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Fractional Integral Inequalities Using Marichev-Saigo-Maeda Fractional Integral Operator

Asha B. Nale[1,*], *Satish K. Panchal*[1], *Vaijanath L. Chinchane*[2] *and Hilal M. Y. Al-Bayatti*[3]

[1] Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad-431 004, India
[2] Department of Mathematics, Deogiri Institute of Engineering and Management Studies, Aurangabad-431005, India
[3] Department of Computer Science, College of Arts and Science, Applied Science University, P.O. Box 5055, Kingdom of Bahrain

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Abstract: Here, we obtain several new fractional integral inequalities using Marichev Saigo Maeda fractional integral operator for synchronous functions concerned with extended Chebyshev functional.

Keywords: Marichev-Saigo-Maeda fractional integral operator, Synchronous functions and inequality.

1 Introduction

The fractional calculus has gained more attention due to its multiple application in the fields of science and engineering. In the area of inequalities, fractional integral inequalities play a vital role in a different fields of mathematics, especially for finding uniqueness of solution in fractional differential equation, fractional partial differential equation, fractional boundary value problem and continuous dependence solution. We start by recalling the Chebyshev functional which has been addressed by several researchers, see [1]:

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\left(\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right), \tag{1}$$

where f and g are two integrable functions which are synchronous on $[a,b]$, $(i.e(f(x)-f(y))(g(x)-g(y))) \geq 0$ for any $x,y \in [a,b]$. We consider the extended Chebyshev's function [2],

$$\begin{aligned} T(f,g,p,q) := &\int_a^b q(x)\int_a^b p(x)f(x)g(x)dx + \int_a^b p(x)dx\int_a^b q(x)f(x)g(x)dx \\ &- \int_a^b p(x)f(x)dx\int_a^b q(x)g(x)dx - \int_a^b q(x)f(x)dx\int_a^b p(x)g(x)dx. \end{aligned} \tag{2}$$

In the last few decades, the researchers have paid valuable attention to Chebyshev and extended Chebyshev functional. Recently, many specialists in several fields have found different results about some known fractional integral inequalities and applications by means of the generalization of the Riemann-Liouville, Hadamard, Erdelyi-Kober, Saigo, generalized fractional integral, k-fractional integral operator, generalized k-fractional integral operator and generalized proportional fractional operators, see [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23]. V. L. Chinchane and D. B. Pachpatte [6, 8] proposed the fractional integral inequalities for extended Chebyshev fractional in case of synchronous function by considering Hadamard and Saigo fractional integral operator respectively. In [24, 25], S. Joshi et al. have recently investigated the Grüss-type inequality and Chebyshev type inequalities by employing Marichev-Saigo-Maeda fractional integral operator. Marichev [26] introduced generalization of the hypergeometric

* Corresponding author e-mail: ashabnale@gmail.com

fractional integral including Saigo operator, (also see [27]). Saigo and Maeda [28] have studied the hypergeometric fractional integral in terms of any complex order with Appell function in the kernel. In [18], S. D. Purohit et al. introduced generalized operators of fractional integration involving Appell's function $F_3(.)$ due to Marichev-Saigo-Maeda. Motivated from [18], [24], [25], [26], [27], [28], [29], our purpose is to obtain some new inequalities for synchronous functions which are related to extended Chebychev functional using Marichev-Saigo-Maeda fractional integral operators.

2 Preliminaries

First, we recall some basic notation, definitions and lemma which are useful later.

Definition 1. *Let $g(\tau), \tau \geq 0$ be a real valued function which is said be in $C_\mu([a,b]), \mu \in \mathbb{R}$ if there exist $\sigma \in \mathbb{R}$ such that $\sigma > \mu$ and $\Phi(\tau) \in C([a,b])$.*

Definition 2. *[18, 25, 28] Let $v, v', \xi, \xi', \vartheta \in \mathbb{C}$, $x > 0$ and $\Re(\vartheta) > 0$, then Marichev-Saigo-Maeda (MSM) fractional integral is defined by*

$$(\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta} f)(x) = \frac{x^{-v}}{\Gamma(\vartheta)} \int_0^x (x-t)^{\vartheta-1} t^{-v'} F_3(v, v', \xi, \xi'; \vartheta; 1-\frac{t}{x}, 1-\frac{x}{t}) f(t) dt. \tag{3}$$

Where $F_3(.)$ is the Appel function defined by [30] as

$$_pF_q(v, v', \xi, \xi'; \gamma; x; y) = \sum_{m,n=0}^{\infty} \frac{(v)_m (v')_m (\xi)_m (\xi)'_n}{(\gamma)_{m+n}} \frac{x^m y^n}{m!n!}, \ \max(x,y) < 1,$$

and $(v)_m = v(v+1)...(v+m-1)$ is Pochhammer symbol.

Lemma 1. *Let $v, v', \xi, \xi', \vartheta, \rho \in \mathbb{C}$, $x > 0$ be such that $\Re(\vartheta) > 0$ and $\Re(\tau) > max\{0, \Re(v - v' - \xi - \vartheta), \Re(v' - \xi')\}$. Then there exists the relation*

$$\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta} x^{\rho-1}(x) = \frac{\Gamma(\rho)\Gamma(\rho+\vartheta-v-v'-\xi)\Gamma(\rho+\xi'-v')}{\Gamma(\rho+\xi')\Gamma(\rho+\vartheta-v-v')\Gamma(\rho+\vartheta-v'-\xi)} x^{\rho-v-v'+\vartheta-1}. \tag{4}$$

If we consider $\rho = 1$ in lemma 1, then we get following relation

$$(\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[1])(x) = \frac{\Gamma(1+\vartheta-v-v'-\xi)\Gamma(1+\xi'-v')}{\Gamma(1+\xi')\Gamma(1+\vartheta-v-v')\Gamma(1+\vartheta-v'-\xi)} x^{-v-v'+\vartheta}. \tag{5}$$

Consider a function,

$$\begin{aligned} \mathfrak{I}(x,t) &= \frac{x^{-v}}{\Gamma(\vartheta)} (x-t)^{\vartheta-1} t^{-v'} F_3(v, v', \xi, \xi'; \vartheta; 1-\frac{t}{x}, 1-\frac{x}{t}) \\ &= \frac{x^{-v}}{\Gamma(\vartheta)} (x-t)^{\vartheta-1} t^{-v'} \left[(1+\frac{v'(\xi)}{\vartheta}) \frac{1-x}{t} + \frac{v(\xi)}{\vartheta} \frac{1-t}{x} + ... \right]. \end{aligned} \tag{6}$$

Clearly, the function $\mathfrak{I}(x,t)$ remains positive because all terms of equation (6) are positive.

3 Main Results

Here, we prove the following lemma.

Lemma 2. *Let f and g be two integrable and synchronous function on $[0,\infty)$ and $u, v : [0,\infty) \to [0,\infty)$. Then for all $x > 0$, $v, v', \xi, \xi', \vartheta, \in \mathbb{C}$, $\Re(\vartheta) > 0$ we have*

$$\begin{aligned} &\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[u(x)] \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[vfg(x)] + \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[v(x)] \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[ufg(x)] \\ &\geq \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[uf(x)] \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[vg(x)] + \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[vf(x)] \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[ug(x)]. \end{aligned} \tag{7}$$

where $v' > -1$, $1 > \max\left\{0, \Re(v+v'+\xi-\vartheta), \Re(v'-\xi')\right\}$ $(\vartheta - v') > \max(1-\xi, 1-v)$.

Proof:- Since f and g are two synchronous on $[0,\infty)$ for all $\tau>0$, $\rho>0$, we have

$$(f(\tau)-f(\rho))\,(g(\tau)-g(\rho))\geq 0. \tag{8}$$

$$f(\tau)g(\tau)-f(\tau)-g(\rho)-f(\rho)g(\tau)+f(\rho)g(\rho)\geq 0. \tag{9}$$

$$f(\tau)g(\tau)+f(\rho)g(\rho)\geq f(\tau)g(\rho)+f(\rho)g(\tau). \tag{10}$$

Multiplying both sides of (10) by $\mathfrak{I}(x,\tau)x(\tau)$ which remains positive for all $\tau\in(0,x)$, $x>0$, then the integrating resulting identity with respect to τ from 0 to x, we have

$$\begin{aligned}
&\frac{x^{-v}}{\Gamma(\eta)}\int_0^x (x-\tau)^{\eta-1}\tau^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})u(\tau)f(\tau)g(\tau)d\tau\\
&+\frac{x^{-v}}{\Gamma(\eta)}\int_0^x (x-\tau)^{\eta-1}\tau^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})u(\tau)f(\rho)g(\rho)d\tau\geq\\
&\frac{x^{-v}}{\Gamma(\eta)}\int_0^x (x-\tau)^{\eta-1}\tau^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})u(\tau)f(\tau)g(\rho)d\tau\\
&\frac{x^{-v}}{\Gamma(\eta)}\int_0^x (x-\tau)^{\eta-1}\tau^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})u(\tau)f(\rho)g(\tau)d\tau,
\end{aligned} \tag{11}$$

so

$$\begin{aligned}
&\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[ugf(x)]+f(\rho)g(\rho)\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[u(x)]\\
&\geq g(\rho)\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[uf(x)]+f(\rho)\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[ug(x)].
\end{aligned} \tag{12}$$

Again, multiplying both sides of (12) by $\mathfrak{I}(x,\rho)y(\rho)$, $\rho\in(0,x)$, $x>0$, then integrating resulting identity with respect to ρ from 0 to x we get

$$\begin{aligned}
&\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}(ufg)(x)\frac{x^{-v}}{\Gamma(\eta)}\int_0^x (x-\rho)^{\eta-1}\rho^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\rho}{x},1-\frac{x}{\rho})v(\rho)d\rho\\
&+\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}u(x)\frac{x^{-v}}{\Gamma(\eta)}\int_0^x (x-\rho)^{\eta-1}\rho^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\rho}{x},1-\frac{x}{\rho})(vfg)(\rho)d\rho\\
&\geq \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}(uf)(x)\frac{x^{-v}}{\Gamma(\eta)}\int_0^x (x-\rho)^{\eta-1}\rho^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\rho}{x},1-\frac{x}{\rho})(vg)(\rho)d\rho\\
&+\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}(ug)(x)\frac{x^{-v}}{\Gamma(\eta)}\int_0^x (x-\rho)^{\eta-1}\rho^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\rho}{x},1-\frac{x}{\rho})(vf)(\rho)d\rho,
\end{aligned} \tag{13}$$

which implies that,

$$\begin{aligned}
&\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[ugf(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[v(x)]+\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[u(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[vgf(x)]\\
&\geq \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[uf(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[vg(x)]+\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[ug(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[vf(x)].
\end{aligned} \tag{14}$$

This completes the proof of inequality 7.

Theorem 1. *Let f and g be two integrable and synchronous functions on $[0,\infty)$, and $r,p,q:[0,\infty)\to[0,\infty)$. Then for all $x>0$, $v,v',\xi,\xi',\vartheta,\in\mathbb{C}$, $\Re(\vartheta)>0$ we have*

$$\begin{aligned}&2\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[r(x)]\left[\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[p(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[qfg(x)]+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[q(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[pfg(x)]\right]+\\&2\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[p(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[q(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[rfg(x)]\geq\\&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[r(x)]\left[\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[pf(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[qg(x)]+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[qf(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[pg(x)]\right]+\\&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[p(x)]\left[\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[rf(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[qg(x)]+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[qf(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[rg(x)]\right]+\\&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[q(x)]\left[\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[rf(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[pg(x)]+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[pf(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[rg(x)]\right].\end{aligned}\tag{15}$$

where $v'>-1$, $1>\max\left\{0,\Re(v+v'+\xi-\vartheta),\Re(v'-\xi')\right\}$, $(\vartheta-v')>\max(1-\xi,1-v)$.

Proof: To prove theorem, put $u=p$, $v=q$, and using lemma 2, we get

$$\begin{aligned}&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[p(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[qfg(x)]+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[q(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[pfg(x)]\geq\\&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[pf(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[qg(x)]+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[qf(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[pg(x)].\end{aligned}\tag{16}$$

Now, multiplying both sides by (16) $\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[r(x)]$, we have

$$\begin{aligned}&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[r(x)]\left[\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[p(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[qfg(x)]+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[q(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[pfg(x)]\right]\geq\\&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[r(x)]\left[\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[pf(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[qg(x)]+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[qf(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[pg(x)]\right],\end{aligned}\tag{17}$$

again, put $u=r$, $v=q$, and using lemma 2, we get

$$\begin{aligned}&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[r(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[qfg(x)]+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[q(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[rfg(x)]\geq\\&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[rf(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[qg(x)]+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[qf(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[rg(x)],\end{aligned}\tag{18}$$

multiplying both sides of (18) by $\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[p(x)]$, we have

$$\begin{aligned}&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[p(x)]\left[\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[r(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[qfg(x)]+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[q(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[rfg(x)]\right]\geq\\&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[p(x)]\left[\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[rf(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[qg(x)]+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[qf(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[rg(x)]\right].\end{aligned}\tag{19}$$

With the same arguments as in equation (17) and (19), we can write

$$\begin{aligned}&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[q(x)]\left[\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[r(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[pfg(x)]+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[p(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[rfg)(x)]\right]\geq\\&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[q(x)]\left[\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[rf(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[pg(x)]+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[pf(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[rg(x)]\right].\end{aligned}\tag{20}$$

Adding the inequalities (17), (19) and (20), we get required inequality (15).

Lemma 3. *Let f and g be two integrable and synchronous functions on $[0,\infty)$, and $u,v:[0,\infty[\to[0,\infty)$. Then for all $x>0$, $v,v',\xi,\xi',\vartheta,\alpha,\alpha',\beta,\beta',\theta\in\mathbb{C}$, $\Re(\vartheta),\Re(\theta)>0$ we have*

$$\begin{aligned}&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[u(x)]\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[vfg(x)]+\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[v(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[ufg(x)]\geq\\&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[uf(x)]\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[vg(x)]+\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[vf(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[ug(x)],\end{aligned}\tag{21}$$

where $v', \alpha' > -1$, $1 > \max\left\{0, \Re(v+v'+\xi-\vartheta), \Re(v'-\xi')\right\}$, $(\vartheta - v') > \max(1-\xi, 1-v)$, $1 > \max\left\{0, \Re(\alpha+\alpha'+\beta-\theta), \Re(\alpha'-\beta')\right\}(\theta-\alpha') > \max(1-\beta, 1-\alpha)$.

Proof:- Now multiplying both sides of (12) by $\mathfrak{I}(x,\rho)v(\rho) = \frac{x^{-\alpha}}{\Gamma(\theta)}(x-\rho)^{\theta-1}\rho^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\theta;1-\frac{\rho}{x},1-\frac{x}{\rho})v(\rho)$ $(\rho \in (0,x), x > 0)$, which (in view of the argument mentioned above in proof of lemma 2) remain positive. Then integrating resulting identity with respect to ρ from 0 to x, we have

$$\begin{aligned}
&\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[ufg(x)]\frac{x^{-\alpha}}{\Gamma(\theta)}\int_0^x (x-\rho)^{\theta-1}\rho^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\theta;1-\frac{\rho}{x},1-\frac{x}{\rho})v(\rho)d\rho \\
&+\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[u(x)]\frac{x^{-\alpha}}{\Gamma(\theta)}\int_0^x (x-\rho)^{\theta-1}\rho^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\theta;1-\frac{\rho}{x},1-\frac{x}{\rho})v(\rho)f(\rho)g(\rho)d\rho \\
&\geq \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[uf(x)]\frac{x^{-\alpha}}{\Gamma(\theta)}\int_0^x (x-\rho)^{\theta-1}\rho^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\theta;1-\frac{\rho}{x},1-\frac{x}{\rho})v(\rho)g(\rho)d\rho \\
&+\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[ug(x)]\frac{x^{-\alpha}}{\Gamma(\theta)}\int_0^x (x-\rho)^{\theta-1}\rho^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\theta;1-\frac{\rho}{x},1-\frac{x}{\rho})v(\rho)f(\rho)d\rho.
\end{aligned} \tag{22}$$

This completes the proof of inequality (21).

Theorem 2. *Let f and g be two integrable and synchronous functions on $[0,\infty)$, and $r,p,q : [0,\infty) \to [0,\infty)$. Then for all $x > 0$, $v,v',\xi,\xi',\vartheta,\alpha,\alpha',\beta,\beta',\theta \in \mathbb{C}$, $\Re(\vartheta), \Re(\theta) > 0$ we have*

$$\begin{aligned}
&\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[r(x)][\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[q(x)]\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[pfg(x)] + 2\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[p(x)]\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[qfg(x)] \\
&+\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[q(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[pfg(x)]] \\
&+\left[\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[p(x)]\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[q(x)] + \mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[p(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[q(x)]\right]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[rfg(x)] \geq \\
&\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[r(x)]\left[\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[pf(x)]\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[qg(x)] + \mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[qf(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[pg(x)]\right] + \\
&\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[p(x)]\left[\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[rf(x)]\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[qg(x)] + \mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[qf(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[rg(x)]\right] + \\
&\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[q(x)]\left[\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[rf(x)]\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[pg(x)] + \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[pf(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[rg(x)]\right].
\end{aligned} \tag{23}$$

where $v', \alpha' > -1$, $1 > \max\left\{0, \Re(v+v'+\xi-\vartheta), \Re(v'-\xi')\right\}$, $(\vartheta - v') > \max(1-\xi, 1-v)$, $1 > \max\left\{0, \Re(\alpha+\alpha'+\beta-\theta), \Re(\alpha'-\beta')\right\}$ $(\theta-\alpha') > \max(1-\beta, 1-\alpha)$.

Proof:- To prove theorem, we put $u = p$, $v = q$ and using lemma 3 we get

$$\begin{aligned}
&\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[p(x)]\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[qfg(x)] + \mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[q(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[pfg(x)] \geq \\
&\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[pf(x)]\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[qg(x)] + \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[qf(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[pg(x)].
\end{aligned} \tag{24}$$

Now, multiplying both sides by (24) $\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[r(x)]$, we have

$$\begin{aligned}
&\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[r(x)]\left[\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[p(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[qfg(x)] + \mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[q(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[pfg(x)]\right] \geq \\
&\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[r(x)]\left[\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[pf(x)]\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[qg(x)] + \mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[qf(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[pg(x)]\right],
\end{aligned} \tag{25}$$

putting $u = r$, $v = q$, and using lemma 3, we get

$$\begin{aligned}&\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\vartheta}[r(x)]\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[qfg(x)]+\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[q(x)]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\vartheta}[rfg(x)]\geq\\&\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\vartheta}[rf(x)]\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[qg(x)]+\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[qf(x)]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\vartheta}[rg(x)],\end{aligned}\tag{26}$$

multiplying both sides by (26) $\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\vartheta}[p(x)]$, we have

$$\begin{aligned}&\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\vartheta}[p(x)]\left[\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\vartheta}[r(x)]\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[qfg(x)]+\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[q(x)]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\vartheta}[rfg(x)]\right]\geq\\&\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\vartheta}[p(x)]\left[\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\vartheta}[rf(x)]\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[qg(x)]+\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[qf(x)]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\vartheta}[rg(x)]\right].\end{aligned}\tag{27}$$

With the same argument as in equation (25) and (27), we obtain

$$\begin{aligned}&\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\vartheta}[q(x)]\left[\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\vartheta}[r(x)]\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[pfg(x)]+\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[p(x)]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\vartheta}[rfg(x)]\right]\geq\\&\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\vartheta}[q(x)]\left[\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\vartheta}[rf(x)]\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[pg(x)]+\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[pf(x)]\mathfrak{I}_{0,x}^{\nu,\nu',\xi,\xi',\vartheta}[rg(x)]\right].\end{aligned}\tag{28}$$

Adding the inequalities (25), (26) and (27), we get the inequality (23).

*Remark.*If f,g,r,p *and* q satisfy the following conditions,

1.The function f and g is asynchronous on $[0,\infty)$.
2.The function r,p,q is negative on $[0,\infty)$.
3.Two of the function r,p,q are positive and the third is negative on $[0,\infty)$.

Then the inequalities 15 and 23 are reversed.

4 Conclusion

In this paper, we presented Marichev-Saigo-Maeda fractional integral operators. Then we established some fractional integral inequalities for extended Chebyshev functional. Several results on fractional integral inequalities have been investigated using different fractional integral operators. Here, we briefly consider some implications of our main results. If we set $\nu' = 0$ in the equation (3) would reduced immediately to Saigo type of fractional integral operators as in following relationship, see [18, 23, 24, 31],

$$\left(\mathfrak{I}_{0,x}^{\nu,0,\xi,\xi',\vartheta}f\right)(x)=\left(\mathfrak{I}_{0,x}^{\vartheta,\nu-\vartheta,-\xi}f\right)(x),\tag{29}$$

where the hypergeometric operators that appear in the right hand side are defined as

$$\mathfrak{I}_{0,x}^{\nu,\nu',\vartheta}f(x)=\frac{x^{-\nu-\xi}}{\Gamma(\vartheta)}\int_0^x(x-t)_2^{\nu-1}F_1(\nu+\xi;\nu;1-\frac{t}{x})f(t)dt,(\vartheta>0,\nu,\xi\in\mathbb{C})\tag{30}$$

Furthermore, we can reduce operator (3) to Erdelyi-Kober and Riemann-Liouville type of fractional integral operators which are special cases of Saigo fractional operator (30). It is noticed that all results established by V. L. Chinchane and D. B. Pachpatte in [8] can be easily obtained by setting $\nu=\alpha,\nu'=\beta,\vartheta=\eta$ in the operator (3). The inequalities investigated in this paper give some contribution in the fields of fractional calculus and Marichev-Saigo-Maeda fractional integral operators. Moreover, they are expected to lead to some applications for finding uniqueness of solutions in fractional differential equations.

Conflict of Interest

The authors declare that they have no conflict of interest.

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