30}$	0.9853	0.9552	2.9672	3.7091	0.7189	0.6583	2.0667	4.9105	0.2854	0.2387	0.5406	1.0511
		$R_1=0^{40},1^{10}$	0.88	0.97	0.87	0.95	0.88	0.96	0.89	0.97	0.88	0.95	0.89	0.97
		$R_2=0^{30},1^{15}$	0.9881	0.9834	2.8403	5.1651	0.7729	0.8073	1.9885	6.7827	0.2219	0.1503	0.4059	0.5349
$n_1=n_2=70$														
60	55	$R_1=5^1,0^3,5^1,0^{55}$	0.93	0.96	0.95	0.95	0.93	0.97	0.95	0.95	0.94	0.93	0.95	0.95
		$R_2= 5^1,0^3,5^1,0^4,5^1,0^{45}$	0.9202	0.9603	3.0656	3.9717	0.8171	0.6219	1.7378	4.9818	0.1947	0.1504	0.4881	0.7414
		$R_1=1^{10},0^{50}$	0.94	0.95	0.93	0.94	0.95	0.96	0.96	0.97	0.95	0.95	0.93	0.95
		$R_2=1^{15},0^{40}$	0.8754	0.8368	2.9860	3.1902	0.6881	0.5855	1.7745	4.5072	0.2556	0.215	0.4826	0.9072
		$R_1=0^{50},1^{10}$	0.92	0.96	0.95	0.95	0.94	0.95	0.95	0.96	0.94	0.95	0.92	0.95
		$R_2=0^{40},1^{15}$	0.9402	1.0270	5.1476	3.8017	0.5871	0.6319	1.6378	4.9519	0.1847	0.1305	0.388	0.4741

500 times for different sampling schemes and different dependence structures. The Gumbel copula parameters is taken to be $\theta = 1$, 2 or equivalently the Kendall's τ association $\tau = 1/3$, $1/5$. The estimated average mean of the parameters and the MSEs are shown in Tables 6 and 8. The coverage probabilities and the mean of lenght of the ACIs and Boot-P CIs with parameters are shown in Tables 7 and 9.

8 Conclusions

In this paper, we considered statistical inference under copula approach of accelerated dependent generalized inverted exponential failure time with progressive hybrid censoring scheme. We are derived the ML and bootstrap estimators. Also, we build different confidence intervals using asymptotic distribution of the MLEs and Bootstrap confidence intervals. For illustrated purposes, we introduce a numerical examples are given and Monte Carlo simulation studies are conducted. From the results, we observe the following points

1. When the effective sample sizes (m_j) are increase the MSEs of the all estimates are decrease.
2. The MSEs are decrease for stronger dependence of competing failure modes and the corresponding MLEs become closer to the true values and the results with $\theta = 2$ is better than the results with $\theta = 1$. these results have shown that, the problem of dependence structure is more important in studying competing risks model.
3. Finally, we observe that, when the sample size is small, the coverage percentages of ACIs are always less than the nominal level. Also, we observe the coverage percentages of ACIs for sample size large is improved.

Hence, we say that the values of ACIs are considered for a large sample size. And, generally coverage percentages of the Boot-P CIs closer to the nominal level. Then, coverage percentages is improved for larger sample size. Therefore, especially in hard to get the exact CIs, the Boot-P CIs have good stability with satisfactory coverage percentages.

Table 8. Mean estimates and MSEs of the parameters with $\theta = 2$, $\alpha_1 = 0.6$, $\alpha_2 = 0.8$ and $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) = (1.324, 2.4865, 0.8227, 1.1142)$

n_1	n_2	m_1	m_2	scheme	λ_{11}	λ_{12}	λ_{21}	λ_{22}	α_1	α_2
40	40	30	25	$R_1 = 5^1, 0^3, 5^1, 0^{25}$	1.1372	4.3223	0.7858	2.0825	0.3935	0.7009
				$R_2 = 5^1, 0^3, 5^1, 0^4, 5^1, 0^{15}$	(0.1878)	(0.2419)	(0.0840)	(0.2231)	(0.0549)	(0.0505)
				$R_1 = 1^{10}, 0^{20}$	1.2063	4.4108	0.8945	2.0235	0.4192	0.7161
				$R_2 = 1^{15}, 0^{10}$	(0.1899)	(0.4194)	(0.2915)	(0.3634)	(0.0474)	(0.0536)
				$R_1 = 0^{20}, 1^{10}$	1.3096	4.4909	0.8705	1.7944	0.4172	0.6637
				$R_2 = 0^{10}, 1^{15}$	(0.6847)	(0.578)	(0.2137)	(0.3811)	(0.0475)	(0.0628)
50	50	40	40	$R_1 = 5^1, 0^3, 5^1, 0^{35}$	1.0495	3.3456	0.7116	2.3152	0.3539	0.7097
				$R_2 = 5^1, 0^3, 5^1, 0^{35}$	(0.1849)	(0.2345)	(0.0758)	(0.2104)	(0.0691)	(0.0307)
				$R_1 = 1^{10}, 0^{30}$	1.095	3.3938	0.7474	2.3254	0.3734	0.6922
				$R_2 = 1^{10}, 0^{30}$	(0.1978)	(0.3851)	(0.1088)	(0.2691)	(0.0598)	(0.0418)
				$R_1 = 0^{30}, 1^{10}$	1.0725	3.0537	0.7107	1.9872	0.3461	0.6335
				$R_2 = 0^{30}, 1^{10}$	(0.1949)	(0.4695)	(0.1581)	(0.3511)	(0.0720)	(0.0494)
60	60	50	45	$R_1 = 5^1, 0^3, 5^1, 0^{45}$	1.0372	3.9507	0.7285	2.3038	0.3624	0.7094
				$R_2 = 5^1, 0^3, 5^1, 0^4, 5^1, 0^{35}$	(0.1837)	(0.3936)	(0.1380)	(0.7812)	(0.0657)	(0.0345)
				$R_1 = 1^{10}, 0^{40}$	1.0057	4.3853	0.7989	2.3788	0.3686	0.7175
				$R_2 = 1^{15}, 0^{30}$	(0.1517)	(0.377)	(0.1777)	(0.0957)	(0.0637)	(0.0382)
				$R_1 = 0^{40}, 1^{10}$	1.0127	3.7368	0.6813	2.2304	0.3097	0.5288
				$R_2 = 0^{30}, 1^{15}$	(0.2186)	(0.686)	(0.0782)	(0.1399)	(0.0885)	(0.0397)
70	70	60	55	$R_1 = 5^1, 0^3, 5^1, 0^{55}$	1.0535	5.0144	0.7336	2.5623	0.3809	0.7465
				$R_2 = 5^1, 0^3, 5^1, 0^4, 5^1, 0^{45}$	(0.1695)	(0.3506)	(0.0829)	(0.0835)	(0.0569)	(0.0444)
				$R_1 = 1^{10}, 0^{50}$	1.0355	4.9895	0.7558	2.5379	0.3758	0.7443
				$R_2 = 1^{15}, 0^{40}$	(0.1608)	(0.3619)	(0.1285)	(0.0668)	(0.0606)	(0.0389)
				$R_1 = 0^{50}, 1^{10}$	1.1535	5.3144	0.7836	2.7623	0.4809	0.7165
				$R_2 = 0^{40}, 1^{15}$	(0.1796)	(0.4506)	(0.1829)	(0.0835)	(0.0569)	(0.0444)

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Conflict of interest

There is no conflict of interest.

References

[1] W. Nelson, *Accelerated testing: statistical models, test plans and data analysis*, NewYork, Wiley(2004).

[2] V. Bagdonavicius and M. Nikulin, *Accelerated life models: modeling and statistical analysis*, Boca Raton, FL: Chapman & Hall/CRC.(2002).

[3] C.M Kim and D.S Bai, Analysis of accelerated life test data under two failure modes, *International Journal of Reliability, Quality and Safety Engineering*, 9, 111-125(2002).

[4] A. A Ismail, A.A Abdel-Ghalyb and E.H El-Khodary, Optimum constant-stress life test plans for Pareto distribution under type-I censoring, *Journal of Statistical Computation and Simulation*. 81, 1835-1845(2011).

[5] R. Miller and W. B. Nelson, Optimum simple step-stress plans for accelerated life testing, *IEEE Trans Reliab*, 32, 59-65(1983).

[6] E. Gouno, A. Sen and N. Balakrishnan, Optimal step-stress test under progressive Type-I censoring, *IEEE Transactions on Reliability*, 53, 388-393(2004).

[7] T.H Fan, W.L Wang and N. Balakrishnan, Exponential progressive step-stress life-testing with link function based on Box Cox transformation, *Journal of Statistical Planning and Inference* 138, 2340-2354(2008).

[8] H.M Ma and W.Q. Meeker, Optimum step-stress accelerated life test plans for log-location scale distributions, *Naval Research Logistics*, 55, 551-562(2008).

[9] S.J Wu , Y.P Lin and S.T Chen, Optimal step-stress test under type I progressive group censoring with random removals. *Journal of Statistical Planning and Inference*, 138, 817-826(2008).

[10] Y. Tangi, Q. Guani , P. Xu , H. Xu, Optimum design for type-I step-stress accelerated life tests of two-parameter Weibull distributions, *Communications in Statistics - Theory and Methods*, 41, 3863-3877(2012).

[11] R. Wang and H. Fei, Statistical inference of Weibull distribution for tampered failure rate model in progressive stress accelerated life testing, *Journal of Systems Science and Complexity*, 17, 237-243(2004).

[12] A.H Abdel-Hamid and E.K Al-Hussaini, Progressive stress accelerated life tests under nite mixture models, *Metrika*, 66, 213-231(2007).

[13] A.M Almarashi and G.A Abd-Elmougod, Accelerated Competing Risks Model from Gompertz lifetime distribution

Table 9. Coverage percentages (95%) and Average Lenght of the parameters with $\theta = 2$, $\alpha_1 = 0.6$, $\alpha_2 = 0.8$ and $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) = (1.324, 2.4865, 0.8227, 1.1142)$

m_1	m_2	scheme	λ_{11}		λ_{12}		λ_{21}		λ_{22}		α_1		α_2	
$n_1 = n_2 = 40$			ACI	Boot	ACI	Boot	ACI	Boot	ACI	Boot	ACI	Boot	ACI	Boot
30	25	R_1=5^1,0^3,5^1,0^{25}	0.89	0.95	0.91	0.94	0.90	0.96	0.92	0.94	0.90	0.95	0.91	0.95
		$R_2 =$	1.3518	1.4074	5.6309	9.5291	0.9262	1.0147	2.6122	6.0073	0.3586	0.2655	0.7300	1.1282
		5^1,0^3,5^1,0^4,5^1,0^{15}												
		$R_1 = 1^{10}, 0^{20}$	0.88	0.96	0.90	0.95	0.88	0.94	0.91	0.95	0.87	0.96	0.91	0.95
		$R_2 = 1^{15}, 0^{10}$	1.3764	1.4064	5.6411	8.9914	1.0447	1.0435	2.3371	5.7385	0.3841	0.2869	0.7704	1.2391
		$R_1 = 0^{20}, 1^{10}$	0.88	0.98	0.90	0.94	0.89	0.95	0.90	0.94	0.91	0.93	0.90	0.94
		$R_2 = 0^{10}, 1^{15}$	1.5055	1.5734	5.6192	9.1952	0.9884	1.1722	2.1005	5.7969	0.3891	0.3037	0.8717	1.6004
$n_1 = n_2 = 50$														
40	40	R_1=5^1,0^3,5^1,0^{35}	0.89	0.92	0.90	0.93	0.89	0.92	0.90	0.95	0.88	0.94	0.91	0.95
		R_2=5^1,0^3,5^1,0^{35}	1.1467	0.8813	3.4559	8.4814	0.7717	0.7214	2.3976	7.6137	0.2749	0.1843	0.5827	1.091
		$R_1 = 1^{10}, 0^{30}$	0.90	0.94	0.89	0.95	0.88	0.93	0.90	0.94	0.89	0.95	0.90	0.96
		$R_2 = 1^{10}, 0^{30}$	1.1618	1.0349	3.6565	7.2132	0.7831	1.0398	2.3867	7.2693	0.2836	0.1994	0.5838	1.0689
		$R_1 = 0^{30}, 1^{10}$	0.88	0.96	0.89	0.95	0.89	0.94	0.90	0.95	0.89	0.94	0.90	0.93
		$R_2 = 0^{30}, 1^{10}$	1.1643	1.0896	3.1103	5.5542	0.7576	0.7729	1.9829	6.2537	0.2699	0.1778	0.5913	1.1929
$n_1 = n_2 = 60$														
50	45	R_1=5^1,0^3,5^1,0^{45}	0.89	0.95	0.90	0.94	0.88	0.94	0.90	0.93	0.88	0.95	0.91	0.96
		$R_2 =$	1.0137	0.8910	3.8942	6.1288	0.7099	0.6412	2.2145	5.4610	0.2518	0.1832	0.5451	0.9188
		5^1,0^3,5^1,0^4,5^1,0^{35}												
		$R_1 = 1^{10}, 0^{40}$	0.88	0.94	0.89	0.93	0.88	0.95	0.90	0.94	0.88	0.94	0.90	0.95
		$R_2 = 1^{15}, 0^{30}$	0.9801	0.8887	4.4054	6.1936	0.7772	0.6573	2.3062	5.4406	0.2556	0.1875	0.5585	0.9505
		$R_1 = 0^{40}, 1^{10}$	0.89	0.95	0.89	0.94	0.88	0.95	0.89	0.95	0.88	0.94	0.90	0.95
		$R_2 = 0^{30}, 1^{15}$	1.0248	1.0228	3.7137	7.8612	0.6869	0.8399	2.1959	7.1683	0.2085	0.13920	0.4345	0.5108
$n_1 = n_2 = 70$														
60	55	R_1=5^1,0^3,5^1,0^{55}	0.95	0.96	0.94	0.95	0.93	0.95	0.92	0.95	0.95	0.95	0.96	0.95
		$R_2 =$	0.9319	0.8004	4.7174	6.0577	0.6489	0.5949	2.3493	5.1144	0.2361	0.1788	0.5324	0.8508
		5^1,0^3,5^1,0^4,5^1,0^{45}												
		$R_1 = 1^{10}, 0^{50}$	0.93	0.97	0.92	0.94	0.95	0.95	0.95	0.97	0.92	0.96	0.95	0.97
		$R_2 = 1^{15}, 0^{40}$	0.9169	0.8073	4.4811	5.2822	0.6667	0.6179	2.2848	4.9449	0.2333	0.1813	0.5357	0.8592
		$R_1 = 0^{50}, 1^{10}$	0.94	0.95	0.95	0.94	0.93	0.95	0.94	0.95	0.93	0.94	0.94	0.95
		$R_2 = 0^{40}, 1^{15}$	0.9346	0,910	4.7237	6.8615	0.6868	0.8399	2.5959	6.1683	0.2585	0.19920	0.5545	0.8808

with type-II censoring scheme, *Thermal Science*, 24, 165-175(2020).

[14] A.M Abouammoh and A.M Alshingiti, Reliability estimation of generalized inverted exponential distribution, *Journal of Statistical Computation and Simulation*, 79, 1301-1315(2009).

[15] H. Krishna, and K. Kumar, Reliability estimation in generalized inverted exponential distribution with progressively type II censored sample, *Journal of Statistical Computation and Simulation*, 83, 1007-1019(2013).

[16] S. Dey and D. Dey, On progressively censored generalized inverted exponential distribution. *Journal of Applied Statistics*, 41, 2557-2576(2014).

[17] S. Dey and B. Pradhan, Generalized inverted exponential distribution under hybrid censoring, *Statistical Methodology*,18, 101-114(2013).

[18] R. Garg, M. Dube and K. Kumar, On randomly censored generalized inverted exponential distribution, *Amer J Math Manage Sci*, 35, 361-379(2016).

[19] H. Krishna, M. Dube and R. Renu Garg, Estimation of P$(Y < X)$ for progressively first-failure-censored generalized inverted exponential distribution, *Journal of Statistical Computation and Simulation*, 87, 2274-2289(2017).

[20] A. A. Soliman, A.E.A hmed, A. A. Farghal and A. A. AL-Shibany, Estimation of Generalized Inverted Exponential Distribution based on Adaptive Type-II Progressive Censoring Data. *Journal of Statistics Applications and Probability*, 9, 215-230(2020).

[21] Y. Tangi, Q. Guani , P. Xu , H. Xu, Optimum design for type-I step-stress accelerated life tests of two-parameter Weibull distributions, *Communications in Statistics-Theory and Methods*, 41, 3863-3877(2012).

[22] N. Balakrishnan and R. Aggarwala, *Progressive Censoring, Theory, Methods, and Applications*, Birkhuser, Boston, (2000).

[23] , N. Balakrishnana and D. Kundu, Hybrid censoring: Models, inferential results and applications, *Computational Statistics and Data Analysis*, 57, 166-209, (2013).

[24] A. Sklar, Functions de repartition a n dimensions et leurs marges, *Publications de lInstitut de Statistique de LUniversitéde Paris*, 8,229-231 (1959).

[25] A.C Davison and D.V Hinkley, *Bootstrap Methods and their Applications*, 2nd, Cambridge University Press, Cambridge United Kingdom (1997).

[26] B. Efron and R.J Tibshirani, *An introduction to the bootstrap*, New York Chapman and Hall,(1993).

[27] B. Efron, *The jackknife, the bootstrap and other resampling plans*, In: CBMS-NSF Regional Conference Series in Applied Mathematics. SIAM, Phiadelphia, PA , 38,(1982).
[28] P. Hall, Theoretical comparison of bootstrap confidence intervals, *Annals of Statistics*, 16, 927 - 953, (1972).
[29] D.G Hoel, A Representation of Mortality Data by Competing Risks, *Biometrics*, 28, 475-488(1972).
[30] N. Balakrishnan and R.A Sandhu, A simple simulation algorithm for generating progressively type-II censored samples, *The American Statistician*, 49,229-230(1995).

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