Applied Mathematics & Information Sciences
An International Journal

On existence-uniqueness of the solution in a nonlinear Biot's model

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Received: 22 Apr. 2012; Revised 21 Nov. 2012; Accepted 27 Nov. 2012
Published online: 1 Jan. 2013

Abstract: The response of elastic porous media under applied loads consists of an instantaneous deformation followed by a time dependent consolidation process associated with the drainage of the pore fluid. In the simulation of the swelling of elastic cartilagineous tissues, the permeability tensor of the porous medium depends on the strain, thus resulting in a nonlinear model. In this paper, we present a nonlinear one dimensional Biot's model and prove the existence and uniqueness of the solution of this model. Appropriate boundary conditions required for the uniqueness of the solution are to be introduced. Then, the Galerkin method is used to prove that the model has a unique weak solution. Finally, two simple numerical examples of 1-D non linear Biot's model are discussed.

Keywords: Biot's model, Partial differential equations, Galerkin's method.

1. Introduction

By electric charges fixed to the solid and counteracted by corresponding charges in fluid, many biological porous media like intervertebral discs exhibit swelling and/or shrinking behavior when, for example, in contact with salt concentrations. For instance, cartilagineous tissues are soft tissues with strong swelling and shrinking properties. They play an important role in joint lubrication and damping of dynamic forces in the human body. The classical Biot model, originally developed for soil consolidation, is able describe the time-dependent interaction between the deformation of an elastic porous material and the fluid flow inside it, consisting of equilibrium equations for an element of soil, stress-strain relations for the soil skeleton and a continuity equation for the pore fluid (see [21–23]).

This problem was first introduced in the study of soils by K. Terzaghi in 1943 [27] and later generalized by Maurice Biot in a number of papers [5–7]. The Biot theory of poroelasticity has been widely used in geomechanics, hydrogeology and petroleum engineering, e.g. [2,15,32]. Some consolidation models have been presented, so far. These models have been studied widely by investigators [5,16,18,13,20,22]. In particular, a lot of effort has been dedicated to the numerical treatment of this model. Several numerical techniques, such as radial point interpolation [29], meshless [14], finite difference [16] and finite element methods [8,18,19], for Biot's consolidation model have been proposed by many investigators. Moreover, the numerical solution to the algebraic systems of equations obtained with such methods can be very difficult, requiring ad hoc advanced preconditioners to converge [4].

In the analysis of biological porous media the deformation behavior can be often assumed elastic with the permeability tensor depending on the strain. This gives rise to a non linear Biot model. Ženišek in [31] has proved the existence and uniqueness for the weak form of Biot's consolidation theory. Also, R.E. Showalter has developed the existence, uniqueness and regularity theory for a general initial boundary value problem for a system of partial differential equations which describes the Biot consolidation model in poroelasticity as well as a coupled quasi static problem in thermoelasticity [26]. In [3], existence and uniqueness of a nonlinear one dimensional wave propagation model in a poro-elastic medium has been studied that this model generally is described by Biot model. In this paper, we will study the existence and uniqueness of a nonlinear one dimensional model for the swelling of cartilaginous tissues. In this model, we assume that the permeability is not constant and it depends on pressure or

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strain. In [21,24], when the coefficient of permeability is constant, this model for one and two dimensional configuration has been studied.

In Section 2, the governing equations are given for Biot's model with the required boundary conditions. In Section 3, a nonlinear one dimensional version of Biot's poroelasticity is presented. Also, some preliminaries about Sobolev spaces are stated in this section. Finally, in Section 4, by Galerkinn's method, existence and uniqueness of our model would be proved.

2. Biot's Problem

We consider Ω as a bounded, open and connected subset of $\mathbb{R}^n, n = 1,2,3$ with a Lipschitz continuous boundary Γ and unit outward normal $\mathbf{n}$. Let $\mathbf{u} : \Omega \to \mathbb{R}^n$ be the displacement of the porous medium and ϵ be the strain for each time $t \in [0,T]$ with $T > 0$. The elastic strain- displacement relationship is defined as:

$$\epsilon = D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$$

We write the stress-strain relation as follows:

$$\sigma = G(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) + \frac{2G\nu}{1-2\nu}(\nabla.\mathbf{u})\mathbf{I}$$

where G and ν are the shear modulus and the Poisson ratio of the elastic material in the absence of a pore fluid and $\mathbf{I}$ is the identity matrix. The shear modulus (or modulus of rigidity) is the ratio of applied shear stress to shear strain; Poisson's ratio describes the compressibility of the material and is defined as the ratio of transverse contraction strain (normal to the applied load) to longitudinal extension strain (in the direction of the applied load). Now, letting $\mu_s = G$ and $\lambda_s = \dfrac{2G\nu}{1-2\nu}$, we have

$$\sigma = 2\mu_s\epsilon + \lambda_s tr\epsilon(\mathbf{u})\mathbf{I}$$

We assume that the fluid transport in the pore space can be described by Darcy's law which relates the fluid flow rate to the gradient of the pore pressure $p : \Omega \to \mathbb{R}^2$:

$$\mathbf{q} = -K\nabla p - \mathbf{f}. \qquad K = \frac{\kappa}{\mu}$$

where K is the coefficient of permeability, κ is the intrinsic permeability of the porous material, μ is the viscosity of the fluid and $\mathbf{f}$ is a body force term. Another equation is related to the continuity equation that is:

$$\nabla.\mathbf{q} + \nabla.\mathbf{u}_t = 0, \quad \text{in } \Omega.$$

Let the sets $\Gamma_{\mathbf{u}}^D$ and $\Gamma_{\mathbf{u}}^N$ (and similarly Γ_p^D and Γ_p^N) be two open subsets of the total boundary $\Gamma = \partial\Omega$ such that

$$\Gamma_\alpha^D \cap \Gamma_\alpha^N = \emptyset \ \text{ and } \overline{\Gamma}_\alpha^{\mathrm{D}} \cap \overline{\Gamma}_\alpha^{\mathrm{N}} = \Gamma \ \text{ for } \alpha = \mathbf{u}, \mathrm{p}.$$

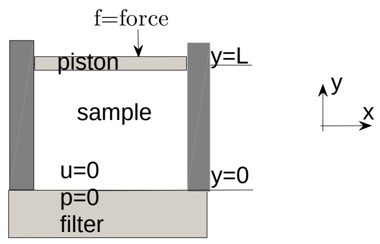

Figure 1 Schematic representation of the confined compression experiment.

Also, assume that $meas(\Gamma_\alpha^D) > 0$ for $\alpha = \mathbf{u}, \mathrm{p}$.

Now, we consider boundary conditions of the following type ([20,22,23]):

$$\begin{aligned} \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_{\mathbf{u}}^D \times (0,T], \\ p &= 0 && \text{on } \Gamma_p^D \times (0,T], \\ \mathbf{n}.(\sigma(\mathbf{u}) - p\mathbf{I}) &= g(t) && \text{on } \Gamma_{\mathbf{u}}^N \times (0,T], \\ \mathbf{n}.\mathbf{q} &= 0 && \text{on } \Gamma_p^N \times (0,T], \end{aligned} \tag{1}$$

which $\mathbf{I}$ is the identity tensor and $g(t)$ is the applied load. Also, we consider the initial boundary condition

$$div(\mathbf{u}) = 0 \ \text{ in } \Omega, \quad \mathrm{t} = 0.$$

3. One Dimensional Model

In this section, we present a one dimensional model of Biot's poroelasticity where the permeability of the porous material is not constant.
Let $u(y,t)$ denotes the vertical solid displacement, $p(y,t)$ the fluid pressure, μ_s and λ_s two material properties and constant, $K(\partial u/\partial y)$ the permeability which is assumed to be dependent on strain $\partial u/\partial y$. We also assume that $K(\partial u/\partial y)$ is a bounded function. i.e. $0 < m \le K(.) \le M$ in Ω where m and M are constant real numbers. Throughout this paper, m and M are considered to be lower and upper bounds of $K(.)$, respectively.

Figures 1 and 2 (see [21]) depict an example of experimental setup. A homogeneous sample is placed frictionless in a holder. At the bottom $y = 0$, the sample is in contact with a glass filter saturated by a sodium chloride solution. An impermeable piston is placed on the top of the sample, $y = L$, where an external mechanical load is applied. The sample was made of hydrogel. A bathing solution flowed through a porous glass filter at the bottom of the sample. We also assume that there is no body force, i.e. $\mathbf{f} = \mathbf{0}$.

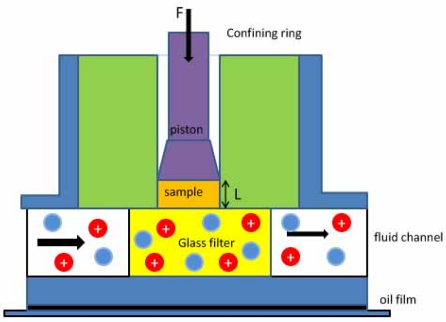

Figure 2 Schematic representation of the confined compression experiment.

The one dimensional Biot's poroelasticity model reads:

$$\begin{aligned}
&(a)\ 0 = (2\mu_s+\lambda_s)\frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial y} && \text{in } [0,L]\times(0,T],\\
&(b)\ 0 = \frac{\partial^2 u}{\partial t\partial y} - \frac{\partial}{\partial y}(K(\frac{\partial u}{\partial y}).\frac{\partial p}{\partial y}) && \text{in } [0,L]\times(0,T],\\
&(c)\ 0 = u && \text{on } \{y=0\}\times(0,T],\\
&(d)\ 0 = p && \text{on } \{y=0\}\times(0,T],\\
&(e)\ g(t) = (2\mu_s+\lambda_s)\frac{\partial u}{\partial y} - p && \text{on } \{y=L\}\times(0,T],\\
&(f)\ 0 = \frac{\partial p}{\partial y} && \text{on } \{y=L\}\times(0,T].
\end{aligned} \tag{2}$$

where $g(t)$ is the load on the piston. In addition, as an initial condition is assumed that

$$p = p_0(y) \ \text{ on } [0,\mathrm{L}]\times\{\mathrm{t}=0\}.$$

We consider that u and p are sufficiently smooth, then, by (2a) and (2e),

$$(2\mu_s+\lambda_s)\frac{\partial u}{\partial y} - p = g(t) \qquad \text{in } [0,\mathrm{L}]\times(0,\mathrm{T}]. \tag{3}$$

This shows that if the permeability $K(.)$ depends on the strain $\frac{\partial u}{\partial y}$, it will also depend on the pressure, i.e.

$$K(\partial u/\partial y) = K((p+g(t))/(2\mu_s+\lambda_s)).$$

In the sequel, for sake of simplicity, we assume that

$$K(\partial u/\partial y) = K(p).$$

Furthermore, by (3),

$$(2\mu_s+\lambda_s)\frac{\partial^2 u}{\partial t\partial y} - \frac{\partial p}{\partial t} = \frac{dg}{dt}. \tag{4}$$

Let $f = -\frac{dg}{dt}$ and therefore by substituting (4) in (2(b)),

$$\frac{dp}{dt} - f = (2\mu_s+\lambda_s)\frac{\partial}{\partial y}(K(p).\frac{\partial p}{\partial y}) \qquad \text{in } [0,\mathrm{L}]\times(0,\mathrm{T}].$$

Thus, we can state the existence and uniqueness theorems for the following initial-boundary value problem:

$$\begin{aligned}
&(a)\quad f = \frac{dp}{dt} - (2\mu_s+\lambda_s)\times\\
&\qquad \times\frac{\partial}{\partial y}(K(p).\frac{\partial p}{\partial y}) && \text{in } [0,L]\times(0,T],\\
&(b)\quad 0 = p && \text{on } \{y=0\}\times(0,T],\\
&(c)\quad 0 = \frac{\partial p}{\partial y} && \text{on } \{y=L\}\times(0,T],\\
&(d)\ p_0(y) = p && \text{on } [0,L]\times\{t=0\}.
\end{aligned} \tag{5}$$

We will prove that the variational problem of (5) has a unique solution and hence by (3) and (2c), the uniqueness of solution is proved for the problem (2). To this end, we first introduce the appropriate function spaces. Let $L^2(\Omega)$ be the Hilbert space of square integrable scalar-valued functions defined on Ω with inner product

$$(f,g) = \int_\Omega fg dx, \qquad f,g\in L^2(\Omega).$$

The symbol $H^m(\Omega)$ denotes the usual Sobolev space,

$$\begin{aligned}H^m(\Omega) = \{q|D^\alpha q\in L^2(\Omega), \alpha = (\alpha_1,\alpha_2,\cdots,\alpha_n),\\ \alpha_i\in\mathbb{Z}^+, |\alpha|\le m\}\end{aligned}$$

equipped with the norm

$$\|q\|_m^2 = \sum_{|\alpha|\le m}\|D^\alpha q\|^2_{L^2(\Omega)}.$$

Furthermore, $H_0^m(\Omega)$ is the closure of the space $C_0^\infty(\Omega)$ in the norm $\|.\|_m$. It can be shown that

$$H_0^1(\Omega) = \{q\in H^1(\Omega) : q = 0 \text{ on } \partial\Omega\}.$$

Also, we define

$$|q|_m^2 = \sum_{|\alpha|=m}\|D^\alpha q\|^2_{L^2(\Omega)},$$

as a semi-norm on $H^1(\Omega)$ that is a norm over the space $H_0^m(\Omega)$, equivalent to the norm $\|.\|_{H^m(\Omega)}$ ([1]). In addition, $H^{-1}(\Omega)$ is the dual space to the space $H_0^1(\Omega)$. We shall also need to consider functions that vanish on a part of the boundary; suppose $\partial\Omega = D\cup N$, a partition of $\partial\Omega$ into disjoint, one then defines

$$H^1_{0,D}(\Omega) = \{q\in H^1(\Omega) : q = 0 \text{ on D}\},$$

and one has $H_0^1(\Omega)\subset H^1_{0,D}(\Omega)\subset H^1(\Omega)$(for more details see [9] Chapter III). In this paper, we have assumed that $D = \{y\in\partial\Omega | y = 0\}$. The following lemma ensures us to consider a semi-norm over the space $H^1(\Omega)$ as a norm over the $H^1_{0,D}(\Omega)$:

Lemma 1. *Let Ω be a connected bounded open subset of $\mathbb{R}^n$. Then the space $H^1_{0,D}(\Omega)$ is a closed subspace of $H^1(\Omega)$. If the $d\gamma-$measure of D is strictly positive, the semi-norm $|.|^2_1$ is a norm over the space $H^1_{0,D}(\Omega)$, equivalent to the norm $\|.\|_{H^1(\Omega)}$.*

Proof.: See [11].

We denote by $H^{-1}_D(\Omega)$ the dual space to the space $H^1_{0,D}(\Omega)$. If $\ell \in H^{-1}_D(\Omega)$, we define the norm

$$\|\ell\|_{H^{-1}_D(\Omega)} = \sup\{< \ell, v >: v \in H^1_{0,D}(\Omega), \|v\|_{H^1_{0,D}(\Omega)} \leq 1\},$$

where $< .,. >$ denotes the duality pairing between $H^{-1}_D(\Omega)$ and $H^1_{0,D}(\Omega)$.

When considering space-time functions $v(\mathbf{x},t)$, in which $(\mathbf{x},t) \in Q_T := (0,T) \times \Omega$, it is natural to introduce the space

$$L^p(0,T;X) = \{v : (0,T) \to X| \quad v \text{ is measurable and } \int_0^T \|v(t)\|_{L^p(0,T;X)} < \infty\},$$

$1 \leq p < \infty$, endowed with the norm

$$\|v(t)\|_{L^p(0,T;X)} = [\int_0^T \|v(t)\|^p_X dt]^{1/p},$$

where X is a Banach space. With these preliminaries, the variational form of (5) reads:
For a given f in $L^2(0,T;H^{-1}_D(\Omega))$ and a given element p_0 of $L^2(\Omega)$, find $p(t) \in L^2(0,T;H^1_{0,D}(\Omega))$ such that:

$$\begin{cases} \frac{d}{dt}(p(t),\phi) + (2\mu_s+\lambda_s)(K(p(t))\frac{\partial p(t)}{\partial y}, \frac{d\phi}{dy}) = (f,\phi), \\ \qquad p(0) = p_0, \end{cases} \tag{6}$$

for all $\phi \in H^1_{0,D}(\Omega)$. We note that the initial condition makes sense only if the solution p is continuous at $t=0$. In fact, it is shown in the following lemma that p is continuous on $[0,T]$.

Lemma 2. *Let $p \in L^2(0,T;H^1_{0,D}(\Omega))$ be a solution of (6). Then $\frac{dp}{dt} \in L^2(0,T;H^{-1}_D(\Omega))$.*

Proof. Let us first consider a function $\phi \in H^1_{0,D}(\Omega)$ so that $\|\phi\|_{H^1_{0,D}(\Omega)} \leq 1$. Take

$$< \ell, \phi >= (2\mu_s+\lambda_s)(K(p(t))\frac{\partial p(t)}{\partial y}, \frac{d\phi}{dy}).$$

Then,

$$| < \ell,\phi > | \leq M(2\mu_s+\lambda_s)|p|_{H^1_{0,D}(\Omega)}|\phi|_{H^1_{0,D}(\Omega)}$$
$$\leq M(2\mu_s+\lambda_s)\|p\|_{H^1_{0,D}(\Omega)}\|\phi\|_{H^1_{0,D}(\Omega)}$$

holds for a.e. $0 \leq t \leq T$. Hence,

$$\|\ell\|_{H^{-1}_D(\Omega)} \leq M(2\mu_s+\lambda_s)\|p\|_{H^1_{0,D}(\Omega)}.$$

Then

$$\int_0^T \|\ell\|^2_{H^{-1}_D(\Omega)} dt \leq (M(2\mu_s+\lambda_s))^2 \times \times \int_0^T \|p\|^2_{H^1_{0,D}(\Omega)} dt < \infty,$$

Since p belongs to $L^2(0,T;H^1_{0,D}(\Omega))$. Hence,

$$\ell \in L^2(0,T;H^{-1}_D(\Omega)).$$

On the other hand, $f \in L^2(0,T;H^{-1}_D(\Omega))$ which imply $\frac{dp}{dt}$ belongs to $L^2(0,T;H^{-1}_D(\Omega))$.

The following lemma yields an important result:

Lemma 3. *Suppose $p \in L^2(0,T;H^1_{0,D}(\Omega))$ with*

$$\frac{dp}{dt} \in L^2(0,T;H^{-1}_D(\Omega)).$$

(i) Then $p \in C([0,T];L^2(\Omega))$ (after possibly being redefined on a set of measure zero).
(ii) The mapping $t \mapsto \|p(t)\|^2_{L^2(\Omega)}$ is absolutely continuous, with

$$\frac{d}{dt}\|p(t)\|^2_{L^2(\Omega)} = 2(\frac{dp(t)}{dt}, p(t))$$

for a.e. $0 \leq t \leq T$.

Proof. see [12].

According to the Lemmas 2 and 3, $p \in C([0,T];L^2(\Omega))$. Then, it is perfectly allowable to prescribe p at $t=0$.

4. Existence - Uniqueness Theorems

Up to now, existence and uniqueness theorems has been verified for some classes of Biot's consolidation models with different dimensions. For instance, theorems of existence and uniqueness for linear Biot's poroelasticity models have been given in [31]. Moreover, discussion on existence and uniqueness of the solution for time-dependent initial-boundary value problems like (6) which $K(.)$ is constant has been given in [12,17,25]. Furthermore, a nonlinear model concerning propagation of elastic waves in a fluid-saturated porous solid has been given in [3]. In this model, the permeability of porous solid was assumed to be depends on space (i.e. points of domain). In this section, we investigate existence and uniqueness of the solution for the problem (6) which the permeability of material is assumed to be depends on strain. The Galerkin's method is used to prove the existence and uniqueness of solutions for the variational problem (6). First, we need the following lemmas:

Lemma 4.*(Gronwall's inequality)*
(i) Let $\eta(.)$ be a nonnegative, absolutely continuous function on $[t_0,T]$ which satisfies for a.e. t, the differential inequality

$$\eta^{'}(t) \leq \xi(t)\eta(t) + \psi(t)$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[t_0,T]$. Then

$$\eta(t) \leq e^{\int_{t_0}^{t} \xi(s)ds}[\eta(t_0) + \int_{t_0}^{t} \psi(s)ds]$$

for all $t_0 \leq t \leq T$.
(ii) In particular, if

$$\eta^{'}(t) \leq \xi(t)\eta(t)$$

on $[t_0,T]$, then $\eta \equiv 0$ provided that $\eta(t_0) = 0$.

Lemma 5. *Consider an initial value problem*

$$y^{'} = f(x, y(x)),$$
$$y(a) = y_0,$$

where $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in its first variable and satisfies a Lipschitz condition in its second variable. Then, there exists a unique solution to this problem.

Proof. See [10].

Now, one can find an approximation solution for the variational problem (6) (with initial condition given in the following lemma) in the space $L^2(0,T;V_h)$ which finite dimensional space

$$V_h = span\{\phi_1, \phi_2, \cdots, \phi_m\} \subset H^1_{0,D}(\Omega),$$

for some $m \in \mathbb{Z}^+$.

Lemma 6. *For each integer $m = 1, 2, \ldots$ there exists unique function $p_m : [0,T] \to H^1_{0,D}(\Omega)$ of the form*

$$p_m(t) = \sum_{j=1}^{m} \alpha_m^j(t)\phi_j, \tag{7}$$

satisfying

$$\alpha_m^j(0) = (p_0, \phi_j) \;\; (j = 1, \cdots, m) \tag{8}$$

and

$$(p_m^{'}, \phi_j) + (2\mu_s + \lambda_s)(K(p_m)\frac{\partial p_m}{\partial y}, \frac{d\phi_j}{dy}) = (f, \phi_j), \tag{9}$$

where $' = \dfrac{d}{dt}$.

Proof. Assuming $p_m(t)$ has the structure (7), we note by (6)

$$\begin{aligned}(f,\phi_j) &= (\frac{d}{dt}p_m, \phi_j) + (2\mu_s + \lambda_s)(K(p_m)\frac{\partial p_m}{\partial y}, \frac{d\phi_j}{dy}) \\ &= \frac{d}{dt}\textstyle\sum_{i=1}^{m} \alpha_m^i(\phi_i, \phi_j) + (2\mu_s + \lambda_s) \\ &\times \textstyle\sum_{i=1}^{m} \alpha_m^i(K(\sum_{l=1}^{m} \alpha_m^l \phi_l)\frac{d\phi_i}{dy}, \frac{d\phi_j}{dy})\end{aligned} \tag{10}$$

for $j = 1, \cdots, m$. On the other hand, Since every Hilbert space has an orthogonal basis, then we can assume the functions $\phi_k = \phi_k(y) \;\; (k = 1, \cdots)$ are smooth, $\{\phi_k\}_{k=1}^{\infty}$ is an orthogonal basis of $H^1_{0,D}(\Omega)$ and an orthonormal basis of $L^2(\Omega)$. Hence, this and (10) yield:

$$(f, \phi_j) = (\alpha_m^j)^{'}(t) + (2\mu_s + \lambda_s)\alpha_m^j(K(\sum_{l=1}^{m} \alpha_m^l \phi_l)\frac{d\phi_j}{dy}, \frac{d\phi_j}{dy}).$$

This is a $m \times m$ nonlinear differential equations system of first order. By the initial condition (8), this system has a unique solution under some conditions that stated in Lemma 5.

Now, we turn to the convergence of the sequence p_m when m tends to infinity. For this purpose, let us to state the following lemma.

Lemma 7. *There exists a constant C, depending on only Ω, T and the coefficient of $K(.)$, such that*

$$\begin{aligned}&\max_{0 \leq t \leq T} \|p_m(t)\|_{L^2(\Omega)} + \|p_m\|_{L^2(0,T;H^1_{0,D}(\Omega))} + \\ &+ \|p_m^{'}\|_{L^2(0,T;H_D^{-1}(\Omega))} \leq C(\|f\|_{L^2(0,T;L^2(\Omega))} + \|p_0\|_{L^2(\Omega)})\end{aligned} \tag{11}$$

for $m = 1, 2, \cdots$

Proof. Multiply (9) by $\alpha_m^j(t)$ and sum over j from 1 to m. In view of (7), we get:

$$(p_m^{'}(t), p_m) + (2\mu_s + \lambda_s)(K(p_m)\frac{\partial p_m}{\partial y}, \frac{\partial p_m}{\partial y}) = (f, p_m) \tag{12}$$

for a.e. $0 \leq t \leq T$. We have

$$|(f, p_m)| \leq \frac{1}{2\epsilon}\|f\|^2_{L^2(\Omega)} + \frac{\epsilon}{2}\|p_m\|^2_{L^2(\Omega)},$$

for any $\epsilon > 0$ and furthermore

$$(p_m^{'}, p_m) = \frac{d}{dt}(\frac{1}{2}\|p_m\|^2_{L^2(\Omega)}).$$

On the other hand,

$$(K(p_m)\frac{\partial p_m}{\partial y}, \frac{\partial p_m}{\partial y}) \geq mC_\Omega\|p_m\|^2_{H^1_{0,D}(\Omega)}$$

holds for all $0 \leq t \leq T$ and $m = 1, \cdots$ where C_Ω denotes the constant of Poincare's inequality. Consequently, by these relations, (12) yields the inequality

$$\begin{aligned}&\frac{d}{dt}(\|p_m\|^2_{L^2(\Omega)}) + m(2\mu_s + \lambda_s)\|p_m\|^2_{H^1_{0,D}(\Omega)} \leq \\ &C_1\|p_m\|^2_{L^2(\Omega)} + C_2\|f\|^2_{L^2(\Omega)}\end{aligned} \tag{13}$$

for a.e. $0 \leq t \leq T$ and appropriate constants C_1 and C_2. By (13),

$$\frac{d}{dt}(\|p_m\|^2_{L^2(\Omega)}) \leq C_1\|p_m\|^2_{L^2(\Omega)} + C_2\|f\|^2_{L^2(\Omega)}$$

holds and thus the differential form of Gronwall's inequality yields the estimate

$$\|p_m(t)\|^2_{L^2(\Omega)} \le e^{C_1 t}(\|p_m(0)\|^2_{L^2(\Omega)})+ + C_2 \int_0^t \|f(s)\|^2_{L^2(\Omega)} ds. \quad (0 \le t \le T) \tag{14}$$

Since $\|p_m(0)\|^2_{L^2(\Omega)} \le \|p_0\|^2_{L^2(\Omega)}$ by the initial condition (8),

$$\max_{0\le t\le T} \|p_m(t)\|^2_{L^2(\Omega)} \le C(\|p_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,T;L^2(\Omega))}).$$

Thus,

$$\max_{0\le t\le T} \|p_m(t)\|_{L^2(\Omega)} \le C(\|p_0\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))}). \tag{15}$$

Now, if we integrate (13) from 0 to T and apply the inequality (15), then

$$\|p_m\|^2_{L^2(0,T;H^1_{0,D}(\Omega))} = \int_0^T \|p_m\|^2_{H^1_{0,D}(\Omega)} dt \le C(\|p_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,T;L^2(\Omega))}). \tag{16}$$

In the last step, fix any $q \in H^1_{0,D}(\Omega)$ with $\|q\|_{H^1_{0,D}(\Omega)} \le 1$ and write $q = q^1 + q^2$ where $q^1 \in span\{\phi_j\}^m_{j=1}$ and $(q^2, \phi_j) = 0 \ (j = 1, \cdots, m)$. By (9), we have

$$(p'_m, q^1) + (K(p_m)\frac{\partial p_m}{\partial y}, \frac{\partial q^1}{\partial y}) = (f, q^1). \tag{17}$$

On the other hand, $(p'_m, q) = (p'_m, q^1)$. Hence, since

$$\|q^1\|_{H^1_{0,D}(\Omega)} \le 1,$$

then, by (17)

$$|(p'_m, q)| = |(f, q^1) - (K(p_m)\frac{\partial p_m}{\partial y}, \frac{\partial q^1}{\partial y})| \le \|f\|_{L^2(\Omega)} + M\|p_m\|_{H^1_{0,D}(\Omega)},$$

holds by Holder's inequality for a.e. $0 \le t \le T$. Then,

$$\|p'_m\|_{H^{-1}_D(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|p_m\|_{H^1_{0,D}(\Omega)}),$$

which implies

$$\|p'_m\|^2_{H^{-1}_D(\Omega)} \le 2C(\|f\|^2_{L^2(\Omega)} + \|p_m\|^2_{H^1_{0,D}(\Omega)}). \tag{18}$$

Thus by (16) and (18),

$$\begin{aligned}\|p'_m\|^2_{L^2(0,T;H^{-1}_D(\Omega))} &\le 2C(\|f\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ \|p_m\|^2_{L^2(0,T;H^1_{0,D}(\Omega))}) \\ &\le C'(\|f\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ \|p_0\|^2_{L^2(\Omega)}).\end{aligned} \tag{19}$$

Then, (15), (16) and (19) complete the proof.

Now, we are ready to state the existence theorem.

Theorem 1 *There exists a solution of (6).*

Proof. According to the energy estimate (11), the sequence $\{p_m\}^\infty_{m=1}$ is bounded in $L^2(0,T; H^1_{0,D}(\Omega))$ and $\{p'_m\}^\infty_{m=1}$ is bounded in $L^2(0,T; H^{-1}_D(\Omega))$. Hence, we can select a subsequence $\{p_{m_i}\}^\infty_{i=1}$ and a function

$$p \in L^2(0,T; H^1_{0,D}(\Omega))$$

with $p' \in L^2(0,T; H^{-1}_D(\Omega))$ such that

$$\begin{aligned} p_{m_i} &\to p \quad \text{weakly in } L^2(0,T; H^1_{0,D}(\Omega)), \\ p'_{m_i} &\to p' \quad \text{weakly in } L^2(0,T; H^{-1}_D(\Omega)), \end{aligned} \tag{20}$$

(see e.g. [30], p. 126). Take a function Ψ in

$$C^1([0,T]; H^1_{0,D}(\Omega)),$$

with $\Psi(T) = 0$, multiply (9) with $\Psi(t)$, integrate over $[0,T]$ and use Green's formula. This gives

$$\begin{aligned} &-\int_0^T (p_m(t), \phi_j)\Psi(t)dt + (2\mu_s + \lambda_s)\times \\ &\times \int_0^T (K(p_m)\frac{\partial p_m}{\partial y}, \frac{d\phi_j}{dy})\Psi(t)dt = \\ &\int_0^T (f(t), \phi_j)\Psi(t)dt + (p_m(0), \phi_j)\Psi(0), \quad 1 \le j \le m. \end{aligned} \tag{21}$$

Now, let us fix an arbitrary integer N and choose a function q in

$$Q_N = span\{\phi_1, \phi_2, \cdots, \phi_N\}.$$

We choose $m \ge N$ and then (21) implies:

$$\begin{aligned} &-\int_0^T (p_m(t), q)\Psi'(t)dt + (2\mu_s + \lambda_s)\times \\ &\int_0^T (K(p_m)\frac{\partial p_m}{\partial y}, \frac{\partial q}{\partial y})\Psi(t)dt = \\ &\int_0^T (f(t), q)\Psi(t)dt + (p_m(0), q)\Psi(0). \end{aligned} \tag{22}$$

We set $m = m_i$ and by virtue of (20), the following limit holds:

$$\lim_{m_i\to\infty} \int_0^T (p_{m_i}(t), q)\Psi'(t)dt = \int_0^T (p(t), q)\Psi'(t)dt.$$

Finally, by hypothesis $p_{m_i}(0) \to p_0$ in $L^2(\Omega)$. Hence, as m_i tends to infinity, the relation (22) becomes:

$$\begin{aligned} &-\int_0^T (p(t), q)\Psi'(t)dt + (2\mu_s + \lambda_s)\times \\ &\int_0^T (K(\frac{\partial u}{\partial y})\frac{\partial p}{\partial y}, \frac{\partial q}{\partial y})\Psi(t)dt = \\ &\int_0^T (f(t), q)\Psi(t)dt + (p(0), q)\Psi(0), \end{aligned} \tag{23}$$

for all $q \in Q_N$ and $\Psi \in C^1([0,T]; H^1_{0,D}(\Omega))$ with $\Psi(T) = 0$. But, N is arbitrary and $\bigcup_{m\ge1} Q_m$ is dense in $H^1_{0,D}(\Omega)$.

Therefore, (23) is also valid for all $q \in H^1_{0,D}(\Omega)$. Hence, in particular,

$$(p'(t), q) + (K(p(t))\frac{\partial p(t)}{\partial y}, \frac{\partial q}{\partial y}) = (f(t), q),$$

for all $q \in H^1_{0,D}(\Omega)$. From Lemma 3, we also see that $p \in C([0,T]; L^2(\Omega))$.

It remains only to show that $p(0) = p_0$. For this, we multiply (6) by a function Ψ like in (23), integrate over $[0,T]$ and use Green's formula. Comparing with (23), we obtain

$$(p(0), q) = (p_0, q), \qquad \forall q \in H^1_{0,D}(\Omega).$$

Hence, $p(0) = p_0$ in the dual space of $H^1_{0,D}(\Omega)$ and also in $L^2(\Omega)$, since $p_0 \in L^2(\Omega)$.

Now, it remains to show the uniqueness of solution for the problem (6).

Theorem 2 *A solution of (6) is unique.*

Proof. Let p_1 and p_2 be two different solutions of the variational form (6). Then by (6),

$$\begin{aligned}(a)\ &(\frac{d}{dt}(p_1 - p_2), q) + (2\mu_s + \lambda_s)(K(p_1)\frac{\partial p_1}{\partial y} - \\ &\qquad K(p_2)\frac{\partial p_2}{\partial y}, \frac{\partial q}{\partial y}) = 0,\\ (b)\ &\qquad (p_1 - p_2)(0) = 0,\end{aligned} \tag{24}$$

for each $q \in H^1_{0,D}(\Omega)$ and a.e. $0 \le t \le T$. Setting $q = p_1 - p_2$ in (24) and using Lemma (3) yields

$$\frac{d}{dt}(\frac{1}{2}\|p_1 - p_2\|^2_{L^2(\Omega)}) + (2\mu_s + \lambda_s)(K(p_1)\frac{\partial p_1}{\partial y} - K(p_2)\frac{\partial p_2}{\partial y}, \frac{\partial(p_1 - p_2)}{\partial y}) = 0. \tag{25}$$

Then, by (25),

$$\begin{aligned}\frac{d}{dt}(\frac{1}{2}\|p_1 - p_2\|^2_{L^2(\Omega)}) &= (2\mu_s + \lambda_s)(-K(p_1)\frac{\partial p_1}{\partial y} + \\ &+ K(p_2)\frac{\partial p_2}{\partial y}, \frac{\partial(p_1 - p_2)}{\partial y})\\ &\le (2\mu_s + \lambda_s)(2M)|(\frac{\partial p_1}{\partial y} - \\ &- \frac{\partial p_2}{\partial y}, \frac{\partial(p_1 - p_2)}{\partial y})| = (2\mu_s + \\ &+ \lambda_s)(2M)|p_1 - p_2|^2_{H^1_{0,D}(\Omega)}\end{aligned} \tag{26}$$

According to the Lemmas 2 and 3, $u \in C([0,T]; L^2(\Omega))$. Moreover, p_1 and p_2 are the different solutions and

$$(p_1 - p_2)(0) = 0.$$

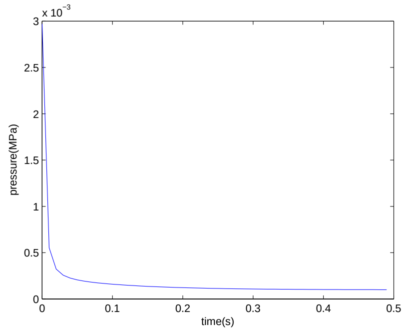

Figure 3 Pore pressure in time at $y = L$.

Then, there exists obviously an interval $(t_k, t_l) \subset [0,T]$ such that $(p_1 - p_2)(t) \neq 0$ on this interval and

$$(p_1 - p_2)(t_k) = 0.$$

On the other hand, by setting

$$\xi(t) := 4M(\mu_s + \lambda_s)|p_1 - p_2|^2_{H^1_{0,D}(\Omega)}/\|p_1 - p_2\|^2_{L^2(\Omega)}$$

and also (26) we have:

$$\begin{cases}\frac{d}{dt}(\|p_1 - p_2\|^2_{L^2(\Omega)}) \le \xi(t)\|p_1 - p_2\|^2_{L^2(\Omega)} \text{ on } [\mathrm{t_k}, \mathrm{t_l}],\\ \qquad (p_1 - p_2)(t_k) = 0.\end{cases} \tag{27}$$

Now, Gronwall's inequality yields $(p_1 - p_2)(t) = 0$ on $[t_k, t_l]$. This contradiction shows that $p_1 - p_2 \equiv 0$ on $[0,T]$. This establishes the proof.

5. Numerical Examples

In this section, we present two numerical examples of non linear Biot's model. In both cases, we set $\mu_s = 0.4$, $\lambda_s = 0.267$ and $L = 1$.

Example 1 We consider the problem (6) with $K(p) = 1 + p$. We use the Euler scheme for time discretisation. Let Δt be the time step and p_n the approximation of the solution p at $t = t_n = n\Delta t$. Then the system (6) resulting from backward Euler is:
Find $p_n \in H^1_{0,D}(\Omega)$ such that

$$(p_n, \phi) + (2\mu_s + \lambda_s)\Delta t(K(p_n)d(p_n)/dy) =$$

$$\Delta t(f(t_n), \phi) - (p_{n-1}, \phi).$$

By letting $\Delta t = .01$, the interval $[0, L]$ is divided to $m = 100$ subintervals and piecewise linear functions considered as nodal basis functions on the $m - 1$ first nodes and fixed function on the last node. We also put $f(t) = -.01\Delta t(L/m)$. Then, by Newton's method the solution of

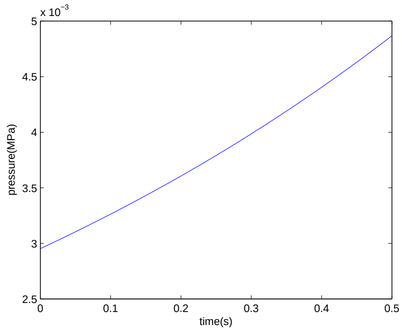

Figure 4 Pore pressure in time at $y = L$.

nonlinear system resulted by discretisation is determined. Figure 3 shows the pore pressure evolution at $y = L$ for 0.5 second.

Example 2 We consider the problem (5) with $K(p) = 1$. Then, by (5(a)):

$$f(t) = \frac{dp}{dt} - (2\mu_s + \lambda_s)\frac{\partial}{\partial y}(K(p)\frac{\partial p}{\partial y}). \tag{28}$$

By the method of separation of variables, we seek a solution in the form $p(t, y) = Q(t)R(y)$. Introducing it into (28), we find out that

$$\frac{\frac{dQ}{dt}}{Q} = c = K(2\mu_s + \lambda_s)\frac{\frac{d^2R}{dy^2} + 1}{R} \tag{29}$$

for some constant c and all $y \in \mathbb{R}, t \geq 0$. Of course, we assumed $f(t) = Q(t)$. By the initial and boundary conditions, the exact solution is as follows:

$$p(t, y) = .001e^{K(2\mu_s+\lambda_s)t}(1 + e^{2L} - e^y - e^{2L-y}). \tag{30}$$

Figure 4 shows the pore pressure evolution at $y = L$ for 0.5 second.

6. Conclusions

We have presented a class of Biot's consolidation nonlinear model. This class is of interest for many of real world applications, including the simulation of the swelling of cartilagineous tissues. The main feature of this model is that the permeability tensor depends on the strain, thus giving rise to a nonlinear model. By Galerkin's method we have shown existence and uniqueness of weak form solution for the nonlinear one dimensional Biot's model. For future works, the existence-uniqueness results can be extended to the solution of two and three dimensional nonlinear Biot's models.

Acknowledgement

AT would like to thank Dr. Kamyar Malakpoor from University of Amsterdam, Amsterdam, Netherland for his useful discussion and encouragement.

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