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# Soliton Pair Propagation in Three-Level Unbalanced Medium

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**Abstract:** The study investigates the soliton-pair propagation in an optical dense three-level atomic media where the dissipation of the atomic system is considered for unbalanced coupling between the allowed atomic transitions and the two classical pump fields. Analytical solutions of the solitary wave-pair are derived and the existence conditions for such wave-pair propagation is highlighted. The allowed soliton-pair velocities are defined.

**Keywords:** Non-linear differential equations, wave propagation, soliton

## 1 Introduction

Nonlinearity represents the origin of several intriguing phenomena in classical and quantum physics. We adress Rogue waves [1,2,3], bifurcation [4,5] Bistability [6,7,8], chaos [9,10,11,12], and solitons [13,14,15] as few examples. Russel discoveres Solitons in 1834 [16] . A Scottish naval architect conducted experiments to define the maximum resourceful plan for canal boats. He observed wave translation. He documented his discovery in a paper adopted as a report representing the first scientific account of solitons in history. The phenomenon was a water wave that shaped in a narrow channel and displayed a few counter-intuitive properties. The wave was stable,i.e. it neither flattened out nor steepened like normal waves. Russel could follow it for a few kilometers. Moreover, it did not merge with other waves, i.e. a small wave moving quicker would instead overtake a large slower one. Through a chain of measurements, Russel managed to define the velocity of such waves , but he couldn't create the appropriate equation. In 1895, Korteweg and De Vries [17] established a nonlinear partial differential equation (i.e. the Korteweg-de Vries equation) that described Russel's solitary waves. Their work maintained anonymity till 1965, but Zabusky and Kruskal [18] numerically solved the KdV equation. In 1967, Gardner, Greene, Kruskal and Miura [19] discovered an inverse scattering transform that facilitated the analytical solution of the KdV equation. In 1973, Robin Bullough [20] presented the first mathematical report concerning the existence of optical solitons. He also proposed the idea of a soliton-based transmission system to increase the performance of optical telecommunications. Solitons present powerful applications in telecommunication. In 1988, Mollenauer [21] and his group transferred soliton pulses over 4000 km. In 1991, Bell research team transferred soliton errors-loose at 2.5 Gb/s for more than 14000 km. A year ago, researchers from Karlsruhe and Lausanne [22] showed a record-high speed optical communication via soliton. Most of the previous analytical studies were devoted to the lossless systems. The previous models frequently neglected the system dissipation. Existence conditions of the soliton propagation appear in case of including the dissipation. The present paper investigates the soliton-pair propagation in an atomic dissipative medium. The soliton pair propagation in absorbing atomic three-level system has been explored in a previous paper [13], which only addressed the case for equal coupling constants between the two coherent fields and the two allowed atomic transitions. The present paper also explores the analytical soliton-pair solutions for unbalanced coupling between the two coherent lights and the atomic transitions. In addition, limiting conditions of the soliton-pair propagation are derived and the speed of the soliton-pair is defined.

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## 2 Model

We consider a three-level system in lambda configuration. The two allowed atomic transitions interact with two laser fields applied to the stocks and pump transitions. The atomic system has an excited state $|0\rangle$ and two ground states $|1\rangle$ and $|2\rangle$. The two allowed atomic transitions are only between the excited state and the ground states. These atomic transitions are excited resonantly by two classical fields with amplitudes $E_1$ and $E_2$ and frequencies $\omega_1$ and $\omega_2$. The expression of the pump field is written as follows

$$E(x,t) = E_1(x,t)e^{k_1 x - i\omega_1 t} + E_2(x,t)e^{k_2 x - i\omega_2 t} \tag{1}$$

where $k_j$ represents the wavenumbers defined by $k_{1,2} = \frac{\omega_{1,2}}{c}$. $c$ designs the vacuum light speed. Here we consider that the amplitudes of the classical light amplitudes are slowly varying in space and time verifying [13, 14, 15]:

$$\left|\frac{\partial E_j}{\partial t}\right| \ll \omega_j |E_j| \text{ and } \left|\frac{\partial E_j}{\partial t}\right| \ll \frac{\omega_j}{c}|E_j| \tag{2}$$

The total Hamiltonian of the system is given by

$$H = \sum_{j=0,1,2} \varepsilon_j |j\rangle\langle j| + \sum_{l=1,2} g_l^{'}(E_l|0\rangle\langle l| + c.c) \tag{3}$$

The first term describes the proper energy of the atomic three-level system. The second term involves the interaction between the two classical fields and the two allowed atomic transitions $|1\rangle$-$|0\rangle$ and $|2\rangle - |0\rangle$. Where $g_1^{'}$ and $g_2^{'}$ are the coupling constants between the atomic transitions $|1\rangle$-$|0\rangle$ and $|2\rangle - |0\rangle$ and respectively the classical light fields $E_1$ and $E_2$. $\varepsilon_j$ is the energy of the atomic level $|j\rangle$ verified in the resonant case: $\omega_1 = \frac{\varepsilon_0 - \varepsilon_1}{\hbar}$ and $\omega_2 = \frac{\varepsilon_0 - \varepsilon_2}{\hbar}$.
We consider that the atomic system is initially prepared in the way that the excited state $|0\rangle$ is almost empty and the population is distributed equally in the ground levels $|1\rangle$ and $|2\rangle$. In addition, the coherence between the ground levels at t=0 is negligible. We explore the soliton-pair propagation in this three-level system with the same velocity, so we can write the following:

$$E_j(x,t) = E_j(x - vt) \tag{4}$$

where $v$ represents the speed of the soliton-pair. Following the same procedure of calculations in [13], we get in the moving frame $z = x - vt$ the following coupled non-linear differential equations

$$\begin{cases} \Gamma\frac{d\alpha_1}{dz} = \alpha_1^3 - \frac{g_1}{2}\alpha_1 + \alpha_2^2\alpha_1 \\ \Gamma\frac{d\alpha_2}{dz} = \alpha_2^3 - \frac{g_2}{2}\alpha_2 + \alpha_1^2\alpha_2 \end{cases} \tag{5}$$

where $\Gamma = \frac{\gamma}{v}$ is the normalized dissipation rate, here we suppose that the spontaneous emission rates from the excited state to the both ground states are the same. $g_j = \frac{g_{Ej}g_j^{'}}{\hbar(c-v)v}$ are the effective normalized coupling constant with $g_{Ej}$ representing the propagation constants of the optical fields inside the atomic media.

$\alpha_j$ are the normalized solitons amplitudes defined by $\alpha_j = \frac{g_j^{'}}{\hbar v}E_j$

## 3 Soliton-pair Solutions

In order to solve the coupled differential equations for the soliton-pair propagation we divide the two differential equations in (5) and we obtain

$$\frac{d\alpha_2}{d\alpha_1} = \frac{\alpha_2^3 - \frac{g_2}{2}\alpha_2 + \alpha_1^2\alpha_2}{\alpha_1^3 - \frac{g_1}{2}\alpha_1 + \alpha_2^2\alpha_1} \tag{6}$$

The above-mentioned differential equation has an implicit solution in the form

$$\frac{1}{2}\frac{(g_1 - g_2)}{g_2}\ln\left(-\frac{g_1}{2}\frac{g_2}{2} + \frac{g_2}{2}\alpha_1^2 + \frac{g_1}{2}\alpha_2^2\right) + \ln(\alpha_1) - \frac{g_1}{g_2}\ln(\alpha_2) = c_1 \tag{7}$$

where $c_1$ is a free constant.
The implicit solution of (7) in general does not have an explicit relation between the two amplitudes of the soliton-pair. However, for unbalanced coupling (when one of the coupling constant $g_j$ is much bigger than the other one) it is possible to derive an explicit relation. In this work we focus on the case where we consider $\frac{g_1}{g_2} << 1$ . Therefore, from (7) we get

$$\alpha_1^2 = k\left(-\frac{g_1}{2}\frac{g_2}{2} + \frac{g_2}{2}\alpha_1^2 + \frac{g_1}{2}\alpha_2^2\right) \tag{8}$$

and

$$\alpha_2^2 = \frac{2}{kg_1}\alpha_1^2 - \frac{g_2}{g_1}\alpha_1^2 + \frac{g_2}{2} \tag{9}$$

where $k$ is a positive constant.
By substituting (9) in the first differential equation of (5) we get a separable first-order differential equation for the amplitude of the first soliton

$$\frac{d\alpha_1}{dz} = A\alpha_1^3 + B\alpha_1$$

whose solution is given by

$$z(\alpha_1) = \frac{1}{B}\ln(\alpha_1) - \frac{1}{2B}\ln(A\alpha_1^2 + B) + K$$

where $A$ and $B$ are constants defined by $A = -\frac{g_2}{\Gamma g_1} + \frac{1}{\Gamma} + \frac{2}{\Gamma k g_1}$ and $B = \frac{1}{2\Gamma}(g_2 - g_1)$. $K$ is a free constant. Note that the above equation gives a relation between moving coordinates $z$ and the amplitude of the first soliton forming the soliton-pair. Therefore

$$\alpha_1 = \pm\sqrt{\frac{B}{e^{-2B(z-K)} - A}}$$

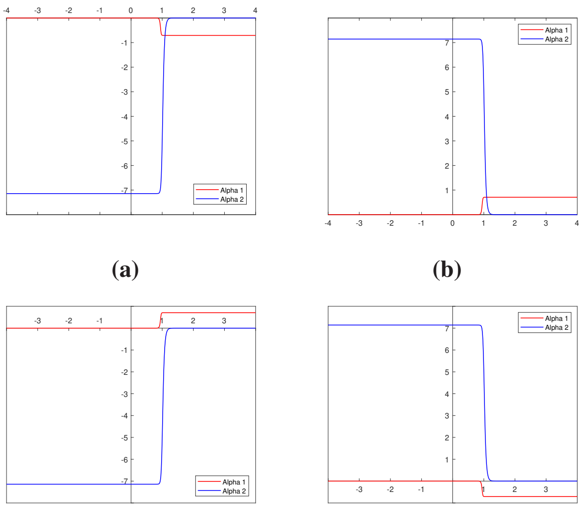
Following the same procedure we will get the following differential equation for $\alpha_2$

$$\frac{d\alpha_2}{dz} = E\alpha_2^3 - G\alpha_2$$

which has two explicit solutions given by

$$\alpha_2 = \pm\sqrt{\frac{G}{e^{2G(z-H)} + E}}$$

where $E$ and $G$ are constants defined by $E = \frac{1}{\Gamma} + \frac{kg_1}{\Gamma(2-kg_2)}$ , $G = \frac{g_2}{2\Gamma} + \frac{kg_1g_2}{\Gamma(4-2kg_2)}$ and $H$ is a free constant.




(a) (b) (c) (d)

**Fig. 1:** Soliton pair shapes $\alpha_1$ and $\alpha_2$ for $g_1 = 1, g_2 = 100, k = 10$ and $\Gamma = H = K = 1$. (a) and (b) represent soliton-pair with same directions of polarization. (c) and (d)represent soliton-pair with opposite directions of polarization.

To avoid any singularity for $\alpha_1$ and $\alpha_2$ we will suppose that $A < 0$, $E > 0$ and $G > 0$. These conditions are satisfied if $k > \frac{2}{g_2-g_1}$ . Accordingly, we obtain the following limits

$$L_1 = \lim_{z\to\infty} \alpha_1 = \sqrt{\frac{B}{-A}}; \quad \lim_{z\to-\infty} \alpha_1 = 0$$

$$\lim_{z\to\infty} \alpha_2 = 0; \quad \lim_{z\to-\infty} \alpha_2 = \sqrt{\frac{G}{E}}$$

Applying the limit to (9) when z approaches $\infty$ we get

$$\left(\frac{2}{kg_1} - \frac{b}{a}\right)L_1^2 + \frac{g_2}{2} = 0 \qquad (10)$$

where $g_1 = \frac{a}{(c-v)v}$ and $g_2 = \frac{b}{(c-v)v}$, $a$ and $b$ are constants given by $a = \frac{g_{E_1}g_1'}{h}$ and $b = \frac{g_{E_2}g_2'}{h}$. Let's define $X = (c-v)v$. $X$ verify $0 \le X \le \frac{c^2}{4}$. Then from (10) we get

$$\left(\frac{4X^2}{ka} - \frac{2bX}{a}\right)L_1^2 + b = 0$$

The quadratic equation

$$\frac{4L_1^2}{ka}X^2 - \frac{2bL_1^2}{a}X + b = 0$$

has two real solutions

$$X_{1,2} = \frac{b}{a}L_1^2 \pm \sqrt{\Delta_1}$$

under the condition that

$$\Delta_1 = \left(\frac{b}{a}\right)^2 L_1^4 - \frac{4L_1^2 b}{ka} \ge 0$$

which is true if

$$L_1 \ge \frac{2}{\sqrt{k}} \qquad (11)$$

Since we know that $X$ should verify $0 \le X_{1,2} \le \frac{c^2}{4}$ which is true for the smallest solution. Therefore the condition for the second solution will give us

$$\frac{b}{a}L_1^2 + \sqrt{\left(\frac{b}{a}\right)^2 L_1^4 - \frac{4L_1^2 b}{ka}} \le \frac{c^2}{4}$$

which implies the two following conditions

$$L_1 \le c\sqrt{\frac{b}{2a}} \qquad (12)$$

and

$$\left(L_1^2 - \frac{bc^2}{4a}\right)^2 \ge L_1^4 - \frac{4aL_1^2}{kb}$$

Let us define $V = L_1^2$. Thus, we will get

$$\left(\frac{4a}{k} - \frac{2bc^2}{4a}\right)V \ge -(\frac{b}{a})^2\frac{c^4}{16}$$

Therefore

$$\frac{2bc^2}{4a} \ge \frac{4a}{k}$$

Finally, we get the auxiliary condition

$$k \ge \frac{8a^2}{c^2 b} \qquad (13)$$

and the condition on $L_1$

$$L_1 \le \frac{\frac{bc^2}{4a}}{\sqrt{\frac{bc^2}{2a} - \frac{4a}{k}}} \qquad (14)$$

The previous conditions (11),(12) and (14) can be written in a double inequality

$$\frac{2}{\sqrt{k}} \leq L_1 \leq \max\left(c\sqrt{\frac{b}{2a}}; \frac{\frac{bc^2}{4a}}{\sqrt{\frac{bc^2}{2a}-\frac{4a}{k}}}\right) \quad (15)$$

The conditions (13) and (15) are the existence conditions for the soliton-pair propagation. In other words if one of the conditions is not satisfied the considered three-level atomic medium can not support the propagation of any soliton-pair. Under the existence conditions (15) and (13), the equation

$$X = v(c-v)$$

has only one possible solution of the soliton-pair velocity which is given by

$$v = \frac{c-\sqrt{c^2-4X}}{2}$$

where

$$X = \frac{b}{a}L_1^2\left(1-\sqrt{1-\frac{4a}{L_1^2 bk}}\right).$$

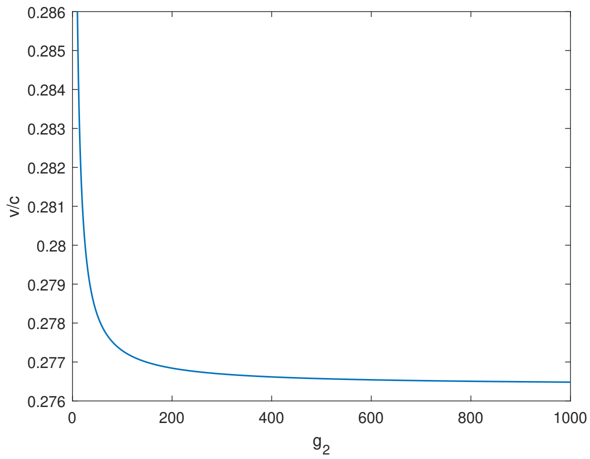



**Fig. 2:** Normalized velocity as function of the coupling with the same parameters as FIG 1.

FIG 2 indicates the following aspects :
(i) velocity is lower than the speed of light.
(ii) When the rate of the coupling $\frac{g_2}{g_1}$ increases, the velocity of the soliton-pair reduces.
(iii) for very high rate of the coupling rate the velocity of the soliton-pair is asymptotic to 0.276 of the light speed.

## 4 Conclusion

We explored the propagation of soliton-pair pulses in three-level atomic media where each soliton from the pair was unequally coupled to the allowed atomic state levels. We have considered an optical dense media with atomic dissipations. We have derived analytical expressions describing the soliton-pair shapes. We have highlighted the conditions of the soliton-pair propagation in such media and defined the allowed soliton-pair velocity.
The present study revealed four possible configurations of the soliton-pair pulses. Two of them can be interpreted as a couple of solitons with same directions of polarization and the others are interpreted as soliton-pair with opposite directions of polarization. Because solitons have stable shapes when propagating in the considered media, they are insensitive to noise and dispersion. The results have potential applications in data transfer with the soliton-pair pulses, where a dissipative three-level medium could be a realistic model for the optical communication media.
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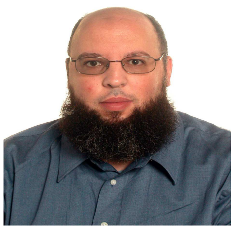

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