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Higher Order Variational Inequalities

Muhammad Aslam Noor[1,*], *Khalida Inayat Noor*[1] *and Hilal M. Y. Al-Bayatti*[2]

[1]Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan
[2]Department of Computer Science, College of Arts and Science, Applied Science University, P.O. Box 5055, East Al-Ekir, Kingdom of Bahrain

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Abstract: In this paper, we prove that the optimality conditions of the higher order strongly convex functions are characterized by a class of variational inequalities, which is called the higher order strongly variational inequality. Auxiliary principle technique is used to suggest an implicit method for solving higher order strongly variational inequalities. Convergence analysis of the proposed method is investigated using the pseudo-monotonicity of the operator. Some special cases also discussed. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

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1 Introduction

Variational inequalities theory, which was introduced by Stampacchia [1] in 1964, contains wealth of new ideas and techniques for investigating a wide class of unrelated problems in a unified framework. Variational inequalities may be viewed as novel generalization of the variational principles, the origin of which can be traced back to Euler, Lagrange and the Bernoulli brothers. Variational principles have played a crucial and important role in the development of various fields of sciences and have appeared as a unifying force. The ideas and techniques of variational inequalities are being applied in a variety of diverse areas of sciences and prove to be productive and innovative. Variational inequalities have been extended and generalized in several directions using novel and new techniques. For the formulation, applications, numerical methods, sensitivity analysis and other aspects of variational inequalities, see [2–11] and the references therein.
In recent years, several extensions and generalizations of the convexity have been considered. Polyak [12] introduced strongly convex functions and studied their applications in optimization programming. Karmardian [13] used the strongly convex functions to discuss the unique existence of a solution of the nonlinear complementarity problems. Mohsen et al [14] introduced the concept of higher order strongly convex functions and studied their properties. Lin and Fukushima [15] discussed the applications of the higher order strongly convex functions in nonlinear programs and mathematical programs with equilibrium constraints. These results can be viewed as significant refinement of the results of Lin and Fukushima [15] and Alabdali et al [16] for higher order strongly uniformly convex functions. Higher order strongly convex functions include the strongly convex functions as special case. With appropriate and suitable choice of non-negative arbitrary functions and constants parameters, one can obtain various new and known classes of convex functions. For more details, see [17–21, 23–25] and the references therein

Noor et al [23, 24] ntroduced the higher order convex functions and studied their properties. We have shown that the minimum of a differentiable higher order strongly convex function on the general biconvex set can be characterized by a class of variational inequality. This results inspired us to consider the higher order strongly variational inequalities. Due to the inherent nonlinearity, the projection method and its variant form can not be used to suggest the iterative methods for solving these variational inequalities. To overcome these draw backs, we use the technique of the auxiliary principle [2, 7–11]

* Corresponding author e-mail: noormaslam@gmail.com

to suggest an implicit method for solving variational inequalities. Convergence analysis of the proposed method is investigated under pseudo-monotonicity, which is a weaker condition than monotonicity. Some special cases are discussed as applications of the results, which represent the improvement and refinement of the thr known results. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

2 Preliminary Results

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot,\cdot\rangle$ and $\|\cdot\|$ be the inner product and norm, respectively. Let $F: K \to R$ be a continuous function.

Definition 1. [18]. *The set K in H is said to be a convex set, if*

$$u+t(v-u)\in K, \qquad \forall u,v\in K, t\in[0,1].$$

Definition 2. *The function F on the convex set K is said to be higher order strongly convex, if there exists a constant $\mu>0$, such that*

$$F(u+t(v-u)) \le (1-t)F(u)+tF(v) \\ -\mu\{t^p(1-t)+t(1-t)^p\}\|v-u\|^p, \\ \forall u,v\in K, t\in[0,1], p>1.$$

The function F is said to be higher order strongly concave, if and only if, $-F$ is higher order strongly convex function.
For the properties of the higher order strongly convex functions in variational inequalities and equilibrium problems, see Noor [8,9,23,24].
I. If $p=2$, then Definition 2 becomes:

Definition 3.*A function F is said to be strongly convex function, if*

$$F(u+t(v-u)) \le (1-t)F(u)+tF(v) \\ -\mu t(1-t)\|v-u\|^2, \forall u,v\in K, t\in[0,1],$$

which were introduced by Polyak [12].

II. If $\mu=0$, then Definition 2 becomes:

Definition 4.*A function F is said to be convex function, if*

$$F(u+t(v-u)) \le (1-t)F(u)+tF(v), \forall u,\in K, t\in[0,1].$$

If the convex function F is differentiable, then $u\in K$ is the minimum of the F, if and only if, $u\in K$ satisfies the inequality

$$\langle F'(u), v-u\rangle \ge 0, \quad \forall v\in K,$$

which is called the variational inequality, introduced and studied by Stampacchia [1] in 1964. For the applications, formulation, sensitivity, dynamical systems, generalizations, and other aspects of the variational inequalities, see [2–11, 22–25] and the references therein.

3 Main Results

In this section, we introduce and consider a new class of variational inequalities, which arises as an optimality condition of differentiable higher order strongly convex functions and this is the main motivation of our next result.

Theorem 1. *Let F be a differentiable higher order strongly convex function with modulus $\mu>0$. If $u\in K$ is the minimum of the function F, then*

$$F(v)-F(u) \ge \mu\|v-u\|^p, \quad \forall u,v\in K. \tag{3.1}$$

Proof. Let $u\in K$ be a minimum of the function F. Then

$$F(u) \le F(v), \forall v\in K. \tag{3.2}$$

Since K is a convex set, so, $\forall u,v\in K, \quad t\in[0,1]$,

$$g(v_t) = u+t(v-u)\in K.$$

Taking $g(v)=g(v_t)$ in (3.2), we have

$$0 \le \lim_{t\to 0}\{\frac{F(u+t(v-u))-F(u)}{t}\} = \langle F'(u), v-u\rangle. \tag{3.3}$$

Since F is differentiable higher order strongly convex function, so $\forall u,v\in K$,

$$F(u+t(v-u)) \le F(u)+t(F(v)-F(u)) \\ -\mu\{t^p(1-t)+t(1-t)^p\}\|v-u)\|^p,,$$

from which, using (3.3), we have

$$F(v)-F(u) \ge \lim_{t\to 0}\{\frac{F(u+t(v-u))-F(u)}{t}\}+\mu\|v-u\|^p \\ = \langle F'(u), v-u\rangle+\mu\|v-u\|^p,$$

the required result (3.1).

Remark: We would like to mention that, if $u\in K$ satisfies the inequality

$$\langle F'(u), v-u\rangle+\mu\|v-u\|^p \ge 0, \quad \forall u,v\in K, \tag{3.4}$$

then $u\in K$ is the minimum of the function F.

*Remark.*The inequality of the type (3.4) is called the higher order strongly variational inequality and appears to a new one. It is well known that the inequalities of the type (3.4) does not arise as a minimum of the differentiable higher order strongly convex function. We now consider a more variational inequality of which (3.4) is a special case.

For a given operator T, consider the problem of finding $u\in K$ for a constant $\mu>0$, such that

$$\langle Tu, v-u\rangle+\mu\|v-u\|^p \ge 0, \quad \forall v\in K, p>1, \tag{3.5}$$

which is called the higher order strongly variational inequality.

We now discus several special cases of the problem (3.5).
(I). If $Tu = F'(u)$, then problem (3.5) is exactly the higher order strongly
variational inequality (3.4).

(II). If $\mu = 0$, then (3.5) is equivalent to finding $u \in K$, such that

$$\langle Tu, v-u\rangle \geq 0, \quad \forall v \in K,$$

which is known as the variational inequality, introduced and studied by Stampacchia [1]. For recent applications, see [2,3,4,5,6,7,8,9,10,11,23,24] and the references therein.

(III). If $p = 1$, then problem (3.5) reduces to the problem of finding $u \in K$ such that

$$\langle Tu, v-u\rangle + \mu\|v-u\| \geq 0, \quad \forall v \in K,$$

which is called the approximate variational inequality and appears to be a new one.

(IV). If $p = 2$, then problem (3.5) reduces to the problem of finding $u \in K$ such that

$$\langle Tu, v-u\rangle + \mu\|v-u\|^2 \geq 0, \quad \forall v \in K,$$

which is called the higher order variational inequality, which appears to be a new one.

We now recall the concept of the monotonicity.

Definition 5. *The operator T is said to be monotone, if*

$$\langle Tu, v-u\rangle \geq 0, \quad \forall v \in K,$$
$$\Longrightarrow$$
$$\langle Tv, v-u\rangle \geq 0, \quad \forall v \in K.$$

Lemma 1. *Let the operator T be monotone. If $u \in K$ is the solution of the problem (3.5), then $u \in K$ satisfies the inequality*

$$\langle Tv, v-u\rangle + \nu\|v-u\|^p \geq 0, \quad \forall v \in K, p > 1. \tag{3.6}$$

Proof. Let $u \in K$ be a solution of the problem(3.5). Then

$$\langle Tu, v-u\rangle + \nu\|v-u\|^p \geq 0, \forall v \in K, p > 1,$$

from which, we have

$$\langle Tu - Tv, v-u\rangle + \langle Tv, v-u\rangle + \nu\|v-u\|^p \geq 0, \forall v \in K.$$

Using the monotonicity of the operator T, it follows that

$$\langle Tv, v-u\rangle + \nu\|v-u\|^p \geq 0, \forall v \in K,$$

which is the required result (3.6).

The inequality of the type (3.6) is called the Minty higher order strongly variational inequality. For suitable and appropriate choice of the parameter μ and p, one can obtain several new and known classes of variational inequalities and optimization problems.

Remark. We would like to emphasize that the converse of Lemma 1 does not hold. However, if the operator T, is hemicontinuous, then one can show that the converse of Lemma 1 holds for $p > 1$ and $\nu = 0$. The variational inequality (3.6) is also call the dual of the inequality (3.6) and plays an important role in the study of variational inequalities.

We note that the projection method and its variant forms can not be used to study the higher order strongly variational inequalities (3.5)due to its inherent structure. These facts motivated us to consider the auxiliary principle technique, which is mainly due to Lions and Stampacchia [7] and Glowinski et al [2] as developed by Noor [6] and Noor et al. [8,9,10]. We again use this technique to suggest some iterative methods for solving the higher order strongly variational inequalities (3.5).

For given $u \in K$ satisfying (3.5), consider the problem of finding $w \in K$, such that

$$\langle \rho Tw, v-w\rangle + \langle w-u, v-w\rangle + \nu\|v-w\|^p \geq 0, \forall v \in K, p > 1. \tag{3.7}$$

The problem (3.7) is called the auxiliary higher order strongly variational inequality. It is clear that the relation (3.7) defines a mapping connecting the problems (3.5) and (3.7).
We not that, if $w(u) = u$, then w is a solution of problem (3.5). This simple observation enables to suggest an iterative method for solving (3.5).

Algorithm 1 *For given $u_0 \in K$, find the approximate solution u_{n+1} by the scheme*

$$\langle \rho Tu_{n+1}, v-u_{n+1}\rangle + \langle u_{n+1}-u_n, v-u_{n+1}\rangle + \nu\|v-u_{n+1}\|^p \geq 0, \forall v \in K, p > 1.$$

The Algorithm 1 is known as the implicit method. Such type of methods have been studied extensively for various classes of variational inequalities. See [6,9,10,11] and the reference therein.
If $\nu = 0$, then Algorithm 1 reduces to:

Algorithm 2 *For given $u_0 \in$, find the approximate solution u_{n+1} by the scheme*

$$\langle \rho Tu_{n+1}, v-u_{n+1}\rangle + \langle u_{n+1}-u_n, v-u_{n+1}\rangle \geq 0, \forall v \in K, \tag{3.8}$$

for solving the variational inequalities(3.6).
For the convergence analysis of Algorithm 1, we need the following concept.

Definition 6. *The operator T is said to be pseudomonotone with respect to $\mu\|v-u\|^p$, if*

$$\langle \rho Tu, v-u\rangle + \mu\|v-u\|^p \geq 0, \forall v \in K, p > 1,$$
$$\Longrightarrow$$
$$\langle \rho Tv, v-u\rangle - \mu\|v-u\|^p \geq 0, \forall v \in K, p > 1$$

We now study the convergence analysis of Algorithm 1.

Theorem 2. *Let $u \in K$ be a solution of (3.5) and u_{n+1} be the approximate solution obtained from Algorithm 1. If T is a pseudomonotone operator with respect to $\nu\|v-u\|^p$, then*

$$\|u_{n+1}-u\|^2 \leq \|u_n-u\|^2 - \|u_{n+1}-u_n\|^2. \tag{3.9}$$

Proof. Let $u \in K$ be a solution of (3.5). Then

$$\langle \rho Tu, v-u\rangle + \nu\|v-u\|^p \geq 0, \forall v \in K,$$

implies that

$$\langle \rho Tv, v-u\rangle - \nu\|v-u\|^p \geq 0, \forall v \in K, \tag{3.10}$$

Now taking $v = u_{n+1}$ in (3.10), we have

$$\langle \rho Tu_{n+1}, u_{n+1}-u\rangle - \nu\|u_{n+1}-u\|^p \geq 0. \tag{3.11}$$

Taking $v = u$ in (3.8), we have

$$\langle \rho Tu_{n+1}, u-u_{n+1}\rangle + \langle u_{n+1}-u_n, u-u_{n+1}\rangle + \nu\|u-u_{n+1}\|^p \geq 0. \tag{3.12}$$

Combining (3.11) and (3.12), we have

$$\langle u_{n+1}-u_n, u_{n+1}-u\rangle \geq 0.$$

Using the inequality

$$2\langle a,b\rangle = \|a+b\|^2 - \|a\|^2 - \|b\|^2, \forall a,b \in H,$$

we obtain

$$\|u_{n+1}-u\|^2 \leq \|u_n-u\|^2 - \|u_{n+1}-u_n\|^2,$$

the required result (3.9).

Theorem 3. *Let the operator T be a pseudomonotone. If u_{n+1} is the approximate solution obtained from Algorithm 1 and $u \in K$ is the exact solution (3.5), then*

$$\lim_{n\to\infty} u_n = u.$$

Proof. Let $u \in K$ be a solution of (3.5). Then, from (3.9), it follows that the sequence $\{\|u-u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. From (3.9), we have

$$\sum_{n=0}^{\infty} \|u_{n+1}-u_n\|^2 \leq \|u_0-u\|^2,$$

from which, it follows that

$$\lim_{n\to\infty} \|u_{n+1}-u_n\| = 0. \tag{3.13}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence u_n converge to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (3.8), taking the limit $n_j \to 0$ and from (3.13), we have

$$\langle T\hat{u}, v-\hat{u}\rangle + \mu\|v-\hat{u})\|^p \geq 0, \quad \forall v \in K, p > 1.$$

This implies that $\hat{u} \in K$ satisfies (3.5)and

$$\|u_{n+1}-u_n\|^2 \leq \|u_n-\hat{u}\|^2.$$

Thus it follows from the above inequality that the sequence u_n has exactly one cluster point $\hat{u}$ and

$$\lim_{n\to\infty} u_n = \hat{u}.$$

In order to implement the implicit Algorithm 1, one uses the predictor-corrector technique. Consequently, Algorithm 1 is equivalent to the following iterative method for solving the general variational inequality (3.5).

Algorithm 3 *For a given $u_0 \in K$, find the approximate solution u_{n+1} by the schemes*

$$\langle \rho Tu_n, v-y_n\rangle + \langle y_n-u_n, v-y_n\rangle + \mu\|v-y_n\|^p \geq 0, \forall v \in K, p > 1$$

$$\langle \rho Ty_n, v-y_n\rangle + \langle u_{n+1}-y_n, v-u_n\rangle + \mu\|v-u_{n+1}\|^p \geq 0, \forall v \in K, p > 1.$$

Algorithm 3 is called the two-step method and appears to be a new one.

We again use the auxiliary principle technique to suggest an other implicit method for solving the variational inequalities (3.5) for a constant $\xi \in [0,1]$.
For given $u \in K$ satisfying (3.5), consider the problem of finding $w \in K$, such that

$$\langle \rho Tw, v-w\rangle + \langle w-(1-\xi)w-\xi u, v-w\rangle + \nu\|v-w\|^p \geq 0, \forall v \in K, p > 1. \tag{3.14}$$

Clearly, if $w(u) = u$, then w is a solution of problem (3.5). This simple observation enables to suggest an iterative method for solving (3.5).

Algorithm 4 *For a given $u_0 \in K$, find the approximate solution u_{n+1} by the schemes*

$$\langle \rho Tu_{n+1}, v-u_{n+1}\rangle + \langle u_{n+1}-(1-\xi)u_{n+1}-\xi u_n, v-u_{n+1}\rangle + \nu\|v-u_{n+1}\|^p \geq 0, \forall v \in K, p > 1.$$

Algorithm 4 is called the unified implicit method.

If $\xi = 1$, then Algorithm 4 collapses to Algorithm 1.

If $\xi = 0$, then Algorithm 4 reduces to:

Algorithm 5 *For a given $u_0 \in K$, find the approximate solution u_{n+1} by the schemes*

$$\langle \rho Tu_{n+1}, v-u_{n+1}\rangle + \langle u_{n+1}-u_{n+1}, v-u_{n+1}\rangle + \nu\|v-u_{n+1}\|^p \geq 0, \forall v \in K, p > 1.$$

Algorithm 5 can be viewed as an extragraident method of Noor [5, 6] and appears to be a new ones. This shows that Algorithm 4 is a more general and unified one. Using the technique of Theorem 2, one consider the convergence criteria of Algorithm 4.

If $\xi = \frac{1}{2}$, then Algorithm 4 becomes:

Algorithm 6*For a given $u_0 \in K$, find the approximate solution u_{n+1} by the schemes*

$$\langle \rho Tu_{n+1}, v-u_{n+1}\rangle + \langle \frac{u_{n+1}-u_n}{2}, v-u_{n+1}\rangle + \nu\|v-u_{n+1}\|^p \geq 0, \forall v \in K, p > 1.$$

Remark. Using the auxiliary principle technique, on can suggest several iterative methods for solving the higher order strongly variational inequalities and related optimization problems. We have only given some glimpse of the higher order strongly variational inequalities. It is an interesting problem to explore the applications of such type of variational inequalities in various fields of pure and applied sciences.

4 Conclusion

In this paper, we have characterized the optimality conditions of higher order strongly differentiable convex functions by a class of variational inequalities. This result motivated to introduce and study a new class of higher order strongly variational inequalities. Using the auxiliary principle technique, some implicit iterative methods are suggested for finding the approximate solution.. Using the pseudo-monotonicity of the operator, convergence criteria is discussed. Some special cases are considered as application of the main results. Comparison of these methods with other methods need further efforts. It is an interesting problem to explore the applications of higher order strongly variational inequalities in various branches of pure and applied sciences

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