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Univariate and Multivariate Double Slash Distribution: Properties and Application

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Abstract: In this paper, we introduce an extension for the slash distribution, called double slash distribution which, is a heavy tailed compared to slash distribution. The univariate and multivariate forms for the proposed model are proposed. Moreover, moments and the invariant property under linear transformations are discussed. A simulation study is performed to investigate asymptotically the bias properties of the estimators. Finally, a real data application is analyzed to obtain the flexibility of the new model.

Keywords: Slash distribution, Moments, Kurtosis, Skewness, Heavy-tailed distribution.

1 Introduction

Heavy-tailed distributions have been the subject of much study in the statistical literature. Heavy-tailed distributions are the distributions which have more observations in the tails. In several areas, there is a clear need for extended forms of these distributions to analyze various types of data. See for example, El-Gohary et al. [1, 2], El-Bassiouny et al. [3–5] , Alizadeh et al. [6], Jehhan et al. [7], Eliwa et al. [8–11], El-Morshedy et al. [12–16], El-Morshedy and Eliwa [17], among others. Unfortunately, the previous distributions are appropriate only for modeling the positive real data sets. Therefore, it is necessary to propose some distributions which are heavy-tailed in both negative and positive ranges. First heavy-tailed alternative distributions to the normal distribution are the student and the slash distributions, which have been very popular in robust statistical analysis (Rogers and Tukey [18], Kafadar [22], Morgenthaler [20], Lange et al. [21], Kafadar [19], Jamshidian [23], and Kashid and Kulkarni [24]). Both of these distributions can be derived by mixing a normally distributed random variable with a nonnegative scale random variable. They both belong to the scale mixture of normal distribution family. It is known that the standard normal distribution has the following density function

$$f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}},\quad -\infty<z<\infty. \tag{1}$$

The slash random variable is defined as the ratio of two independent random variables: Let the standard normal random variable Z be independent of the uniform random variable U on $(0,1)$. Then the random variable $S=ZU^{-1/q}$ is said to have slash normal distribution with the following density:

$$\Psi(s;q)=q\int_0^1 t^q f(st)dt,\qquad -\infty<s<\infty. \tag{2}$$

Where $q>0$ is the shape parameter. For $q=1$, the distribution is called the standard slash normal distribution and it has the following density:

$$\Psi(s;1)=\begin{cases}\frac{1}{\sqrt{2\pi}s^2}\left(1-e^{-\frac{s^2}{2}}\right), & \text{if } s\neq 0\\ \frac{1}{2\sqrt{2\pi}}, & \text{if } s=0.\end{cases} \tag{3}$$

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The standard slash normal density has heavier tails than those of the normal. In literature, many authors studied multivariate and skew multivariate extensions of the slash distribution such as Tan and Peng [25], Wang and Genton [26], GÓmez et al. [27], Arslan [28], Arslan and Genc [29], and EL-Bassiouny and El-Morshedy [30].

The main objective of this paper is to generalize the known family of the slash distributions to be more fitted to the data.

2 Univariate Double Slash Distribution

Theorem 2.1.: Let X have slash distribution with shape parameter q_1, symbolically we write $X \sim SL(q_1)$, and U has uniform distribution over the interval $(0,1)$. Assume that X and U are independent random variables. Define a new random variable $Y = \mu + \sigma X U^{-1/q_2}$, where $q_1, q_2, \sigma > 0$ and $-\infty < \mu < \infty$. The random variable Y has the univariate double slash distribution, symbolically we write $DSL(\mu, \sigma, q_1, q_2)$. The probability density function (pdf) of Y is given by

$$f(y;\mu,\sigma,q_1,q_2) = \frac{q_1 q_2}{\sigma\sqrt{2\pi}} \int_0^1 \left(\int_0^1 e^{-\frac{(y-\mu)^2}{2\sigma^2}\upsilon^2 t^2} t^{q_1}\, dt \right) \upsilon^{q_2} d\upsilon \qquad , -\infty < y < \infty. \tag{4}$$

Proof: Since X and U are independent, then the joint pdf of (X,U) will be

$$g(x,u) = \frac{q_1}{\sqrt{2\pi}} \int_0^1 e^{-\frac{x^2 t^2}{2}} t^{q_1}\, dt \qquad , -\infty < x < \infty.$$

From the transformation $x = \left(\frac{y-\mu}{\sigma}\right) u^{1/q_2}$, the joint pdf of (Y,U) is given by

$$h(y,u) = \frac{q_1 u^{1/q_2}}{\sigma\sqrt{2\pi}} \int_0^1 e^{-\frac{(y-\mu)^2}{2\sigma^2} u^{1/q_2} t^2} t^{q_1}\, dt \quad , -\infty < y < \infty, 0 < u < 1,$$

where $\frac{u^{1/q_2}}{\sigma}$ is the value of the Jacobean. Then the marginal pdf of Y is given by

$$f(y) = \frac{q_1}{\sigma\sqrt{2\pi}} \int_0^1 \left(\int_0^1 e^{-\frac{(y-\mu)^2}{2\sigma^2} u^{1/q_2} t^2} t^{q_1}\, dt \right) u^{1/q_2} du. \tag{5}$$

Using the transformation $\nu = u^{1/q_2}$ in (5), then the pdf of Y will be found as claimed.

If we put $\mu = 0$ and $\sigma = 1$ in (4), then we get the standard form of the univariate double slash distribution $DSL(0,1,q_1,q_2)$.

Remarks:

1. The double slash distribution is much more flexible with its shape parameters than the ordinary slash distribution. Heavy tails and a lower peak of the distribution are associated with smaller q_1 and q_2, see Fig (1) and Fig (2).

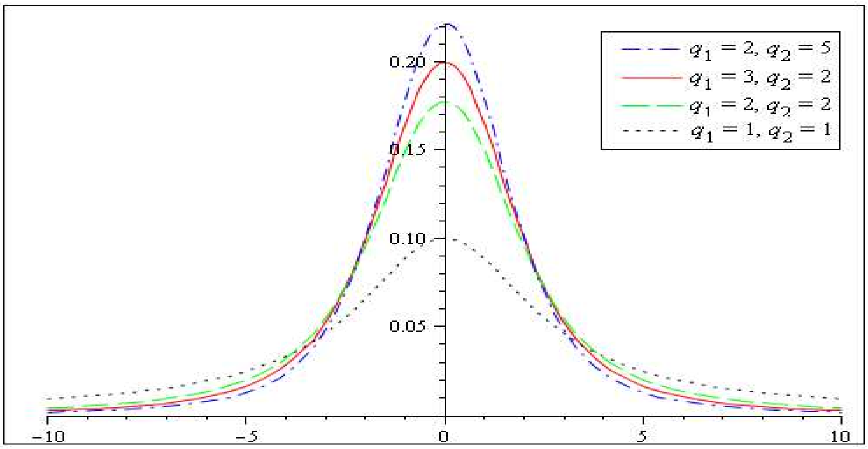
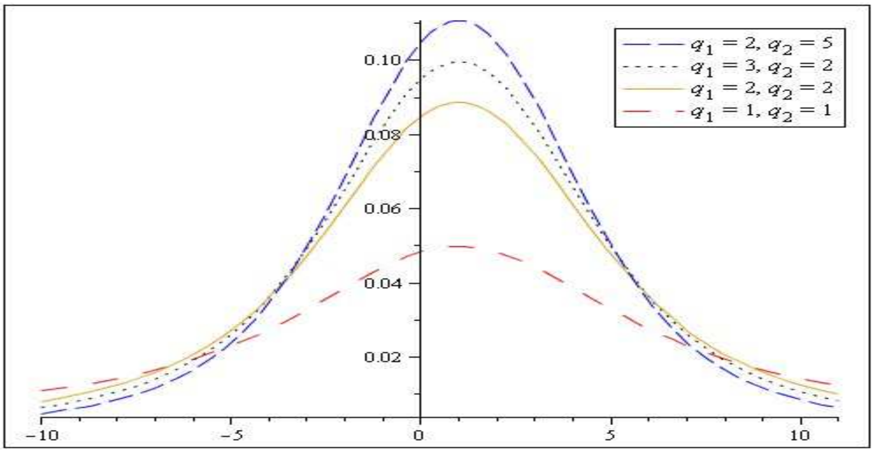


Fig (1): Plot of the pdf of $DSL(0,1,q_1,q_2)$ for different values of q_1, q_2.



Fig(2): Plot of the pdf of $DSL(1, 2, q_1, q_2)$ for different values of q_1, q_2.

2.From Fig (3) and Fig (4), one can easily see that, the double slash distribution is heavier in tails than the slash distribution and the standard normal distribution.

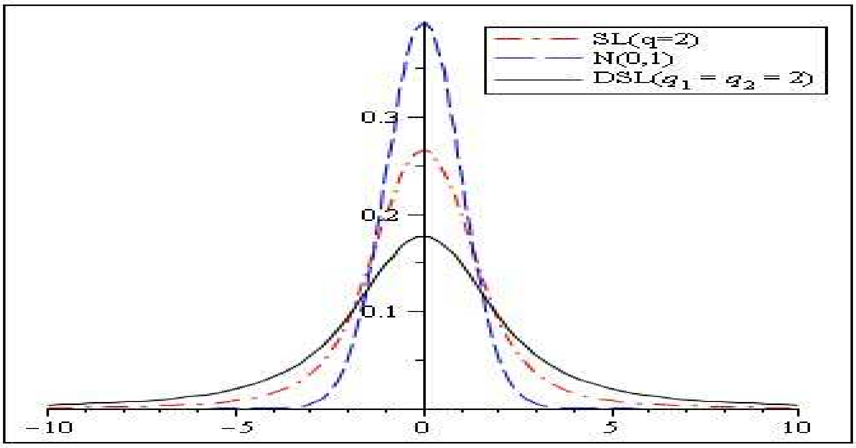
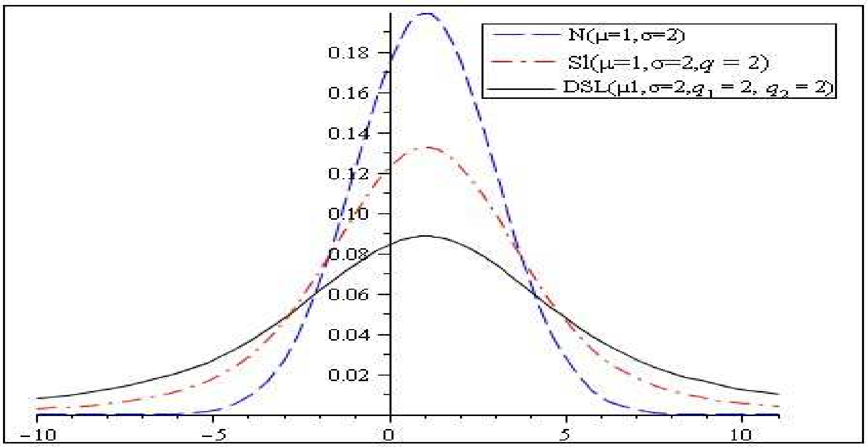


Fig (3): Plot of $N(0, 1)$, $SL(2)$ and $DSL(0, 1, 2, 2)$.



Fig(4): Plot of the normal distribution, slash distribution and double slash distribution.

Special Cases:

1. If $q_1 \to \infty$, then the pdf in (4) tends to the pdf of the slash distribution with shape parameter q_2.
2. If $q_2 \to \infty$, then the pdf in (4) tends to the pdf of the slash distribution with shape parameter q_1.
3. If $q_1 \to \infty$ and $q_2 \to \infty$, then the pdf in (4) tends to the pdf of standard normal distribution given in (1).

3 Statistical Properties for Univariate Double Slash Distribution

3.1 Moments

The moments of the random variable $Y = \mu + \sigma X U^{-1/q_2}$ are given in the following proposition.

Proposition 3.1. The rth moment of the random variable $Y \sim DSL(\mu, \sigma, q_1, q_2)$ is given by

$$E(Y^r) = \sum_{k=0}^{r} \binom{r}{k} \sigma^k \mu^{r-k} E(U^{-k/q_2}) E(X^k), \; r = 1, 2, ..., \tag{6}$$

where $E(X^k)$ and $E(U^{-k/q_2})$ are the kth moment of a slash random variable $X \sim SL(q_1)$ and a uniform random variable $U \sim U(0,1)$ are respectively (see, Wang and Genton [26]) given by

$$E(X^k) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{[(k-1)(k-3)....3.1]q_1}{(q_1-k)} & \text{if } k \text{ is even}, q_1 > k, \end{cases} \tag{7}$$

$$E(U^{-k/q_2}) = \frac{q_2}{q_2 - k}, \qquad q_2 > k. \tag{8}$$

Proof: From the definition of the random variable Y, one can easily get

$$\begin{aligned} E(Y^r) &= E\left(\left(\mu + \sigma U^{-1/q_2} X\right)^r\right) \\ &= E\left(\sum_{k=0}^{r} \binom{r}{k} \left(\sigma U^{-1/q_2} X\right)^k \mu^{r-k}\right) \\ &= \sum_{k=0}^{r} \binom{r}{k} \mu^{r-k} \sigma^k E\left(U^{-k/q_2} X^k\right). \end{aligned}$$

Since Y and U are independent, then (6) follows immediately.

3.2 Skewness and kurtosis

The first four moments about the origin of the random variable $Y = \mu + \sigma X U^{-1/q_2}$ are given by

$$\mu_1^{'} = E(Y) = \mu, \tag{9}$$

$$\mu_2^{'} = E(Y^2) = \mu^2 + \frac{\sigma^2 q_1 q_2}{(q_1 - 2)(q_2 - 2)}, q_1, q_2 > 2, \tag{10}$$

$$\mu_3^{'} = E(Y^3) = \mu^3 + \frac{3\mu\sigma^2 q_1 q_2}{(q_1 - 2)(q_2 - 2)}, q_1, q_2 > 2, \tag{11}$$

and

$$\mu_4^{'} = E(Y^4) = \mu^4 + \frac{6\mu^2\sigma^2 q_1 q_2}{(q_1 - 2)(q_2 - 2)} + \frac{3\sigma^4 q_1 q_2}{(q_1 - 4)(q_2 - 4)}, q_1, q_2 > 4, \tag{12}$$

respectively. Then the first four moments about the mean of the random variable Y are given by

$$\mu_1 = \mu_1^{'} = \mu, \tag{13}$$

$$\begin{aligned} Var(X) = \mu_2 &= \mu_2^{'} - (\mu_1^{'})^2 \\ &= \frac{\sigma^2 q_1 q_2}{(q_1-2)(q_2-2)} \quad , q_1, q_2 > 2, \end{aligned} \tag{14}$$

$$\begin{aligned} \mu_3 &= \mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2\left(\mu_1^{'}\right)^3 \\ &= 0 \end{aligned} \tag{15}$$

and

$$\begin{aligned} \mu_4 &= \mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}\left(\mu_1^{'}\right)^2 - 3\left(\mu_1^{'}\right)^4 \\ &= \frac{3\sigma^4 q_1 q_2}{(q_1-4)(q_2-4)} \quad , q_1, q_2 > 4. \end{aligned} \tag{16}$$

Thus the skewness γ_1 and kurtosis γ_2 are given by

$$\gamma_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} = 0 \tag{17}$$

and

$$\begin{aligned} \gamma_2 &= \frac{\mu_4}{(\mu_2)^2} \\ &= \frac{3(q_1-2)^2(q_2-2)^2}{q_1 q_2 (q_1-4)(q_2-4)} \ , q_1, q_2 > 4. \end{aligned} \tag{18}$$

3.3 Unimodality

The pdf given in (4) has a unimodal. One can show this by verifying this inequality

$$\mu < y_1 \leq y_2 \Rightarrow f(y_1) \geq f(y_2),$$

and this is as, since

$$\begin{aligned} & y_1 \leq y_2 \Rightarrow y_1 - \mu \leq y_2 - \mu. \\ & \Rightarrow \frac{(y_1-\mu)\upsilon t}{\sigma} \leq \frac{y_2-\mu)\upsilon t}{\sigma}. \\ & \Rightarrow \frac{(y_1-\mu)^2 \upsilon^2 t^2}{2\sigma^2} \leq \frac{(y_2-\mu)^2 \upsilon^2 t^2}{2\sigma^2}. \\ & \Rightarrow -\frac{(y_1-\mu)^2 \upsilon^2 t^2}{2\sigma^2} \geq -\frac{(y_2-\mu)^2 \upsilon^2 t^2}{2\sigma^2}. \\ & \Rightarrow e^{-\frac{(y_1-\mu)^2}{2\sigma^2}\upsilon^2 t^2} \geq e^{-\frac{(y_2-\mu)^2}{2\sigma^2}\upsilon^2 t^2}. \\ & \Rightarrow \int_0^1 e^{-\frac{(y_1-\mu)^2\upsilon^2 t^2}{2\sigma^2}} t^{q_1}\,dt \geq \int_0^1 e^{-\frac{(y_2-\mu)^2\upsilon^2 t^2}{2\sigma^2}} t^{q_1}\,dt. \\ & \Rightarrow \frac{q_1 q_2}{\sigma\sqrt{2\pi}} \int_0^1 \left(\int_0^1 e^{-\frac{(y_1-\mu)^2\upsilon^2 t^2}{2\sigma^2}} t^{q_1}\,dt\right) \upsilon^{q_2}\,d\upsilon \geq \frac{q_1 q_2}{\sigma\sqrt{2\pi}} \int_0^1 \left(\int_0^1 e^{-\frac{(y_2-\mu)^2\upsilon^2 t^2}{2\sigma^2}} t^{q_1}\,dt\right) \upsilon^{q_2}\,d\upsilon. \\ & \Rightarrow f(y_1) \geq f(y_2), \end{aligned}$$

since $\frac{q_1 q_2}{\sigma\sqrt{2\pi}} \int_0^1 \left(\int_0^1 e^{-\frac{(y-\mu)^2\upsilon^2 t^2}{2\sigma^2}} t^{q_1}\,dt\right) \upsilon^{q_2}\,d\upsilon \geq 0$. Thus from the inequality and the symmetry of the distribution, the pdf given in (4) is a unimodal.

4 Multivariate Double Slash Distribution

In several scientific practical situations, multivariate lifetime data arise frequently. So, it is very important to consider various multivariate models that could be used to model such multivariate lifetime data. See for example, El-Bassiouny et al. [31], El-Morshedy et al. [32, 33], El-Gohary et al. [34], Mohamed et al. [35], Eliwa and El-Morshedy [36–38], among others. In this section we define a multivariate double slash distribution and derive its pdf. We show that the multivariate double slash distribution is invariant under linear transformations. Furthermore, the moments and marginal distributions are discussed. In the sequel, we denote the $k-$ dimensional multivariate normal distribution with mean vector μ and covariance matrix $\mathbf{\Sigma}$ by $N_k(\mu,\mathbf{\Sigma})$, its pdf by $\phi_k(\mathbf{x};\mu,\mathbf{\Sigma})$. And the standard uniform distribution on the interval $(0,1)$ by $U(0,1)$. Wang and Genton [26] have defined the multivariate slash distribution as the distribution of the random vector

$$\mathbf{X}=\mu+\mathbf{\Sigma}^{1/2}ZU^{-1/q}, \tag{19}$$

where $Z\sim N_k(0,\mathbf{I_k})$ is independent of $U\sim U(0,1)$. The pdf of the random vector $\mathbf{X}$ in (19) is

$$\Psi_k(\mathbf{x};\mu,\mathbf{\Sigma},q)=q\int_0^1 t^{q+k-1}\phi_k(\mathbf{x}t;\mu t,\mathbf{\Sigma})dt \quad ,x\in\mathbb{R}^k. \tag{20}$$

When $\mu=\mathbf{0}$ and $\mathbf{\Sigma}=\mathbf{I}_k$, $\mathbf{X}$ in (19) has a standard form of a multivariate slash distribution, symbolically we write $SL_k(\mathbf{0},\mathbf{I_k},q)$.

Theorem 6.1. Let $\mathbf{X}\sim SL_k(\mathbf{0},\mathbf{I}_k,q_1)$ and $U\sim U(0,1)$ are independent. A k-dimensional continuous random vector $\mathbf{Y}=(Y_1,Y_2,.....,Y_k)$ is said to have a multivariate double slash distribution with location vector $\mu\in\mathbb{R}^k$, positive definite scale matrix $\mathbf{\Sigma}$ and tail parameters q_1 , $q_2>0$, written $\mathbf{Y}\sim DSL_k(\mu,\mathbf{\Sigma},q_1,q_2)$, if it can be written in the form $\mathbf{Y}=\mu+\mathbf{\Sigma}^{1/2}XU^{-1/q_2}$. The pdf of the random vector $\mathbf{Y}$ is

$$f_k(\mathbf{y};\mu,\mathbf{\Sigma},q_1,q_2)=q_1q_2\int_0^1\left(\int_0^1 t^{q_1+k-1}\phi_k(\mathbf{y}vt;\mu vt,\mathbf{\Sigma})dt\right)v^{q_2+k-1}dv \quad ,y\in\mathbb{R}^k. \tag{21}$$

Proof: Since $\mathbf{X}$ and U are independent, then the jpdf of $(\mathbf{X},U)$ will be

$$g(\mathbf{x},u)=q_1\int_0^1 t^{q_1+k-1}\phi_k(\mathbf{x}t;\mathbf{0},\mathbf{I_k})dt \quad ,\mathbf{x}\in\mathbb{R}^k.$$

From the transformation $\mathbf{x}=u^{1/q_2}\left(\frac{\mathbf{y}-\mu}{\mathbf{\Sigma}^{1/2}}\right)$, the jpdf of (Y,U) is given by

$$h(\mathbf{y},u)=\frac{q_1u^{k/q_2}}{|\mathbf{\Sigma}|^{1/2}}\int_0^1 t^{q_1+k-1}\phi_k(u^{1/q_2}(\frac{\mathbf{y}-\mu}{\mathbf{\Sigma}^{1/2}})t;\mathbf{0},\mathbf{I_k})dt \quad ,y\in\mathbb{R}^k.$$

where $\frac{u^{k/q_2}}{|\mathbf{\Sigma}|^{1/2}}$ is the value of the jacobian. Then the marginal pdf of Y is given by

$$f_k(\mathbf{y};\mu,\mathbf{\Sigma},q_1,q_2)=\frac{q_1}{|\mathbf{\Sigma}|^{1/2}}\int_0^1\left(\int_0^1 t^{q_1+k-1}\phi_k(u^{1/q_2}(\frac{\mathbf{y}-\mu}{\mathbf{\Sigma}^{1/2}})t;\mathbf{0},\mathbf{I_k})dt\right)u^{k/q_2}du. \tag{22}$$

Using the transformation $v=u^{1/q_2}$ in (22), and

$$\frac{1}{|\mathbf{\Sigma}|^{1/2}}\phi_k(\frac{\mathbf{y}-\mu}{\mathbf{\Sigma}^{1/2}}vt;\mathbf{0},\mathbf{I_k})=\phi_k(\mathbf{y}vt;\mu vt,\mathbf{\Sigma}),$$

then the pdf of $\mathbf{Y}$ will be found as claimed.

If we put $\mu=\mathbf{0}$ and $\mathbf{\Sigma}=\mathbf{I_k}$ in (21) , then we get the standard form of a multivariate double slash distribution $DSL_k(\mathbf{0},\mathbf{I_k},q_1,q_2)$.

Special Cases:

1. If $k=1$ in (21), then the pdf in (21) tends to the pdf of the univariate double slash distribution given in (4).

2.If $k = 2$ in (21), then we obtain the bivariate double slash distribution, and its pdf will be

$$f_2(\mathbf{y};\mu,\mathbf{\Sigma},q_1,q_2) = q_1q_2\int_0^1\left(\int_0^1 t^{q_1+1}\phi_2(\mathbf{y}vt;\mu vt,\mathbf{\Sigma})dt\right)v^{q_2+1}dv \quad , y\in\mathbb{R}^2. \tag{23}$$

3.If $q_1\to\infty$, then the pdf in (21) tends to the pdf of the multivariate slash distribution with shape parameter $q_2 > 0$.
4.If $q_2\to\infty$, then the pdf in (21) tends to the pdf of the multivariate slash distribution with shape parameter $q_1 > 0$.
5.If $q_1\to\infty$ and $q_2\to\infty$, then the pdf in (21) tends to the pdf of the multivariate standard normal distribution.

5 Statistical Properties for Multivariate Double Slash Distribution

5.1 Moments

The expectation, variance and the first two moments of the multivariate double slash distribution are given in the following proposition.

Proposition 7.1. If $\mathbf{Y}=\mu+\mathbf{\Sigma}^{1/2}\mathbf{X}U^{-1/q_2}$ has $DSL_k(\mathbf{y};\mu,\mathbf{\Sigma},q_1,q_2)$, then its expectation and variance are given by

$$E(\mathbf{Y})=\mu, \tag{24}$$

$$Var(\mathbf{Y})=\frac{\mathbf{\Sigma}q_1q_2}{(q_1-2)(q_2-2)} \quad , q_1,q_2>2. \tag{25}$$

Proof: The moments of slash and uniform random variables are given in (11) and (12) respectively. Since $\mathbf{X}$ and U are independent, then the first two moments of $\mathbf{Y}$ is

$$\begin{aligned}E(\mathbf{Y}) &= E(\mu+\mathbf{\Sigma}^{1/2}\mathbf{X}U^{-1/q_2})\\ &= \mu+\mathbf{\Sigma}^{1/2}E(\mathbf{X}U^{-1/q_2}) = \mu\ ,\end{aligned} \tag{26}$$

$$\begin{aligned}E(\mathbf{Y}^2)&=E(\left(\mu+\mathbf{\Sigma}^{1/2}\mathbf{X}U^{-1/q_2}\right)^2)\\ &= E(\mu^2+2\mu\mathbf{\Sigma}^{1/2}\mathbf{X}U^{-1/q_2}+\mathbf{\Sigma}\left(\mathbf{X}U^{-1/q_2}\right)^2)\\ &= \mu^2+\frac{\mathbf{\Sigma}q_1q_2}{(q_1-2)(q_2-2)}.\end{aligned} \tag{27}$$

From (26) and (27), one can easily get (25).

5.2 Marginal distributions

Since the marginal distributions of a multivariate slash distribution are still slash distributions (see Wang and Genton [26]), the marginal distributions of a double slash distribution are also still double slash distributions. The following proposition states this fact.

Proposition 8.1.1. The marginal distributions of a double slash distribution are still double slash.

Proof: It suffices to show without loss of generality that

$$\int f_k(y_1,..,y_k;\mathbf{0},\mathbf{I_k},q_1,q_2)dy_{s+1}..dy_k = f_s(y_1,..,y_s;\mathbf{0},I_\mathbf{s},q_1,q_2) \quad ,y_1,..,y_s \in \mathbb{R}. \tag{28}$$

For every $0 \le s \le k$. Substitution of the formula (21) in the left hand of the above gives

$$\begin{aligned} LHS &= \int f_k(y_1,..,y_k;\mathbf{0},I_k,q_1,q_2)dy_{s+1}..dy_k, \\ &= q_1q_2\int_0^1\left(\int_0^1 t^{q_1+k-1}\int \phi_k(vty_1,..,vty_k;0,I_k)dy_{s+1}..dy_k dt\right)v^{q_2+k-1}dv. \end{aligned}$$

With substitution $x_{s+1} = vty_{s+1},...,x_k = vty_k$ for $v,t > 0$ one has

$$\int \phi_k(vty_1,..vty_k;\mathbf{0},\mathbf{I}_k)dy_{s+1}..dy_k = (vt)^{s-k}\int \phi_k(vty_1,..,vty_s,x_{s+1},..,x_k;\mathbf{0},\mathbf{I}_k)dx_{s+1}..dx_k.$$

Because the marginals of the normal distribution are still normal, we have

$$\int \phi_k(vty_1,..,vty_s,x_{s+1},..,x_k;\mathbf{0},\mathbf{I_k})dx_{s+1}..dx_k = \phi_s(vty_1,..,vty_s;\mathbf{0},\mathbf{I_s}).$$

The last two equalities yield the desired equality.

5.3 Linear combinations

Since the distribution of a linear function of slash random vector is still slash (see Wang and Genton [26]), the distribution of a linear function of double slash random vector is also still double slash, i.e. the multivariate double slash distribution is invariant under linear transformation. The following proposition states this fact.

Proposition 8.2.1. If $Y \sim DSL_k(\mathbf{y};\mu,\mathbf{\Sigma},q_1,q_2)$, then its linear transformation $\mathbf{W} = \mathbf{b}+\mathbf{AY} \sim DSL_k(\mathbf{b}+\mathbf{A}\mu,\mathbf{A\Sigma A}^T,q_1,q_2)$, $\mathbf{b}$ is a vector in $\mathbb{R}^k$, and $\mathbf{A}$ is a nonsingular matrix.

Proof: From the transformation, we have $\mathbf{Y} = \mathbf{A}^{-1}(\mathbf{W}-\mathbf{b})$, therefore, the Jacobian determinant of the transformation is $|\mathbf{A}|^{-1}$, hence the pdf of $\mathbf{W}$ is

$$\begin{aligned} f(\mathbf{w}) &= |\mathbf{A}|^{-1} f_k(\mathbf{A}^{-1}(\mathbf{w}-\mathbf{b});\mu,\mathbf{\Sigma},q_1,q_2) \\ &= |\mathbf{A}|^{-1} q_1q_2\int_0^1\left(\int_0^1 t^{q_1+k-1}\phi_k(\mathbf{A}^{-1}(\mathbf{w}-\mathbf{b})vt;\mu vt,\mathbf{\Sigma})dt\right)v^{q_2+k-1}dv. \end{aligned}$$

We have

$$|\mathbf{A}|^{-1}\phi_k(\mathbf{A}^{-1}(\mathbf{w}-\mathbf{b})vt;\mu vt,\mathbf{\Sigma}) = \phi_k(\mathbf{w}vt;(\mathbf{b}+\mathbf{A}\mu)vt,\mathbf{A\Sigma A}^T). \tag{29}$$

Hence from (21) and (29), the pdf of W is

$$\begin{aligned} f(\mathbf{w}) &= q_1q_2\int_0^1\left(\int_0^1 t^{q_1+k-1}\phi_k(\mathbf{w}vt;(\mathbf{b}+\mathbf{A}\mu)vt,\mathbf{A\Sigma A}^T)dt\right)v^{q_2+k-1}dv \\ &= f_k(\mathbf{w};\mathbf{b}+\mathbf{A}\mu,\mathbf{A\Sigma A}^T,q_1,q_2). \end{aligned}$$

This shows that $\mathbf{W}$ has a multivariate double slash distribution $DSL_k(\mathbf{b}+\mathbf{A}\mu,\mathbf{A\Sigma A}^T,q_1,q_2)$. It implies that the multivariate double slash distribution is invariant under linear transformation.

6 Likelihood Estimation

Proposition 5.1. Let $y_1, y_2, ..., y_n$ be a data set modeled by the $DSL(\mu, \sigma, q_1, q_2)$ distribution in the location scale form. Then the estimation of μ and σ^2 are given by

$$\hat{\mu} = \frac{\sum_{i=1}^{n} \omega_i y_i}{\sum_{i=1}^{n} \omega_i} \tag{30}$$

and

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \omega_i \left(y_i - \hat{\mu} \right)^2, \tag{31}$$

where

$$\omega_i(s) = \frac{\int_0^1 \left(\int_0^1 e^{-\frac{(s\upsilon t)^2}{2}} t^{q_1+2} dt \right) \upsilon^{q_2+2} d\upsilon}{\int_0^1 \left(\int_0^1 e^{-\frac{(s\upsilon t)^2}{2}} t^{q_1} dt \right) \upsilon^{q_2} d\upsilon} \quad , s = \left| y_i - \hat{\mu} \right| / \hat{\sigma}. \tag{32}$$

Proof: The log-likelihood function is given by

$$\begin{aligned} L(\mu, \sigma, q_1, q_2) &= \log \Pi_{i=1}^{n} f(y_i; \mu, \sigma, q_1, q_2) \\ &= n \log \left[\frac{q_1 q_2}{\sqrt{2\pi}} \right] - n \log \sigma + \sum_{i=1}^{n} \log \int_0^1 \left(\int_0^1 e^{-\frac{(y_i-\mu)^2}{2\sigma^2} \upsilon^2 t^2} t^{q_1} dt \right) \upsilon^{q_2} d\upsilon. \end{aligned}$$

Taking partial derivatives of the log-likelihood function with respect to μ and σ, assuming the shape parameters are fixed, and equating the derivatives to 0, we get

$$\frac{\partial L(\mu, \sigma, q_1, q_2)}{\partial \mu} = 0,$$

$$\sum_{i=1}^{n} \frac{\frac{(y_i-\mu)}{\sigma^2} \int_0^1 \left(\int_0^1 e^{-\frac{(y_i-\mu)^2}{2\sigma^2} \upsilon^2 t^2} t^{q_1+2} dt \right) \upsilon^{q_2+2} d}{\int_0^1 \left(\int_0^1 e^{-\frac{(y_i-\mu)^2}{2\sigma^2} \upsilon^2 t^2} t^{q_1} dt \right) \upsilon^{q_2} d\upsilon} = 0.$$

Using (32), we get

$$\frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \mu) \, \omega_i(s) = 0,$$

$$\sum_{i=1}^{n} y_i \omega_i(s) = \mu \sum_{i=1}^{n} \omega_i(s),$$

thus (30) is obtained.

$$\frac{\partial L(\mu, \sigma, q_1, q_2)}{\partial \sigma} = 0,$$

$$-\frac{n}{\sigma} + \sum_{i=1}^{n} \frac{\frac{(y_i-\mu)^2}{\sigma^3} \int_0^1 \left(\int_0^1 e^{-\frac{(y_i-\mu)^2}{2\sigma^2} \upsilon^2 t^2} t^{q_1+2} dt \right) \upsilon^{q_2+2} d\upsilon}{\int_0^1 \left(\int_0^1 e^{-\frac{(y_i-\mu)^2}{2\sigma^2} \upsilon^2 t^2} t^{q_1} dt \right) \upsilon^{q_2} d\upsilon} = 0.$$

Using (32), we obtain

$$\frac{n}{\sigma} = \frac{1}{\sigma^3} \sum_{i=1}^{n} (y_i - \mu)^2 \, \omega_i(s),$$

thus (31) is obtained.

7 Application

We perform a simulation study to investigate bias properties of the estimators asymptotically. All computations were performed using R program and all program codes are available from the author on request.

7.1 Simulation results

Because of the complexity of the log-likelihood function, one cannot derive the information matrix. It is impossible to find a theoretical asymptotic property of the maximum likelihood estimators. Therefore, we investigate the properties of the estimators numerically. We perform simulations to investigate the properties (bias and variance) of the estimators depending on the shape parameters. We first generate 500 samples of different sizes from the double slash distribution for fixed shape parameters. Then, use the iterative forms of the estimators given in (30) and (31) to compute the estimates. The Means and variances of the location-scale estimates of 500 samples of sizes n = 20, 50, 250, 500 from the double slash distribution with $\mu = 1$ and $\sigma = 2$, and (q_1, q_2) are equal to $(2, 3)$, $(3, 3)$, $(3, 4)$, $(3, 6)$ and $(5, 2)$ as given in Table (1).

The mean and variances of the estimates are given in Table (1).

Table 1. Simulation results for double slash distribution.

n	$M(\hat{\mu})$	$V(\hat{\mu})$	$M(\hat{\sigma})$	$V(\hat{\sigma})$
	$(q_1, q_2) = (2, 3)$			
20	1.0089801	0.8766603	2.112191	0.3714785
50	1.0057378	0.4084151	1.971972	0.1079974
250	0.9931468	0.0633544	2.057733	0.0227480
500	1.0107949	0.0322603	2.006742	0.0177269
	$(q_1, q_2) = (3, 3)$			
20	0.9353532	0.7641253	2.034942	0.3193231
50	1.0073877	0.2864782	1.989331	0.1555472
250	0.9895778	0.0550117	1.991173	0.0184434
500	0.9875621	0.0278744	1.959431	0.0094265
	$(q_1, q_2) = (3, 4)$			
20	1.0231992	0.6865062	1.963236	0.2462273
50	1.0069928	0.2522465	1.933337	0.0876646
250	0.9986048	0.0548670	2.023148	0.4731074
500	1.0050103	0.0242011	2.025628	0.0079936
	$(q_1, q_2) = (3, 6)$			
20	1.0947817	0.5813642	1.973516	0.2241017
50	0.9685235	0.1995623	1.975382	0.0782994
250	0.9906496	0.0453823	1.954289	0.0146792
500	0.9985277	0.0213544	2.010562	0.0078255
	$(q_1, q_2) = (5, 2)$			
20	1.0227201	0.7215455	2.009685	0.2966756
50	0.9945268	0.2695377	2.023486	0.1013172
250	1.0210454	0.0636307	2.036190	0.0200127
500	1.0122063	0.0288960	2.034938	0.0112197

Table (1) tells us that the estimates $\hat{\mu}$ and $\hat{\sigma}$ seem asymptotically unbiased. As the sample size increase, the variance of the estimates approaches to 0, as expected.

7.2 Real data

In estimating the location and dispersion of a univariate data set, we usually use the sample mean and the sample variance, respectively. These statistics are easily computed and efficient at the normal situation. However, if the data follow a heavy-tailed distribution, they can give unreliable information about the location and scale parameters. In that case a robust method can be used. In order to see the performance of the GSl estimators in the location-scale case, we consider

Rosner data [39]. The data set consists of 10 monthly diastolic blood pressure measurements and as follows: 90, 93, 86, 92, 95, 83, 75, 40, 88, 80. We note that the observation 40 is far from the other observations. Thus, it is a (possible) outlier. The sample mean of the data is 82.2, and the standard deviation is 19.1. They are influenced by the outlier badly. In order to summarize the location and scale of the data more accurately, we must apply a robust method. Note that the mean without the outlier, which is robust by an appropriate rejection rule, is 86.9 and the median is 87. The location.lms function in S-Plus gives the LMS-estimates of location and scale as 90.8 and 5.1, respectively. We can also find estimates by modeling Rosner data with some heavy-tailed distributions such as exponential power (EP) and generalized t (GT) and, of course, the GSl. We use nlminb function in S-Plus to find the estimates. The fitted distributions are compared using some criteria, namely the maximized log-likelihood (**-L**), Akaike Information Criterion (AIC), correct Akaike information criterion (CAIC), bayesian information Criterion (BIC) and Hannan-Quinn information criterion (HQIC). The results are in Table (2).

Table 2. The MLE, log-likelihood, AIC, BIC, CAIC, and HQIC values.

Model	$\hat{p}$	$\hat{q}_1$	$\hat{q}_2$	$\hat{\mu}$	$\hat{\sigma}$	-L	AIC	BIC	CAIC	HQIC
N	$--$	$--$	$--$	85.3	10.2	41.5	87.0	87.6	88.7	86.3
EP	1	$--$	$--$	87.5	9.4	39.3	82.6	83.2	84.3	81.9
Sl	$--$	6	$--$	88.2	8.9	38.9	81.8	82.4	83.5	81.1
GT	184.9	$--$	$--$	86.5	6.7	37.8	79.6	80.2	81.3	78.9
GSl	1.5	6	$--$	90.5	5.3	35.8	75.6	76.2	77.3	74.9
DSl	$--$	6	6	90.8	5.1	34.9	73.0	73.6	74.7	72.3

It is clear that from Table (2) that the DSl distribution is the best distribution for fitting this data among all the tested distributions, as the DSl has the smallest valu of -L, AIC, CAIC, BIC, and HQIC.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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