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Trigonometric Conformable Fractional Quantitative Approximation of Stochastic Processes

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Abstract: Here we consider very general stochastic positive linear operators induced by general positive linear operators that are acting on continuous functions in the trigonometric sense. These are acting on the space of real conformable fractionally differentiable stochastic processes. Under some very mild, general and natural assumptions on the stochastic processes we produce related trigonometric conformable fractional stochastic Shisha-Mond type inequalities of L^q-type $1 \leq q < \infty$ and corresponding trigonometric conformable fractional stochastic Korovkin type theorems. These are regarding the trigonometric stochastic q-mean conformable fractional convergence of a sequence of stochastic positive linear operators to the stochastic unit operator for various cases. All convergences are derived with rates and are given via the trigonometric conformable fractional stochastic inequalities involving the stochastic modulus of continuity of the α-th conformable fractional derivatives of the engaged stochastic process, $\alpha \in (n, n+1)$, $n \in \mathbb{Z}_+$. The impressive fact is that only two basic real Korovkin test functions assumptions, one of them trigonometric, are enough for the conclusions of our trigonometric conformable fractional stochastic Korovkin theorems. We give applications to stochastic Bernstein operators in the trigonometric sense.

Keywords: Stochastic positive linear operator, trigonometric conformable fractional stochastic Korovkin theory and trigonometric conformable fractional inequalities, trigonometric conformable fractional stochastic Shisha-Mond inequality, stochastic modulus of continuity, stochastic process.

1 Introduction

Motivation for this work comes from [1], [2], [3], [4], [5]. This work continues our earlier work [6], now at the stochastic conformable fractional level.

In section 2 we talk briefly about conformable fractional calculus. In the section 3, we talk about the q-mean $(1 \leq q < \infty)$ first modulus of continuity of a stochastic process and its upper bounds. There we describe completely our setting by introducing our stochastic positive linear operator M, see (19), which is based on the positive linear operator $\widetilde{L}$ from $C([-\pi, \pi])$ into itself. The operator M is acting on a wide space of conformable fractional differentiable real valued stochastic processes X. See there Assumptions 1 - 4. We first give the main trigonometric pointwise conformable fractional stochastic Shisha-Mond type inequalities ([3]), see Theorems 5, 6, and their several corollaries covering important trigonometric special cases.

We continue with trigonometric conformable fractional q-mean uniform Shisha-Mond type inequalities, see Theorems 7, 8, and their interesting corollaries. All this theory is regarding the trigonometric conformable fractional stochastic convergence of operators M to I (stochastic unit operator) given quantitatively with rates. An extensive trigonometric application about the stochastic Bernstein operators follows in full details. Based on our Shisha-Mond type inequalities of our main Theorems 5 - 8 we derive trigonometric pointwise and uniform Stochastic Korovkin theorems ([7]) on stochastic processes, see Theorems 9 - 12. The amazing fact here is, that basic conditions on operator $\widetilde{L}$ regarding two simple real valued functions, one of them trigonometric, that are not stochastic, are able to imply conformable fractional stochastic convergence on all stochastic processes we are dealing with; see Concepts 1 and Assumptions 1 - 4 on $[-\pi, \pi]$.

This work is based on [8].

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2 Background - I

Here we follow [9], see also [10].

We need

Definition 1.([9]) *Let $a,b \in \mathbb{R}$. The left conformable fractional derivative starting from a of a function $f:[a,\infty) \to \mathbb{R}$ of order $0<\alpha\leq 1$ is defined by*

$$\left(T_{\alpha}^{a} f\right)(t)=\lim_{\varepsilon \to 0} \frac{f\left(t+\varepsilon(t-a)^{1-\alpha}\right)-f(t)}{\varepsilon}. \tag{1}$$

If $\left(T_{\alpha}^{a} f\right)(t)$ exists on (a,b), then

$$\left(T_{\alpha}^{a} f\right)(a)=\lim_{t \to a+}\left(T_{\alpha}^{a} f\right)(t). \tag{2}$$

The right conformable fractional derivative of order $0<\alpha\leq 1$ terminating at b of $f:(-\infty,b] \to \mathbb{R}$ is defined by

$$\left({}_{\alpha}^{b}T f\right)(t)=-\lim_{\varepsilon \to 0} \frac{f\left(t+\varepsilon(b-t)^{1-\alpha}\right)-f(t)}{\varepsilon}. \tag{3}$$

If $\left({}_{\alpha}^{b}T f\right)(t)$ exists on (a,b), then

$$\left({}_{\alpha}^{b}T f\right)(b)=\lim_{t \to b-}\left({}_{\alpha}^{b}T f\right)(t). \tag{4}$$

Note that if f is differentiable then

$$\left(T_{\alpha}^{a} f\right)(t)=(t-a)^{1-\alpha} f'(t), \tag{5}$$

and

$$\left({}_{\alpha}^{b}T f\right)(t)=-(b-t)^{1-\alpha} f'(t). \tag{6}$$

In the higher order case we can generalize things as follows:

Definition 2.([9]) *Let $\alpha \in (n,n+1]$, and set $\beta=\alpha-n$. Then, the left conformable fractional derivative starting from a of a function $f:[a,\infty)\to\mathbb{R}$ of order α, where $f^{(n)}(t)$ exists, is defined by*

$$\left(\mathbf{T}_{\alpha}^{a} f\right)(t)=\left(T_{\beta}^{a} f^{(n)}\right)(t). \tag{7}$$

The right conformable fractional derivative of order α terminating at b of $f:(-\infty,b]\to\mathbb{R}$, where $f^{(n)}(t)$ exists, is defined by

$$\left({}_{\alpha}^{b}\mathbf{T} f\right)(t)=(-1)^{n+1}\left({}_{\beta}^{b}T f^{(n)}\right)(t). \tag{8}$$

If $\alpha=n+1$ then $\beta=1$ and $\mathbf{T}_{n+1}^{a} f=f^{(n+1)}$.

If n is odd, then ${}_{n+1}^{b}\mathbf{T} f=-f^{(n+1)}$, and if n is even, then ${}_{n+1}^{b}\mathbf{T} f=f^{(n+1)}$.

When $n=0$ (or $\alpha\in(0,1]$), then $\beta=\alpha$, and (7), (8) collapse to $\{$(1), (2)$\}$, $\{$(3), (4)$\}$ respectively.

We make

Remark.([11], p. 155) We notice the following: let $\alpha\in(n,n+1]$ and $f\in C^{n+1}([a,b])$, $n\in\mathbb{N}$. Then ($\beta:=\alpha-n$, $0<\beta\leq 1$)

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(x)=\left(T_{\beta}^{a} f^{(n)}\right)(x)=(x-a)^{1-\beta} f^{(n+1)}(x), \tag{9}$$

and

$$\left({}_{\alpha}^{b}\mathbf{T}(f)\right)(x)=(-1)^{n+1}\left({}_{\beta}^{b}T f^{(n)}\right)(x)=$$

$$(-1)^{n+1}(-1)(b-x)^{1-\beta} f^{(n+1)}(x)=(-1)^{n}(b-x)^{1-\beta} f^{(n+1)}(x). \tag{10}$$

Consequently we get that

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(x),\ \left({}_{\alpha}^{b}\mathbf{T}(f)\right)(x)\in C([a,b]).$$

Furthermore it is obvious that

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(a)=\left({}_{\alpha}^{b}\mathbf{T}(f)\right)(b)=0, \tag{11}$$

when $0<\beta<1$, i.e. when $\alpha\in(n,n+1)$.

3 Background - II

We need

Definition 3. *Here $(\Omega, \mathscr{F}, P)$ is a probability space, $\omega \in \Omega$. We define the relative q-mean first modulus of continuity of stochastic process $X(t,\omega)$ by*

$$\Omega_1 (X, \delta)_{L^q,[c,d]} :=$$

$$\sup \left\{ \left(\int_{\Omega} |X(x,\omega) - X(y,\omega)|^q P(d\omega) \right)^{\frac{1}{q}} : x, y \in [c,d] \subseteq [a,b], \ |x-y| \leq \delta \right\}, \tag{12}$$

where $\delta > 0$, $1 \leq q < \infty$, $t \in [a,b] \subset \mathbb{R}$.

Definition 4. *Let $1 \leq q < \infty$. Let $X(x,\omega)$ be a stochastic process. We call X a q-mean uniformly continuous stochastic process over $[a,b]$, iff $\forall \ \varepsilon > 0 \ \exists \ \delta > 0$: whenever $|x-y| \leq \delta$; $x, y \in [a,b]$ implies that*

$$\int_{\Omega} |X(x,s) - X(y,s)|^q P(ds) \leq \varepsilon. \tag{13}$$

We denote it as $X \in C_{\mathbb{R}}^{U_q}([a,b])$.

It holds

Proposition 1. *([6]) Let $X \in C_{\mathbb{R}}^{U_q}([a,b])$, then $\Omega_1(X,\delta)_{L^q,[a,b]} < \infty$, any $\delta > 0$.*

Also it holds

Proposition 2. *([6]) Let $X(t,\omega)$ be a stochastic process from $[a,b] \times (\Omega, \mathscr{F}, P)$ into $\mathbb{R}$. Then following are true ($[c,d] \subseteq [a,b]$):*
(i) $\Omega_1(X,\delta)_{L^q,[c,d]}$ is nonnegative and nondecreasing in $\delta > 0$,
(ii) $\lim_{\delta \downarrow 0} \Omega_1(X,\delta)_{L^q,[c,d]} = \Omega_1(X,0)_{L^q,[c,d]} = 0$, iff $X \in C_{\mathbb{R}}^{U_q}([c,d])$,
(iii) $\Omega_1(X,\delta_1+\delta_2)_{L^q,[c,d]} \leq \Omega_1(X,\delta_1)_{L^q,[c,d]} + \Omega_1(X,\delta_2)_{L^q,[c,d]}$, $\delta_1, \delta_2 > 0$,
(iv) $\Omega_1(X,m\delta)_{L^q,[c,d]} \leq m\Omega_1(X,\delta)_{L^q,[c,d]}$, $\delta > 0$, $m \in \mathbb{N}$,
(v) $\Omega_1(X,\lambda\delta)_{L^q,[c,d]} \leq \lceil \lambda \rceil \Omega_1(X,\delta)_{L^q,[c,d]} \leq (\lambda+1)\Omega_1(X,\delta)_{L^q,[c,d]}$, $\lambda > 0$, $\delta > 0$, $\lceil \cdot \rceil$ is the ceiling of the number,
(vi) $\Omega_1(X+Y,\delta)_{L^q,[c,d]} \leq \Omega_1(X,\delta)_{L^q,[c,d]} + \Omega_1(Y,\delta)_{L^q,[c,d]}$, $\delta > 0$,
(vii) $\Omega_1(X,\cdot)_{L^q,[c,d]}$ is continuous on $\mathbb{R}_+$ for $X \in C_{\mathbb{R}}^{U_q}([c,d])$.

We give

Remark.(to Proposition 2) By Proposition 2 (v) we get

$$\omega_1 (X, |x-y|)_{L^q,[c,d]} \leq \left\lceil \frac{|x-y|}{\delta} \right\rceil \omega_1 (X,\delta)_{L^q,[c,d]}, \tag{14}$$

$\forall \ x, y \in [c,d]$, any $\delta > 0$.

We give

*Remark.*Let $\alpha \in (n, n+1]$, $n \in \mathbb{Z}_+$, $X(\cdot,\omega) \in C^{n+1}([a,b])$, $\forall \ \omega \in \Omega$. We assume that $\left| X^{(n+1)}(t,\omega) \right| \leq M^*$, $\forall \ (t,\omega) \in [a,b] \times \Omega$, where $M^* > 0$. Let $\delta > 0$, $1 \leq q < \infty$. Then

$$\Omega_1 (\mathbf{T}_{\alpha}^c X, \delta)_{L^q,[c,d]} = \sup \left\{ \left(\int_{\Omega} |(\mathbf{T}_{\alpha}^c X)(x,\omega) - (\mathbf{T}_{\alpha}^c X)(y,\omega)|^q P(d\omega) \right)^{\frac{1}{q}} : \right.$$

$$\left. x, y \in [c,d] \subseteq [a,b], \ |x-y| \leq \delta \right\} \leq$$

$$\sup\left\{\left(\int_{\Omega}\left(\left|\left(\mathbf{T}_{\alpha}^{c}X\right)(x,\omega)\right|+\left|\left(\mathbf{T}_{\alpha}^{c}X\right)(y,\omega)\right|\right)^{q}P(d\omega)\right)^{\frac{1}{q}}:\right.$$

$$\left. x,y\in[c,d]\subseteq[a,b],\ |x-y|\leq\delta\right\}\underset{(\beta:=\alpha-n)}{\overset{(9)}{\leq}}$$

$$\sup\left\{\left(\int_{\Omega}\left[(x-c)^{1-\beta}\left|X^{(n+1)}(x,\omega)\right|+(y-c)^{1-\beta}\left|X^{(n+1)}(y,\omega)\right|\right]^{q}P(d\omega)\right)^{\frac{1}{q}}:\right.$$

$$\left. x,y\in[c,d]\subseteq[a,b],\ |x-y|\leq\delta\right\}\leq$$

$$M^{*}\sup\left\{\left(\int_{\Omega}\left((x-c)^{1-\beta}+(y-c)^{1-\beta}\right)^{q}P(d\omega)\right)^{\frac{1}{q}}:\right.$$

$$\left. x,y\in[c,d]\subseteq[a,b],\ |x-y|\leq\delta\right\}\leq 2M^{*}(b-a)^{1-\beta}.$$

That is

$$\Omega_{1}\left(\mathbf{T}_{\alpha}^{c}X,\delta\right)_{L^{q},[c,d]}\leq 2M^{*}(b-a)^{1-\beta}, \tag{15}$$

for any $[c,d]\subseteq[a,b]$.

Similarly, it holds

$$\Omega_{1}\left({}_{\alpha}^{d}\mathbf{T}X,\delta\right)_{L^{q},[c,d]}\leq 2M^{*}(b-a)^{1-\beta}, \tag{16}$$

for any $[c,d]\subseteq[a,b]$.

In particular it holds

$$\sup_{t\in[a,b]}\Omega_{1}\left(\mathbf{T}_{\alpha}^{t}X,\delta\right)_{L^{q},[t,b]}\leq 2M^{*}(b-a)^{1-\beta}, \tag{17}$$

and

$$\sup_{t\in[a,b]}\Omega_{1}\left({}_{\alpha}^{t}\mathbf{T}X,\delta\right)_{L^{q},[a,t]}\leq 2M^{*}(b-a)^{1-\beta}. \tag{18}$$

Above, it is not strange to assume that $\mathbf{T}_{\alpha}^{c}X, {}_{\alpha}^{d}\mathbf{T}X, \mathbf{T}_{\alpha}^{t}X, {}_{\alpha}^{t}\mathbf{T}X$ are stochastic processes, see Remark 2.

We need

Concepts 1 *Let $\widetilde{L}$ be a positive linear operator from $C([a,b])$ into itself. Let $X(t,\omega)$ be a stochastic process from $[a,b]\times(\Omega,\mathscr{F},P)$ into $\mathbb{R}$, where $(\Omega,\mathscr{F},P)$ is a probability space. Let $\alpha\in(n,n+1]$, $n\in\mathbb{Z}_{+}$, and $X(\cdot,\omega)\in C^{n+1}([a,b])$, $\forall\ \omega\in\Omega$. Also we assume for each $t\in[a,b]$ that $X^{(k)}(t,\cdot)$ is measurable over $(\Omega,\mathscr{F})$ for all $k=1,\ldots,n+1$. Define*

$$M(X)(t,\omega):=\widetilde{L}(X(\cdot,\omega))(t),\ \ \forall\ \omega\in\Omega,\ \forall\ t\in[a,b], \tag{19}$$

and assume that it is a random variable in ω. Clearly M is a positive linear operator on stochastic processes.

We make

Assumption 1 *i) For any $t\in[a,b]$ we assume that $\mathbf{T}_{\alpha}^{t}X(z,\omega)$ is continuous in $z\in[t,b]$, uniformly with respect to $\omega\in\Omega$. I.e. $\forall\ \varepsilon>0\ \exists\ \delta>0:$ whenever $|z_1-z_2|\leq\delta$; $z_1,z_2\in[t,b]$, then $\left|\mathbf{T}_{\alpha}^{t}X(z_1,\omega)-\mathbf{T}_{\alpha}^{t}X(z_2,\omega)\right|\leq\varepsilon$, $\forall\ \omega\in\Omega$.*

We denote this by $\mathbf{T}_{\alpha}^{t}X\in C_{\mathbb{R}}^{U}([t,b])$, the space of continuous in x, uniformly with respect to ω, stochastic processes over $[t,b]$.

ii) For any $t\in[a,b]$ we assume that ${}_{\alpha}^{t}\mathbf{T}X(z,\omega)$ is continuous in $z\in[a,t]$, uniformly with respect to $\omega\in\Omega$. I.e. $\forall\ \varepsilon>0$ $\exists\ \delta>0:$ whenever $|z_1-z_2|\leq\delta$; $z_1,z_2\in[a,t]$, then $\left|{}_{\alpha}^{t}\mathbf{T}X(z_1,\omega)-{}_{\alpha}^{t}\mathbf{T}X(z_2,\omega)\right|\leq\varepsilon$, $\forall\ \omega\in\Omega$.

We denote this by ${}_{\alpha}^{t}\mathbf{T}X\in C_{\mathbb{R}}^{U}([a,t])$, the space of continuous in x, uniformly with respect to ω, stochastic processes over $[a,t]$.

Remark. Assumption 1 implies:

i) $\mathbf{T}_{\alpha}^{t}X(\cdot,\omega)\in C([t,b])$, $\forall\ \omega\in\Omega$, and $\mathbf{T}_{\alpha}^{t}X$ is q-mean uniformly continuous in $z\in[t,b]$, that is $\mathbf{T}_{\alpha}^{t}X\in C_{\mathbb{R}}^{U_q}([t,b])$, for any $1\leq q<\infty$.

ii) ${}_{\alpha}^{t}\mathbf{T}X(\cdot,\omega)\in C([a,t])$, $\forall\ \omega\in\Omega$, and ${}_{\alpha}^{t}\mathbf{T}X$ is q-mean uniformly continuous in $z\in[a,t]$, that is ${}_{\alpha}^{t}\mathbf{T}X\in C_{\mathbb{R}}^{U_q}([a,t])$, for any $1\leq q<\infty$.

We need

Definition 5. *Denote by*

$$(EX)(t) := \int_{\Omega} X(t,\omega) P(d\omega), \ \ \forall\, t \in [a,b], \tag{20}$$

the expectation operator.

We make

Assumption 2 *We assume that*

$$\left(E\left|X^{(k)}\right|^{q}\right)(t) < \infty, \ \ \forall\, t \in [a,b], \tag{21}$$

$q > 1$, *for all* $k = 0,1,...,n$.

We make

Assumption 3 *We assume that*

$$\left(E\left|X^{(k)}\right|\right)(t) < \infty, \ \ \forall\, t \in [a,b], \tag{22}$$

for all $k = 0,1,...,n$.

We give

Remark. By the Riesz representation theorem ([12]) we have that there exists μ_t unique, completed Borel measure on $[a,b]$ with

$$m_t := \mu_t([a,b]) = \widetilde{L}(1)(t) > 0, \tag{23}$$

such that

$$\widetilde{L}(f)(t) = \int_{[a,b]} f(x)\, d\mu_t(x), \ \ \forall\, t \in [a,b], \forall\, f \in C([a,b]). \tag{24}$$

Consequently we have that

$$M(X)(t,\omega) = \int_{[a,b]} X(x,\omega)\, d\mu_t(x), \ \ \forall\ (t,\omega) \in [a,b]\times\Omega, \tag{25}$$

and X as in Concepts 1.

Here $\chi_{[\gamma,\delta]}(s)$ stands for the characteristic function on $[\gamma,\delta] \subseteq [a,b]$.

Notice that ($r > 0$)

$$\int_{[t,b]} (s-t)^r \mu_t(ds) = \int_{[a,b]} \chi_{[t,b]}(s)\, |s-t|^r \mu_t(ds) = \widetilde{L}\left(|\cdot - t|^r \chi_{[t,b]}(\cdot)\right)(t), \tag{26}$$

and

$$\int_{[a,t]} (t-s)^r \mu_t(ds) = \int_{[a,b]} \chi_{[a,t]}(s)\, |s-t|^r \mu_t(ds) = \widetilde{L}\left(|\cdot - t|^r \chi_{[a,t]}(\cdot)\right)(t). \tag{27}$$

Let $\alpha \in (n, n+1]$, $n \in \mathbb{N}$ and $k = 1,...,n$. Then by Hölder's inequality we obtain

$$\left|\int_{[a,b]} (x-t)^k d\mu_t(x)\right| \leq \int_{[a,b]} |x-t|^k d\mu_t(x) \leq$$

$$\left(\int_{[a,b]} |x-t|^{\alpha+1} d\mu_t(x)\right)^{\left(\frac{k}{\alpha+1}\right)} \left(\mu_t([a,b])\right)^{\left(\frac{\alpha+1-k}{\alpha+1}\right)}. \tag{28}$$

Therefore it holds

$$\left\|\widetilde{L}\left((\cdot - t)^k\right)(t)\right\|_{\infty,[a,b]} \leq \left\|\widetilde{L}\left(|\cdot - t|^k\right)(t)\right\|_{\infty,[a,b]} \leq$$

$$\left\|\widetilde{L}\left(|\cdot - t|^{\alpha+1}\right)(t)\right\|_{\infty,[a,b]}^{\left(\frac{k}{\alpha+1}\right)} \left\|\widetilde{L}(1)\right\|_{\infty,[a,b]}^{\left(\frac{\alpha+1-k}{\alpha+1}\right)}, \tag{29}$$

all $k = 1,...,n$.

Also, we observe that

$$C([a,b]) \ni |\cdot - t|^{\alpha+1} \chi_{[a,t]}(\cdot) \leq |\cdot - t|^{\alpha+1}, \ \ \forall\, t \in [a,b], \tag{30}$$

and

$$C([a,b]) \ni |\cdot - t|^{\alpha+1} \chi_{[t,b]}(\cdot) \leq |\cdot - t|^{\alpha+1}, \ \ \forall\, t \in [a,b]. \tag{31}$$

By positivity of $\widetilde{L}$ we obtain

$$\left\| \widetilde{L}\left(|\cdot - t|^{\alpha+1} \chi_{[a,t]}(\cdot) \right)(t) \right\|_{\infty,[a,b]} \leq \left\| \widetilde{L}\left(|\cdot - t|^{\alpha+1} \right)(t) \right\|_{\infty,[a,b]} < \infty, \tag{32}$$

by $\widetilde{L}\left(|\cdot - t|^{\alpha+1} \right)(t)$ being continuous in $t \in [a,b]$, see p. 388 of [13],
and

$$\left\| \widetilde{L}\left(|\cdot - t|^{\alpha+1} \chi_{[t,b]}(\cdot) \right)(t) \right\|_{\infty,[a,b]} \leq \left\| \widetilde{L}\left(|\cdot - t|^{\alpha+1} \right)(t) \right\|_{\infty,[a,b]}. \tag{33}$$

Above (26)-(29) and (32), (33) were used to derive convergences in our earlier results mentioned next.

In this work we denote by $\widetilde{L}\left(\chi_{[a,t]}(\cdot)\right)(t) := \mu_t([a,t]) \leq \widetilde{L}(1)(t)$, and by $\widetilde{L}\left(\chi_{[t,b]}(\cdot)\right)(t) := \mu_t([t,b]) \leq \widetilde{L}(1)(t)$.

We make

Assumption 4 *Assume that* $\left|X^{(n+1)}(t,\omega)\right| \leq M^*$, $\forall\, (t,\omega) \in [a,b] \times \Omega$, *where* $M^* > 0$. *And suppose that* $X^{(n+1)}(\cdot,\omega)$ *is continuous over* $[a,b]$, *uniformly with respect to* $\omega \in \Omega$.

Clearly, Assumption 4 implies Assumption 1, see also Remark 2.

We mention the following pointwise result on the quantitative stochastic conformable fractional approximation regarding stochastic processes:

Theorem 1. *([8]) Let* $n \in \mathbb{Z}_+$, $\alpha \in (n,n+1)$; $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *with* $\beta := \alpha - n > \frac{1}{q}$; *set* $\lambda_1 := \left(\frac{\Gamma(np+1)\Gamma(p(\beta-1)+1)}{\Gamma(p(\alpha-1)+2)}\right)^{(q-1)}$. *Suppose Concepts 1, Assumptions 1 and 2.*

Then, $\forall\, t \in [a,b]$, *we have:*

$$\begin{aligned}
&(E(|M(X) - X|^q)(t))^{\frac{1}{q}} \leq ((E|X|^q)(t))^{\frac{1}{q}} \left| \widetilde{L}(1)(t) - 1 \right| + \\
&\sum_{k=1}^{n} \frac{\left(\left(E\left|X^{(k)}\right|^q\right)(t)\right)^{\frac{1}{q}}}{k!} \left| \widetilde{L}\left((\cdot - t)^k\right)(t) \right| + \frac{2^{\frac{1}{p}} \lambda_1^{\frac{1}{q}}}{(q+1)^{\frac{\alpha}{q(\alpha+1)}}\, n!} \\
&\left\{ \left[\Omega_1\left(\mathbf{T}^t_\alpha X, \left(\frac{1}{(q+1)} \widetilde{L}\left(|\cdot - t|^{q(\alpha+1)} \chi_{[t,b]}(\cdot) \right)(t) \right)^{\frac{1}{q(\alpha+1)}} \right)_{L^q,[t,b]} \right.\right. \\
&\left(\widetilde{L}\left(\chi_{[t,b]}(\cdot)\right)(t)\right)^{\frac{1}{p}} \left(\widetilde{L}\left(|\cdot - t|^{q(\alpha+1)} \chi_{[t,b]}(\cdot) \right)(t) \right)^{\frac{\alpha}{q(\alpha+1)}} \\
&\left.\left[\left(\widetilde{L}\left(\chi_{[t,b]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}} (q+1)^{\frac{\alpha}{(\alpha+1)}} + 1 \right]^{\frac{1}{q}} \right] + \\
&\left[\Omega_1\left({}^t_\alpha\mathbf{T} X, \left(\frac{1}{(q+1)} \widetilde{L}\left(|\cdot - t|^{q(\alpha+1)} \chi_{[a,t]}(\cdot) \right)(t) \right)^{\frac{1}{q(\alpha+1)}} \right)_{L^q,[a,t]} \right. \\
&\left(\widetilde{L}\left(\chi_{[a,t]}(\cdot)\right)(t)\right)^{\frac{1}{p}} \left(\widetilde{L}\left(|\cdot - t|^{q(\alpha+1)} \chi_{[a,t]}(\cdot) \right)(t) \right)^{\frac{\alpha}{q(\alpha+1)}} \\
&\left.\left.\left[\left(\widetilde{L}\left(\chi_{[a,t]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}} (q+1)^{\frac{\alpha}{(\alpha+1)}} + 1 \right]^{\frac{1}{q}} \right] \right\}
\end{aligned} \tag{34}$$

We mention also the following pointwise results, the L_1-quantitative stochastic conformable fractional approximation of stochastic processes.

Theorem 2.(*[8]*) *Let $n \in \mathbb{Z}_+$ and $\alpha \in (n, n+1)$, $\beta := \alpha - n$. Suppose Concepts 1, Assumptions 1, 3. Then, $\forall\, t \in [a,b]$, we have:*

$$E\left(|M(X)-X|\right)(t) \leq \left(E|X|\right)(t)\left|\widetilde{L}(1)(t)-1\right| +$$

$$\sum_{k=1}^{n}\frac{\left(E\left|X^{(k)}\right|\right)(t)}{k!}\left|\widetilde{L}\left((\cdot-t)^{k}\right)(t)\right| + \frac{1}{\prod_{j=1}^{n}(\beta+j)}$$

$$\left\{\left[\frac{\left(\widetilde{L}\left(\chi_{[a,t]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}}{\alpha-n}+\frac{1}{\alpha+1}\right]\left(\widetilde{L}\left(|\cdot-t|^{\alpha+1}\chi_{[a,t]}(\cdot)\right)(t)\right)^{\frac{\alpha}{\alpha+1}}\right.$$

$$\Omega_1\left({}_{\alpha}^{t}\mathbf{T}X, \left(\widetilde{L}\left(|\cdot-t|^{\alpha+1}\chi_{[a,t]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[a,t]} +$$

$$\left[\frac{\left(\widetilde{L}\left(\chi_{[t,b]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}}{\alpha-n}+\frac{1}{(\alpha+1)}\right]\left(\widetilde{L}\left(|\cdot-t|^{\alpha+1}\chi_{[t,b]}(\cdot)\right)(t)\right)^{\frac{\alpha}{\alpha+1}}$$

$$\left.\Omega_1\left(\mathbf{T}_{\alpha}^{t}X, \left(\widetilde{L}\left(|\cdot-t|^{\alpha+1}\chi_{[t,b]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[t,b]}\right\}. \tag{35}$$

Uniform estimates follow:
Theorem 1 implies

Theorem 3.(*[8]*) *Let $n \in \mathbb{Z}_+$, $\alpha \in (n, n+1)$; $p, q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, with $\beta := \alpha - n > \frac{1}{q}$; set $\lambda_1 := \left(\frac{\Gamma(np+1)\Gamma(p(\beta-1)+1)}{\Gamma(p(\alpha-1)+2)}\right)^{(q-1)}$. Suppose Concepts 1, Assumptions 2 and 4. Then*

$$\left\|E\left(|M(X)-X|^{q}\right)(t)\right\|_{\infty}^{\frac{1}{q}} \leq \left\|E\left(|X|^{q}\right)\right\|_{\infty}^{\frac{1}{q}}\left\|\widetilde{L}(1)-1\right\|_{\infty} +$$

$$\sum_{k=1}^{n}\frac{\left\|E\left(\left|X^{(k)}\right|^{q}\right)\right\|_{\infty}^{\frac{1}{q}}}{k!}\left\|\widetilde{L}\left((\cdot-t)^{k}\right)(t)\right\|_{\infty} + \frac{\lambda_1^{\frac{1}{q}}\left(2\left\|\widetilde{L}(1)\right\|_{\infty}\right)^{\frac{1}{p}}}{(q+1)^{\frac{\alpha}{q(\alpha+1)}}\,n!}$$

$$\left[\left\|\widetilde{L}(1)\right\|_{\infty}^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}}$$

$$\left\{\left[\sup_{t\in[a,b]}\Omega_1\left(\mathbf{T}_{\alpha}^{t}X, \frac{1}{(q+1)}\left\|\widetilde{L}\left(|\cdot-t|^{q(\alpha+1)}\chi_{[t,b]}(\cdot)\right)(t)\right\|_{\infty}^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[t,b]}\right.\right.$$

$$\left.\left\|\widetilde{L}\left(|\cdot-t|^{q(\alpha+1)}\chi_{[t,b]}(\cdot)\right)(t)\right\|_{\infty}^{\frac{\alpha}{q(\alpha+1)}}\right] +$$

$$\left[\sup_{t\in[a,b]}\Omega_1\left({}_{\alpha}^{t}\mathbf{T}X, \frac{1}{(q+1)}\left\|\widetilde{L}\left(|\cdot-t|^{q(\alpha+1)}\chi_{[a,t]}(\cdot)\right)(t)\right\|_{\infty}^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[a,t]}\right. \tag{36}$$

$$\left.\left.\left\|\widetilde{L}\left(|\cdot-t|^{q(\alpha+1)}\chi_{[a,t]}(\cdot)\right)(t)\right\|_{\infty}^{\frac{\alpha}{q(\alpha+1)}}\right]\right\}.$$

Theorem 2 implies:

Theorem 4.([8]) *Let $n \in \mathbb{Z}_{+}$, $\alpha \in (n, n+1)$, $\beta := \alpha - n$. Suppose Concepts 1, Assumptions 3, 4. Then*

$$\|E(|M(X) - X|)\|_{\infty} \leq \|E(|X|)\|_{\infty} \left\|\widetilde{L}(1) - 1\right\|_{\infty} +$$

$$\sum_{k=1}^{n} \frac{\left\|E\left(\left|X^{(k)}\right|\right)\right\|_{\infty}}{k!} \left\|\widetilde{L}\left((\cdot - t)^{k}\right)(t)\right\|_{\infty} +$$

$$\frac{1}{\prod_{j=1}^{n} (\beta + j)} \left[\frac{\left\|\widetilde{L}(1)\right\|_{\infty}^{\frac{1}{\alpha+1}}}{\alpha - n} + \frac{1}{\alpha + 1}\right]$$

$$\left\{\left\|\widetilde{L}\left(|\cdot - t|^{\alpha+1} \chi_{[a,t]}(\cdot)\right)(t)\right\|_{\infty}^{\frac{\alpha}{\alpha+1}}\right.$$

$$\sup_{t \in [a,b]} \Omega_1 \left({}_{\alpha}^{t}\mathbf{T}X, \left\|\widetilde{L}\left(|\cdot - t|^{\alpha+1} \chi_{[a,t]}(\cdot)\right)(t)\right\|_{\infty}^{\frac{1}{\alpha+1}}\right)_{L^1,[a,t]} +$$

$$\left\|\widetilde{L}\left(|\cdot - t|^{\alpha+1} \chi_{[t,b]}(\cdot)\right)(t)\right\|_{\infty}^{\frac{\alpha}{\alpha+1}}$$

$$\left.\sup_{t \in [a,b]} \Omega_1 \left(\mathbf{T}_{\alpha}^{t}X, \left\|\widetilde{L}\left(|\cdot - t|^{\alpha+1} \chi_{[t,b]}(\cdot)\right)(t)\right\|_{\infty}^{\frac{1}{\alpha+1}}\right)_{L^1,[t,b]}\right\}. \tag{37}$$

Remark. Next we specify $[a,b]$ as $[-\pi, \pi]$. Clearly then $\widetilde{L} : C([-\pi, \pi]) \to C([-\pi, \pi])$ is the positive linear operator on hand.

Here $\alpha \in (n, n+1)$, $n \in \mathbb{Z}_{+}$, $k = 1, ..., n$. Next we use of Hölder's inequality we notice that

$$\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{k}\right)(t) = \int_{[-\pi,\pi]} \left(\sin\left(\frac{|x - t|}{4}\right)\right)^{k} d\mu_t(x) \leq$$

$$\left(\int_{[-\pi,\pi]} \left(\sin\left(\frac{|x - t|}{4}\right)\right)^{\alpha+1} d\mu_t(x)\right)^{\frac{k}{\alpha+1}} \left(\mu_t([-\pi, \pi])\right)^{\frac{\alpha+1-k}{\alpha+1}} = \tag{38}$$

$$\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{k}{\alpha+1}} \left(\widetilde{L}(1)(t)\right)^{\frac{\alpha+1-k}{\alpha+1}}.$$

That is

$$\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{k}\right)(t) \leq \tag{39}$$

$$\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{k}{\alpha+1}} (L(1)(t))^{\frac{\alpha+1-k}{\alpha+1}},$$

for $k = 1, ..., n$; true also for $q(\alpha + 1)$ instead of $(\alpha + 1)$, for any $1 < q < \infty$.

Next $\|\cdot\|_{\infty}$ denotes $\|\cdot\|_{\infty,[-\pi,\pi]}$.

Consequently, it holds

$$\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{k}\right)(t)\right\|_{\infty} \leq \tag{40}$$

$$\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{k}{\alpha+1}} \left\|\widetilde{L}(1)(t)\right\|_{\infty}^{\frac{\alpha+1-k}{\alpha+1}},$$

for $k = 1, \ldots, n$; true also for $q(\alpha+1)$ instead of $(\alpha+1)$, for any $1 < q < \infty$.

In this work we use a lot the following well known inequality:

$$|z| \leq \pi \sin\left(\frac{|z|}{2}\right), \ \forall z \in [-\pi, \pi]. \tag{41}$$

Notice that, for any $t \in [-\pi, \pi]$, we have $C([-\pi, \pi]) \ni |\cdot - t| \chi_{[-\pi,t]}(\cdot) \leq |\cdot - t| \in C([-\pi, \pi])$, therefore

$$C([-\pi,\pi]) \ni \left(\sin\left(\frac{|\cdot - t| \chi_{[-\pi,t]}(\cdot)}{4}\right)\right)^{\alpha+1} \leq \left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{\alpha+1} \in C([-\pi,\pi]). \tag{42}$$

Consequently, by positivity of $\widetilde{L}$ we obtain

$$\left\| \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t| \chi_{[-\pi,t]}(\cdot)}{4}\right)\right)^{\alpha+1}\right)(t) \right\|_{\infty} \leq \left\| \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{\alpha+1}\right)(t) \right\|_{\infty}. \tag{43}$$

Similarly, for any $t \in [-\pi, \pi]$, we have $C([-\pi, \pi]) \ni |\cdot - t| \chi_{[t,\pi]}(\cdot) \leq |\cdot - t| \in C([-\pi, \pi])$, thus

$$C([-\pi,\pi]) \ni \left(\sin\left(\frac{|\cdot - t| \chi_{[t,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1} \leq \left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{\alpha+1} \in C([-\pi,\pi]). \tag{44}$$

Hence

$$\left\| \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t| \chi_{[t,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}\right)(t) \right\|_{\infty} \leq \left\| \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{\alpha+1}\right)(t) \right\|_{\infty}. \tag{45}$$

So, if the right hand side of (43), (45) goes to zero, so do their left hand sides. Above in (42)-(45), one can use $q(\alpha+1)$ instead of $(\alpha+1)$, $1 < q < \infty$.

A further detailed analysis reveals:

We have that $(1 \leq q < \infty)$

$$\begin{gathered}
\widetilde{L}\left(|\cdot - t|^{q(\alpha+1)} \chi_{[t,\pi]}(\cdot)\right)(t) \overset{(24)}{=} \int_{[t,\pi]} (x-t)^{q(\alpha+1)} d\mu_t(x) = \\
2^{q(\alpha+1)} \int_{[t,\pi]} \left(\frac{x-t}{2}\right)^{q(\alpha+1)} d\mu_t(x) \overset{(41)}{\leq} \\
(2\pi)^{q(\alpha+1)} \int_{[t,\pi]} \left(\sin\left(\frac{x-t}{4}\right)\right)^{q(\alpha+1)} d\mu_t(x) = \\
(2\pi)^{q(\alpha+1)} \int_{[t,\pi]} \left(\sin\left(\frac{|x-t|}{4}\right)\right)^{q(\alpha+1)} d\mu_t(x) = \\
(2\pi)^{q(\alpha+1)} \int_{[-\pi,\pi]} \left(\sin\left(\frac{|x-t|}{4}\right)\right)^{q(\alpha+1)} \chi_{[t,\pi]}(x) \, d\mu_t(x) = \\
(2\pi)^{q(\alpha+1)} \int_{[-\pi,\pi]} \left(\sin\left(\frac{|x-t|}{4} \chi_{[t,\pi]}(x)\right)\right)^{q(\alpha+1)} d\mu_t(x) \overset{(24)}{=} \\
(2\pi)^{q(\alpha+1)} \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4} \chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t).
\end{gathered} \tag{46}$$

That is, we have obtained

$$\widetilde{L}\left(|\cdot - t|^{q(\alpha+1)} \chi_{[t,\pi]}(\cdot)\right)(t) \leq (2\pi)^{q(\alpha+1)} \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4} \chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t). \tag{47}$$

Similarly, it holds

$$\widetilde{L}\left(|\cdot-t|^{q(\alpha+1)}\chi_{[-\pi,t]}(\cdot)\right)(t)\leq(2\pi)^{q(\alpha+1)}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t). \tag{48}$$

Above inequalities (47), (48) are valid for any $1\leq q<\infty$.

Furthermore, we observe that

$$\begin{aligned}&\left|\widetilde{L}\left((\cdot-t)^{k}\right)(t)\right|=\left|\int_{[-\pi,\pi]}(x-t)^{k}d\mu_{t}(x)\right|\leq\int_{[-\pi,\pi]}|x-t|^{k}d\mu_{t}(x)=\\&2^{k}\int_{[-\pi,\pi]}\left(\frac{|x-t|}{2}\right)^{k}d\mu_{t}(x)\overset{(41)}{\leq}(2\pi)^{k}\int_{[-\pi,\pi]}\left(\sin\left(\frac{|x-t|}{4}\right)\right)^{k}d\mu_{t}(x)\\&=(2\pi)^{k}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\right)\right)^{k}\right)(t).\end{aligned} \tag{49}$$

That is

$$\left|\widetilde{L}\left((\cdot-t)^{k}\right)(t)\right|\leq(2\pi)^{k}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\right)\right)^{k}\right)(t), \tag{50}$$

$\forall\, t\in[-\pi,\pi]$, all $k=1,\dots,n$.

4 Main Results

Next we give our first main result on the trigonometric quantitative stochastic conformable fractional approximation of stochastic processes, a pointwise result.

Theorem 5. *All as in Theorem 1 for $[a,b]=[-\pi,\pi]$. Then, $\forall\, t\in[-\pi,\pi]$, we have*

$$\begin{aligned}&\left(E\left(|M(X)-X|^{q}\right)(t)\right)^{\frac{1}{q}}\leq\left(\left(E|X|^{q}\right)(t)\right)^{\frac{1}{q}}\left|\widetilde{L}(1)(t)-1\right|+\\&\sum_{k=1}^{n}\frac{\left(\left(E\left|X^{(k)}\right|^{q}\right)(t)\right)^{\frac{1}{q}}}{k!}(2\pi)^{k}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\right)\right)^{k}\right)(t)+\frac{2^{\alpha+\frac{1}{p}}\lambda_{1}^{\frac{1}{q}}\pi^{\alpha}}{(q+1)^{\frac{\alpha}{q(\alpha+1)}}n!}\cdot\\&\left\{\left[\left\|\Omega_{1}\left(\mathbf{T}_{\alpha}^{t}X,2\pi\left(\frac{1}{(q+1)}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{1}{q(\alpha+1)}}\right)\right\|_{L^{q},[t,\pi]}\right.\right.\\&\left(\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{1}{p}}\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{\alpha}{q(\alpha+1)}}\\&\left.\left[\left(\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}}\right]+\\&\left[\left\|\Omega_{1}\left({}_{\alpha}^{t}\mathbf{T}X,2\pi\left(\frac{1}{(q+1)}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{1}{q(\alpha+1)}}\right)\right\|_{L^{q},[-\pi,t]}\right.\\&\left(\widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{1}{p}}\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{\alpha}{q(\alpha+1)}}\\&\left.\left.\left[\left(\widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}}\right]\right\}.\end{aligned} \tag{51}$$

Proof. Apply Theorem 1, upper bound right hand side of (34) by using (50), (47) and (48).

We continue with the trigonometric L_1-quantitative stochastic conformable fractional pointwise approximation of stochastic processes.

Theorem 6. *All as in Theorem 2 for $[a,b]=[-\pi,\pi]$. Then, $\forall\, t\in[-\pi,\pi]$, we have:*

$$E\left(\left|M\left(X\right)-X\right|\right)\left(t\right)\leq\left(E\left|X\right|\right)\left(t\right)\left|\widetilde{L}\left(1\right)\left(t\right)-1\right|+$$

$$\sum_{k=1}^{n}\frac{\left(E\left|X^{(k)}\right|\right)\left(t\right)}{k!}\left(2\pi\right)^{k}\widetilde{L}\left(\left(\sin\left(\frac{\left|\cdot-t\right|}{4}\right)\right)^{k}\right)\left(t\right)+\frac{\left(2\pi\right)^{\alpha}}{\prod_{j=1}^{n}\left(\beta+j\right)}$$

$$\left\{\left[\frac{\left(\widetilde{L}\left(\chi_{[-\pi,t]}\left(\cdot\right)\right)\left(t\right)\right)^{\frac{1}{\alpha+1}}}{\alpha-n}+\frac{1}{\left(\alpha+1\right)}\right]\left(\widetilde{L}\left(\left(\sin\left(\frac{\left|\cdot-t\right|}{4}\chi_{[-\pi,t]}\left(\cdot\right)\right)\right)^{\alpha+1}\right)\left(t\right)\right)^{\frac{\alpha}{\alpha+1}}\right.$$

$$\Omega_{1}\left({}_{\alpha}^{t}\mathbf{T}X,2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{\left|\cdot-t\right|}{4}\chi_{[-\pi,t]}\left(\cdot\right)\right)\right)^{\alpha+1}\right)\left(t\right)\right)^{\frac{1}{\alpha+1}}\right)_{L^{1},[-\pi,t]}+$$

$$\left[\frac{\left(\widetilde{L}\left(\chi_{[t,\pi]}\left(\cdot\right)\right)\left(t\right)\right)^{\frac{1}{\alpha+1}}}{\alpha-n}+\frac{1}{\left(\alpha+1\right)}\right]\left(\widetilde{L}\left(\left(\sin\left(\frac{\left|\cdot-t\right|}{4}\chi_{[t,\pi]}\left(\cdot\right)\right)\right)^{\alpha+1}\right)\left(t\right)\right)^{\frac{\alpha}{\alpha+1}}\tag{52}$$

$$\left.\Omega_{1}\left(\mathbf{T}_{\alpha}^{t}X,2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{\left|\cdot-t\right|}{4}\chi_{[t,\pi]}\left(\cdot\right)\right)\right)^{\alpha+1}\right)\left(t\right)\right)^{\frac{1}{\alpha+1}}\right)_{L^{1},[t,\pi]}\right\}.$$

Proof. Apply Theorem 2, upper bound right hand side of (35) by using (50), (47) and (48) for $q=1$.

We continue with trigonometric conformable fractional uniform estimates ($\|\cdot\|_{\infty,[-\pi,\pi]}:=\|\cdot\|_{\infty}$) in L_q-mean ($1\leq q<\infty$).

Theorem 7. *All as in Theorem 3 for $[a,b]=[-\pi,\pi]$. Then*

$$\left\|E\left(\left|M\left(X\right)-X\right|^{q}\right)\right\|_{\infty}^{\frac{1}{q}}\leq\left\|E\left(\left|X\right|^{q}\right)\right\|_{\infty}^{\frac{1}{q}}\left\|\widetilde{L}\left(1\right)-1\right\|_{\infty}+$$

$$\sum_{k=1}^{n}\frac{\left\|E\left(\left|X^{(k)}\right|^{q}\right)\right\|_{\infty}^{\frac{1}{q}}}{k!}\left(2\pi\right)^{k}\left\|\widetilde{L}\left(\left(\sin\left(\frac{\left|\cdot-t\right|}{4}\right)\right)^{k}\right)\left(t\right)\right\|_{\infty}+$$

$$\frac{2^{\alpha+\frac{1}{p}}\lambda_{1}^{\frac{1}{q}}\pi^{\alpha}\left\|\widetilde{L}\left(1\right)\right\|_{\infty}^{\frac{1}{p}}}{\left(q+1\right)^{\frac{\alpha}{q(\alpha+1)}}n!}\left[\left\|\widetilde{L}\left(1\right)\right\|_{\infty}^{\frac{1}{\alpha+1}}\left(q+1\right)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}}\tag{53}$$

$$\left[\sup_{t\in[-\pi,\pi]}\Omega_{1}\left(\mathbf{T}_{\alpha}^{t}X,2\pi\left\|\frac{1}{\left(q+1\right)}\widetilde{L}\left(\left(\sin\left(\frac{\left|\cdot-t\right|}{4}\chi_{[t,\pi]}\left(\cdot\right)\right)\right)^{q(\alpha+1)}\right)\left(t\right)\right\|_{\infty}^{\frac{1}{q(\alpha+1)}}\right)_{L^{q},[t,\pi]}\right.$$

$$\left.\left\|\widetilde{L}\left(\left(\sin\left(\frac{\left|\cdot-t\right|}{4}\chi_{[t,\pi]}\left(\cdot\right)\right)\right)^{q(\alpha+1)}\right)\left(t\right)\right\|_{\infty}^{\frac{\alpha}{q(\alpha+1)}}\right]+$$

$$\left[\sup_{t\in[-\pi,\pi]}\Omega_1\left({}_{\alpha}^{t}\mathbf{T}X,2\pi\left\|\frac{1}{(q+1)}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right\|_{\infty}^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[-\pi,t]}\right.$$

$$\left.\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right\|_{\infty}^{\frac{\alpha}{q(\alpha+1)}}\right]\Bigg\}.$$

Proof. By (51).

Theorem 8. *All as in Theorem 4 for $[a,b]=[-\pi,\pi]$. Then*

$$\|E(|M(X)-X|)\|_{\infty}\leq\|E(|X|)\|_{\infty}\left\|\widetilde{L}(1)-1\right\|_{\infty}+$$

$$\sum_{k=1}^{n}\frac{\left\|E\left(\left|X^{(k)}\right|\right)\right\|_{\infty}}{k!}(2\pi)^{k}\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\right)\right)^{k}\right)(t)\right\|_{\infty}+$$

$$\frac{(2\pi)^{\alpha}}{\prod_{j=1}^{n}(\beta+j)}\left[\frac{\left\|\widetilde{L}(1)\right\|_{\infty}^{\frac{1}{\alpha+1}}}{\alpha-n}+\frac{1}{\alpha+1}\right] \tag{54}$$

$$\left\{\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{\alpha}{\alpha+1}}\right.$$

$$\sup_{t\in[-\pi,\pi]}\Omega_1\left({}_{\alpha}^{t}\mathbf{T}X,2\pi\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{1}{\alpha+1}}\right)_{L^1,[-\pi,t]}+$$

$$\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{\alpha}{\alpha+1}}$$

$$\left.\sup_{t\in[-\pi,\pi]}\Omega_1\left(\mathbf{T}_{\alpha}^{t}X,2\pi\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{1}{\alpha+1}}\right)_{L^1,[t,\pi]}\right\}.$$

Proof. By (52).

We continue with interesting pointwise corollaries.

Corollary 1. *All as in Theorem 1 for $[a,b]=[-\pi,\pi]$. Further assume that $\widetilde{L}(1)(t_0)=1$, and $X^{(k)}(t_0,\omega)=0$, $\forall\ \omega\in\Omega$, all $k=1,\ldots,n$, for a fixed $t_0\in[-\pi,\pi]$. Then*

$$\left(E\left(|M(X)-X|^{q}\right)(t_0)\right)^{\frac{1}{q}}\leq\frac{2^{\alpha+\frac{1}{p}}\lambda_1^{\frac{1}{q}}\pi^{\alpha}}{(q+1)^{\frac{\alpha}{q(\alpha+1)}}n!} \tag{55}$$

$$\left\{\left[\Omega_1\left(\mathbf{T}_{\alpha}^{t_0}X,2\pi\left(\frac{1}{(q+1)}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t_0|}{4}\chi_{[t_0,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t_0)\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[t_0,\pi]}\right.\right.$$

$$\left(\widetilde{L}\left(\chi_{[t_0,\pi]}(\cdot)\right)(t_0)\right)^{\frac{1}{p}}\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t_0|}{4}\chi_{[t_0,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t_0)\right)^{\frac{\alpha}{q(\alpha+1)}}$$

$$\left[\left(\widetilde{L}\left(\chi_{[t_0,\pi]}(\cdot)\right)(t_0)\right)^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}}+$$

$$\Omega_1\left({}_{\alpha}^{t_0}\mathbf{T}X, 2\pi\left(\frac{1}{(q+1)}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t_0|}{4}\chi_{[-\pi,t_0]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t_0)\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[-\pi,t_0]}$$

$$\left(\widetilde{L}\left(\chi_{[-\pi,t_0]}(\cdot)\right)(t_0)\right)^{\frac{1}{p}}\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t_0|}{4}\chi_{[-\pi,t_0]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t_0)\right)^{\frac{\alpha}{q(\alpha+1)}}$$

$$\left[\left(\widetilde{L}\left(\chi_{[-\pi,t_0]}(\cdot)\right)(t_0)\right)^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}}\Bigg]\Bigg\}.$$

Proof. By (51).

Corollary 2. *All as in Theorem 1 for $[a,b]=[-\pi,\pi]$. Further assume that $\widetilde{L}(1)=1$ and $\frac{1}{q}<\alpha<1$ (i.e. $n=0$). Then, $\forall$ $t\in[-\pi,\pi]$, we have*

$$\left(E\left(|M(X)-X|^q\right)(t)\right)^{\frac{1}{q}}\leq\frac{2^{\alpha+\frac{1}{p}}\pi^{\alpha}}{\left(p(\alpha-1)+1\right)^{\frac{1}{p}}(q+1)^{\frac{\alpha}{q(\alpha+1)}}} \tag{56}$$

$$\Bigg\{\Bigg[\Omega_1\left(T_{\alpha}^{t}X, 2\pi\left(\frac{1}{(q+1)}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[t,\pi]}$$

$$\left(\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{1}{p}}\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{\alpha}{q(\alpha+1)}}$$

$$\left[\left(\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}}+$$

$$+\Bigg[\Omega_1\left({}_{\alpha}^{t}TX, 2\pi\left(\frac{1}{(q+1)}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[-\pi,t]}$$

$$\left(\widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{1}{p}}\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{\alpha}{q(\alpha+1)}}$$

$$\left[\left(\widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}}\Bigg]\Bigg\}.$$

Proof. By (51).

Corollary 3. *All as in Theorem 2 for $[a,b]=[-\pi,\pi]$. Further assume that $\widetilde{L}(1)(t_0)=1$, and $X^{(k)}(t_0,\omega)=0$, $\forall$ $\omega\in\Omega$, all $k=1,\ldots,n$, for a fixed $t_0\in[-\pi,\pi]$. Then*

$$E\left(|M(X)-X|\right)(t_0)\leq\frac{(2\pi)^{\alpha}}{\prod_{j=1}^{n}(\beta+j)} \tag{57}$$

$$\left\{\left[\frac{\left(\widetilde{L}\left(\chi_{[-\pi,t_0]}(\cdot)\right)(t_0)\right)^{\frac{1}{\alpha+1}}}{\alpha-n}+\frac{1}{(\alpha+1)}\right]\right.$$

$$\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t_0|}{4}\chi_{[-\pi,t_0]}(\cdot)\right)\right)^{\alpha+1}\right)(t_0)\right)^{\frac{\alpha}{\alpha+1}}$$

$$\Omega_1\left({}_{\alpha}^{t_0}\mathbf{T}X,2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t_0|}{4}\chi_{[-\pi,t_0]}(\cdot)\right)\right)^{\alpha+1}\right)(t_0)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[-\pi,t_0]}+$$

$$\left[\frac{\left(\widetilde{L}\left(\chi_{[t_0,\pi]}(\cdot)\right)(t_0)\right)^{\frac{1}{\alpha+1}}}{\alpha-n}+\frac{1}{(\alpha+1)}\right]$$

$$\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t_0|}{4}\chi_{[t_0,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t_0)\right)^{\frac{\alpha}{\alpha+1}}$$

$$\left.\Omega_1\left(\mathbf{T}_{\alpha}^{t_0}X,2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t_0|}{4}\chi_{[t_0,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t_0)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[t_0,\pi]}\right\}.$$

Proof. By (52).

Corollary 4. *All as in Theorem 2 for $[a,b]=[-\pi,\pi]$. Further assume that $\widetilde{L}(1)=1$ and $0<\alpha<1$. Then, $\forall\, t\in[-\pi,\pi]$, we have*

$$E(|M(X)-X|)(t)\leq(2\pi)^{\alpha} \tag{58}$$

$$\left\{\left[\frac{\left(\widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}}{\alpha}+\frac{1}{(\alpha+1)}\right]\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}}\right.$$

$$\Omega_1\left({}_{\alpha}^{t}TX,2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[-\pi,t]}+$$

$$\left[\frac{\left(\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}}{\alpha}+\frac{1}{(\alpha+1)}\right]\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}}$$

$$\left.\Omega_1\left(T_{\alpha}^{t}X,2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[t,\pi]}\right\}.$$

Proof. By (52).

Corollary 5. *All as in Theorem 2 for $[a,b]=[-\pi,\pi]$. Further assume that $\widetilde{L}(1)=1$ and $\alpha=\frac{1}{2}$. Then, $\forall\, t\in[-\pi,\pi]$, we have*

$$E(|M(X)-X|)(t)\leq\sqrt{2\pi} \tag{59}$$

$$\left\{\left[2\left(\widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{2}{3}}+\frac{2}{3}\right]\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{1}{3}}\right.$$

$$\Omega_1\left({}_{\frac{1}{2}}^{t}TX,2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{2}{3}}\right)_{L^1,[-\pi,t]}+$$

$$\left[2\left(\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{2}{3}}+\frac{2}{3}\right]\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{1}{3}}$$

$$\Omega_1\left(T^t_{\frac{1}{2}}X, 2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{2}{3}}\right)_{L^1,[t,\pi]}\Bigg\}.$$

Proof. By (58).

We continue with interesting uniform corollaries:

Corollary 6. *All as in Theorem 3 for $[a,b] = [-\pi,\pi]$. Further assume that $\widetilde{L}(1) = 1$ and $\frac{1}{q} < \alpha < 1$ (i.e. $n = 0$). Then*

$$\|E(|M(X) - X|^q)\|_\infty^{\frac{1}{q}} \le \frac{2^{\alpha+\frac{1}{p}}\pi^\alpha \left\|\widetilde{L}(1)\right\|_\infty^{\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}(q+1)^{\frac{\alpha}{q(\alpha+1)}}}$$

$$\left[\left\|\widetilde{L}(1)\right\|_\infty^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{\alpha+1}} + 1\right]^{\frac{1}{q}} \tag{60}$$

$$\left[\sup_{t\in[-\pi,\pi]}\Omega_1\left(T^t_\alpha X, 2\pi\left\|\frac{1}{(q+1)}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right\|_\infty^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[t,\pi]}\right.$$

$$\left.\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right\|_\infty^{\frac{\alpha}{q(\alpha+1)}}\right] +$$

$$\left[\sup_{t\in[-\pi,\pi]}\Omega_1\left({}^t_\alpha TX, 2\pi\left\|\frac{1}{(q+1)}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right\|_\infty^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[-\pi,t]}\right.$$

$$\left.\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right\|_\infty^{\frac{\alpha}{q(\alpha+1)}}\right]\Bigg\}.$$

Proof. By (53).

Corollary 7. *All as in Theorem 4 for $[a,b] = [-\pi,\pi]$. Further assume that $\widetilde{L}(1) = 1$, and $0 < \alpha < 1$ (i.e. $n = 0$). Then*

$$\|E(|M(X) - X|)\|_\infty \le (2\pi)^\alpha\left[\frac{\left\|\widetilde{L}(1)\right\|_\infty^{\frac{1}{\alpha+1}}}{\alpha} + \frac{1}{\alpha+1}\right]$$

$$\Bigg\{\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right\|_\infty^{\frac{\alpha}{\alpha+1}}$$

$$\sup_{t\in[-\pi,\pi]}\Omega_1\left({}^t_\alpha TX, 2\pi\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right\|_\infty^{\frac{1}{\alpha+1}}\right)_{L^1,[-\pi,t]} \tag{61}$$

$$+\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right\|_\infty^{\frac{\alpha}{\alpha+1}}$$

$$\sup_{t\in[-\pi,\pi]}\Omega_1\left(T^t_\alpha X, 2\pi\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right\|_\infty^{\frac{1}{\alpha+1}}\right)_{L^1,[t,\pi]}\Bigg\}.$$

Proof. By (54).

Corollary 8. *All as in Theorem 4 for $[a,b]=[-\pi,\pi]$. Further assume that $\widetilde{L}(1)=1$, and $\alpha=\frac{1}{2}$. Then*

$$\|E(|M(X)-X|)\|_{\infty}\le\sqrt{2\pi}\left[2\left\|\widetilde{L}(1)\right\|_{\infty}^{\frac{2}{3}}+\frac{2}{3}\right]$$

$$\left\{\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right\|_{\infty}^{\frac{1}{3}}\right.$$

$$\sup_{t\in[-\pi,\pi]}\left\|\Omega_1\left({}_{\frac{1}{2}}^{t}TX,2\pi\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right\|_{\infty}^{\frac{2}{3}}\right)\right\|_{L^1,[-\pi,t]} \tag{62}$$

$$+\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right\|_{\infty}^{\frac{1}{3}}$$

$$\left.\sup_{t\in[-\pi,\pi]}\left\|\Omega_1\left(T_{\frac{1}{2}}^{t}X,2\pi\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right\|_{\infty}^{\frac{2}{3}}\right)\right\|_{L^1,[t,\pi]}\right\}.$$

Proof. By (61).

5 Application

Consider the Bernstein polynomials on $[-\pi,\pi]$ for $f\in C([-\pi,\pi])$:

$$B_N(f)(x)=\sum_{k=0}^{N}\binom{N}{k}f\left(-\pi+\frac{2\pi k}{N}\right)\left(\frac{x+\pi}{2\pi}\right)^{k}\left(\frac{\pi-x}{2\pi}\right)^{N-k}, \tag{63}$$

$N\in\mathbb{N}$, any $x\in[-\pi,\pi]$. There are positive linear operators from $C([-\pi,\pi])$ into itself.

Setting $g(t)=f(2\pi t-\pi)$, $t\in[0,1]$, we have $g(0)=f(-\pi)$, $g(1)=f(\pi)$, and

$$(B_Ng)(t)=\sum_{k=0}^{N}\binom{N}{k}g\left(\frac{k}{N}\right)t^k(1-t)^{N-k}=(B_Nf)(x),\quad x\in[-\pi,\pi]. \tag{64}$$

Here $x=\varphi(t)=2\pi t-\pi$ is an $1-1$ and onto map from $[0,1]$ onto $[-\pi,\pi]$. Clearly here $g\in C([0,1])$.

Notice also that

$$\left(B_N\left((\cdot-x)^2\right)\right)(x)=\left[\left(B_N\left((\cdot-t)^2\right)\right)(t)\right](2\pi)^2=\frac{(2\pi)^2}{N}t(1-t)$$
$$=\frac{(2\pi)^2}{N}\left(\frac{x+\pi}{2\pi}\right)\left(\frac{\pi-x}{2\pi}\right)=\frac{1}{N}(x+\pi)(\pi-x)\le\frac{\pi^2}{N},\quad\forall\,x\in[-\pi,\pi].$$

I.e.

$$\left(B_N\left((\cdot-x)^2\right)\right)(x)\le\frac{\pi^2}{N},\quad\forall\,x\in[-\pi,\pi]. \tag{65}$$

In particular

$$(B_N1)(x)=1,\quad\forall\,x\in[-\pi,\pi]. \tag{66}$$

Define the corresponding application of M by

$$\widetilde{B}_N(X)(t,\omega):=B_N(X(\cdot,\omega))(t)=$$

$$\sum_{k=0}^{N}\binom{N}{k}X\left(-\pi+\frac{2\pi k}{N},\omega\right)\left(\frac{t+\pi}{2\pi}\right)^{k}\left(\frac{\pi-t}{2\pi}\right)^{N-k}, \tag{67}$$

$\forall\, N\in\mathbb{N}, \forall\, t\in[-\pi,\pi], \forall\, \omega\in\Omega$, where X is a stochastic process. Clearly $\widetilde{B}_N$ is a stochastic process.

We give

Proposition 3. *Let $X(t,\omega)$ be a stochastic process from $[-\pi,\pi]\times(\Omega,\mathscr{F},P)$ into $\mathbb{R}$, where $(\Omega,\mathscr{F},P)$ is a probability space. Here $0<\alpha<1$ (i.e. $n=0$) and $X(\cdot,\omega)\in C^1([-\pi,\pi])$, $\forall\,\omega\in\Omega$ and $X^{(1)}(t,\cdot)$ is measurable over $(\Omega,\mathscr{F})$, $\forall\, t\in[-\pi,\pi]$. For any $t\in[-\pi,\pi]$ we assume that $T_\alpha^t X(z,\omega)$ is continuous in $z\in[t,\pi]$, uniformly with respect to $\omega\in\Omega$. And for any $t\in[-\pi,\pi]$ we assume that ${}_\alpha^t TX(z,\omega)$ is continuous in $z\in[-\pi,t]$, uniformly with respect to $\omega\in\Omega$. Finally, we assume that $(E|X|)(t)<\infty$, $\forall\, t\in[-\pi,\pi]$. Then, for any $t\in[-\pi,\pi]$, we have:*

$$E\left(\left|\widetilde{B}_N(X)-X\right|\right)(t)\le(2\pi)^{\alpha}$$
$$\Bigg\{\left[\frac{\left(B_N\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}}{\alpha}+\frac{1}{(\alpha+1)}\right]\left(B_N\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}}$$
$$\Omega_1\left({}_\alpha^t TX,2\pi\left(B_N\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[-\pi,t]}+$$
$$\left[\frac{\left(B_N\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}}{\alpha}+\frac{1}{(\alpha+1)}\right]\left(B_N\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}}$$
$$\Omega_1\left(T_\alpha^t X,2\pi\left(B_N\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[t,\pi]}\Bigg\}, \tag{68}$$

$\forall\, N\in\mathbb{N}$.

Proof. By Corollary 4.

We give

Proposition 4. *All as in Proposition 3 with $\alpha=\frac{1}{2}$. Then, for any $t\in[-\pi,\pi]$, we have:*

$$E\left(\left|\widetilde{B}_N(X)-X\right|\right)(t)\le\sqrt{2\pi}$$
$$\Bigg\{\left[2\left(B_N\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{2}{3}}+\frac{2}{3}\right]\left(B_N\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{1}{3}}$$
$$\Omega_1\left({}_{\frac{1}{2}}^t TX,2\pi\left(B_N\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{2}{3}}\right)_{L^1,[-\pi,t]}+$$
$$\left[2\left(B_N\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{2}{3}}+\frac{2}{3}\right]\left(B_N\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{1}{3}}$$
$$\Omega_1\left(T_{\frac{1}{2}}^t X,2\pi\left(B_N\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{2}{3}}\right)_{L^1,[t,\pi]}\Bigg\}, \tag{69}$$

$\forall\, N\in\mathbb{N}$.

Proof. By Proposition 3.

We continue with

Proposition 5. *All as in Proposition 3 with $\alpha = \frac{1}{2}$. Then, for any $t \in [-\pi, \pi]$, we have:*

$$E\left(\left|\widetilde{B}_N(X) - X\right|\right)(t) \leq \frac{5\sqrt{2\pi}}{3}\left(B_N\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{1}{3}}$$

$$\left[\Omega_1\left({}_{\frac{1}{2}}^{t}TX, 2\pi\left(B_N\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{2}{3}}\right)_{L^1,[-\pi,t]} + \right.$$

$$\left.\Omega_1\left(T_{\frac{1}{2}}^{t}X, 2\pi\left(B_N\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{2}{3}}\right)_{L^1,[t,\pi]}\right], \tag{70}$$

$\forall\, N \in \mathbb{N}$.

Proof. By (69) and the positivity of B_N, see also (42) and (44).

We make

Remark. By $|\sin x| < |x|$, $\forall\, x \in \mathbb{R} - \{0\}$, in particular $\sin x \leq x$, for $x \geq 0$, we get

$$\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{\frac{3}{2}} \leq \left(\frac{|\cdot - t|}{4}\right)^{\frac{3}{2}} = \frac{1}{8}|\cdot - t|^{\frac{3}{2}}.$$

Hence

$$\left\|B_N\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{\frac{3}{2}}\right)(t)\right\|_\infty \leq \frac{1}{8}\left\|B_N\left(|\cdot - t|^{\frac{3}{2}}\right)(t)\right\|_\infty. \tag{71}$$

We observe that

$$B_N\left(|\cdot - t|^{\frac{3}{2}}\right)(t) = \sum_{k=0}^{N}\left|t + \pi - \frac{2\pi k}{N}\right|^{\frac{3}{2}}\binom{N}{k}\left(\frac{t+\pi}{2\pi}\right)^k\left(\frac{\pi - t}{2\pi}\right)^{N-k}$$

(by discrete Hölder's inequality)

$$\leq \left[\sum_{k=0}^{N}\left(t + \pi - \frac{2\pi k}{N}\right)^2\binom{N}{k}\left(\frac{t+\pi}{2\pi}\right)^k\left(\frac{\pi - t}{2\pi}\right)^{N-k}\right]^{\frac{3}{4}}$$

$$= \left(B_N\left((\cdot - t)^2\right)(t)\right)^{\frac{3}{4}} \overset{(65)}{\leq} \frac{\pi^{\frac{3}{2}}}{N^{\frac{3}{4}}}, \quad \forall\, t \in [-\pi, \pi]. \tag{72}$$

Consequently it holds

$$\left\|B_N\left(|\cdot - t|^{\frac{3}{2}}\right)(t)\right\|_\infty \leq \frac{\pi^{\frac{3}{2}}}{N^{\frac{3}{4}}}, \tag{73}$$

and

$$\left\|B_N\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{\frac{3}{2}}\right)(t)\right\|_\infty \leq \frac{\pi^{\frac{3}{2}}}{8N^{\frac{3}{4}}}, \quad \forall\, N \in \mathbb{N}. \tag{74}$$

We further have

Proposition 6. *All as in Proposition 3 with $\alpha = \frac{1}{2}$. Then, for any $t \in [-\pi, \pi]$ we have:*

$$E\left(\left|\widetilde{B}_N(X) - X\right|\right)(t) \leq \frac{5\sqrt{2}\pi}{6\sqrt[4]{N}}$$

$$\left[\Omega_1\left({}_{\frac{1}{2}}^{t}TX, \frac{\pi^2}{2\sqrt{N}}\right)_{L^1,[-\pi,t]} + \Omega_1\left(T_{\frac{1}{2}}^{t}X, \frac{\pi^2}{2\sqrt{N}}\right)_{L^1,[t,\pi]}\right], \tag{75}$$

$\forall\, N \in \mathbb{N}$.

As $N \to +\infty$, we get $E\left(\left|\widetilde{B}_N(X) - X\right|\right)(t) \to 0$.

Proof. By positivity of B_N, (70) and (74). See also Proposition 1.

Consequently we obtain

Proposition 7. *All as in Proposition 3 with $\alpha = \frac{1}{2}$. Assume further that $\left|X^{(1)}(t,\omega)\right| \leq M^*$, $\forall\,(t,\omega) \in [-\pi,\pi] \times \Omega$, where $M^* > 0$. Then*

$$\left\|E\left(\left|\widetilde{B}_N(X) - X\right|\right)\right\|_\infty \leq \frac{5\sqrt{2}\pi}{6\sqrt[4]{N}} \tag{76}$$

$$\left[\sup_{t\in[-\pi,\pi]} \Omega_1\left({}_{\frac{1}{2}}^{t}TX, \frac{\pi^2}{2\sqrt{N}}\right)_{L^1,[-\pi,t]} + \sup_{t\in[-\pi,\pi]} \Omega_1\left(T_{\frac{1}{2}}^{t}X, \frac{\pi^2}{2\sqrt{N}}\right)_{L^1,[t,\pi]}\right],$$

$\forall\, N \in \mathbb{N}$.

As $N \to +\infty$, then $\left\|E\left(\left|\widetilde{B}_N(X) - X\right|\right)\right\|_\infty \to 0$, i.e. $\widetilde{B}_N \to I$ (stochastic unit operator) in 1-mean.

Proof. By (75) and Remark 3, see (17), (18).

6 Trigonometric conformable fractional stochastic Korovkin results

In this section $\widetilde{L}$, M are meant as sequences of operators.

We give first pointwise results:

Theorem 9. *Here all as in Theorem 5. Assume further that $\widetilde{L}(1)(t) \to 1$ and $\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{q(\alpha+1)}\right)(t) \to 0$, pointwise in $t \in [-\pi,\pi]$.*

Then $E\left(|M(X) - X|^q\right)(t) \to 0$, pointwise in $t \in [-\pi,\pi]$, that is $M \to I$ (stochastic unit operator) in q-mean-pointwise with rates, quantitatively.

Proof. We use (51), we take into account $\widetilde{L}(1)(t) \to 1$, (39); and $\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t), \widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t) \leq \widetilde{L}(1)(t)$, which $\widetilde{L}(1)(t)$ is bounded, and by (42), (44) and the positivity of $\widetilde{L}$ we get that

$\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)$, $\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t) \leq \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{q(\alpha+1)}\right)(t) \to 0.$

Finally, we use Proposition 2 (ii) for the $\Omega_1(\cdot,\cdot)$'s to go to zero.

We continue with

Theorem 10. *Here all as in Theorem 6. Assume further that $\widetilde{L}(1)(t) \to 1$ and $\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{(\alpha+1)}\right)(t) \to 0$, pointwise in $t \in [-\pi,\pi]$.*

Then $E\left(|M(X) - X|\right)(t) \to 0$, pointwise in $t \in [-\pi,\pi]$, that is $M \to I$ in 1-mean-pointwise with rates, quantitatively.

Proof. Based on (52), similar to the proof of Theorem 9, just take $q = 1$ there.

Next we give uniform results:

Theorem 11. *Here all as in Theorem 7. Assume further that $\widetilde{L}(1) \to 1$, uniformly, and $\left\| \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4} \right) \right)^{q(\alpha+1)} \right)(t) \right\|_{\infty} \to 0$.*

Then $\| E(|M(X) - X|^q) \|_{\infty} \to 0$ over $[-\pi, \pi]$, that is $M \to I$ in the q-mean, quantitatively with rates.

Proof. We use (53), we take into account $\widetilde{L}(1) \to 1$ uniformly, (40); $\left\| \widetilde{L}(1) \right\|_{\infty}$ is bounded, use of (43), (45) and Remark 3, see there (17), (18).

Next we give the L_1-mean uniform result

Theorem 12. *Here all as in Theorem 8. Assume further that $\widetilde{L}(1) \to 1$, uniformly, and $\left\| \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4} \right) \right)^{(\alpha+1)} \right)(t) \right\|_{\infty} \to 0$.*

Then $\| E(|M(X) - X|) \|_{\infty} \to 0$ over $[-\pi, \pi]$, that is $M \to I$ in the 1-mean, quantitatively with rates.

Proof. Use of (54), similar to the proof of Theorem 11, just take $q = 1$ there.

We finish with

Remark. An amazing fact/observation follows: In all trigonometric convergence results here, see Theorems 9-12, the forcing conditions for convergences are based only on $\widetilde{L}$ and basic real valued continuous functions on $[-\pi, \pi]$ and are not related to stochastic processes, but they are giving trigonometric convergence results on stochastic processes!

Conflict of Interest

The authors declare that they have no conflict of interest.

References

[1] G. A. Anastassiou, Korovkin inequalities for stochastic processes, *J. Math. Anal. Appl.* **157**(2), 366-384 (1991).
[2] G. A. Anastassiou, *Moments in probability and approximation theory, Pitman/Longman*, # 287, UK, 1993.
[3] O. Shisha and B. Mond, The degree of approximation to periodic functions by linear positive operators, *J Approx Theor.* **1**, 335-339 (1968).
[4] M. Weba, Korovkin systems of stochastic processes, *Math. Z.* **192**(1), 73-80 (1986).
[5] M. Weba, A quantitative Korovkin theorem for random functions with multivariate domains, *J. Approx. Theor.* **61**(1), 74-87 (1990).
[6] G. Anastassiou, Stochastic Korovkin theory given quantitatively, Facta Universit. (Nis), *Ser. Math. Inf.* **22**(1), 43-60 (2007).
[7] P. P. Korovkin, *Linear operators and approximation theory*, Hindustan Publ. Corp., Delhi, India, 1960.
[8] G. Anastassiou, *Conformable fractional approximation of stochastic processes*, submitted for publication, 2020.
[9] T. Abdeljawad, On conformable fractional calculus, *J. Comput. Appl. Math.* **279**, 57-66 (2015).
[10] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, A new definition of fractional derivative, *J. Comput. Appl. Math.* **264**, 65-70 (2014).
[11] G. Anastassiou, *Nonlinearity: ordinary and fractional approximations by sublinear and Max-product operators*, Springer, Heidelberg, New York, 2018.
[12] H.L. Royden, *Real analysis*, second edition, MacMillan Publishing Co. Inc., New York, 1968.
[13] G. Anastassiou, *Quantitative approximations*, Chapman & Hall/CRC, Boca Raton, New York, 2001.