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The Theory of the Edge Ideal of a Graph Together with Formal Local Cohomology Modules

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Abstract: There exists a systematic study of the formal local cohomology modules. In this paper, we discuted and studied about the formal local cohomology module. For the definition of this module is used the definition of the local cohomology module with respect to an ideal, where this module is mencioned in the text. Moreover, we involve the theory of graphs within of such module achieving an application for the edge ideal of a graph.

Keywords: inverse limit, local cohomology modules, formal local cohomology modules, edge ideal of a graph

1 Introduction

Throughout this paper, R is a commutative ring with non-zero identity. The local cohomology theory of Grothendieck has proved to be an important tool in commutative algebra. The theory of local cohomology if has developed so much six decades after its introduction by Grothendieck. There exists a relation between local cohomology, given by [2], and formal local cohomology, given by [5]. Not so much is known about these modules.

The motivation of this work is precisely the formal local cohomology of P. Schenzel. Let $\mathfrak{a}$ denote an ideal of a local ring $(R,\mathfrak{m})$. For a finitely generated R-module M, let $\mathrm{H}^i_{\mathfrak{a}}(M)$, for $0 \leq i \in \mathbb{Z}$, denote the local cohomology module of M with respect to $\mathfrak{a}$ (cf. [2] for the basic definitions). There are the following integers related to these local cohomology modules, such as

$$\operatorname{grade}(\mathfrak{a},M) = \inf\left\{i \in \mathbb{Z} \mid \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\right\} \text{ and,}$$

$$\operatorname{cd}(\mathfrak{a},M) = \sup\left\{i \in \mathbb{Z} \mid \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\right\},$$

called the grade, respectively the cohomological dimension, of M with respect to $\mathfrak{a}$. In general we have the bounds:

$$\operatorname{height}_M(\mathfrak{a}) \leq \operatorname{cd}(\mathfrak{a},M) \leq \dim(M).$$

In the case of $\mathfrak{m}$ the maximal ideal it follows that

$$\operatorname{grade}(\mathfrak{m},M) = \operatorname{depth}(M) \text{ and,}$$

$$\operatorname{cd}(\mathfrak{m},M) = \dim(M).$$

From the local cohomology we obtain the formal local cohomology, and the purpose here is to obtain results for the formal local cohomology module. Here we consider the asymptotic behavior of the family of local cohomology modules given by $\left\{\mathrm{H}^i_{\mathfrak{m}}\left(\frac{M}{\mathfrak{a}^n M}\right)\right\}_{n\in\mathbb{N}}$ for an integer $i \in \mathbb{Z}$. By the natural homomorphisms these families form a projective system. Their projective limit given by

$$\varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}\left(\frac{M}{\mathfrak{a}^n M}\right)$$

is called the i-th formal local cohomology module of M with respect to $\mathfrak{m}$. Not so much is know about these modules.

The principal subject of the paper is a systematic study of the formal local cohomology modules.

In the Section 2 we present notions of local cohomology modules and some prerequisites, which will be used throughout of the paper.

In the Section 3 we put some preliminaries of the theory of graphs which involving the edge ideal of a graph G, according to [6] and [1]. Associated to the graph G is a monomial ideal

$$I(G) = (v_{i-1}v_i : v_{i-1}v_i \text{ is an edge of } G),$$

in the polynomial ring $R = K[v_1, v_2, \ldots, v_s]$ over a field K, called the **edge ideal** of G, where $\{v_1, v_2, \ldots, v_s\}$ is the set

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of vertices of G. The preliminaries of the theory of graphs were introduced in this Section 3 together with the concepts suitable for the work. Here, we mean by a graph G, a finite simple graph with the vertex set $V(G)$ and with no isolated vertices.

In Section 4 we provide a theorem about the formal local cohomology module, applied on an edge ideal. We observe that the formal local cohomology module is used for to characterize some values. For example, we have the result: "Let $\mathfrak{a}$ denote an ideal of a local ring $(R,\mathfrak{m})$. Then

$$\dim\left(\frac{M}{\mathfrak{a}M}\right) = \sup\left\{ i\in\mathbb{Z} \mid \varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}\left(\frac{M}{\mathfrak{a}^n M}\right)\neq 0\right\},$$

for a finitely generated R-module M."

Moreover, the description of

$$\inf\left\{ i\in\mathbb{Z} \mid \varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}\left(\frac{M}{\mathfrak{a}^n M}\right)\neq 0\right\},$$

is used for to define the formal grade, i.e., we have that

$$\mathrm{fgrade}(\mathfrak{a},M) = \inf\left\{ i\in\mathbb{Z} \mid \varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}\left(\frac{M}{\mathfrak{a}^n M}\right)\neq 0\right\},$$

for an ideal $\mathfrak{a}$ of R and a finitely generated R-module M.

2 Some prerequisites

Let I be an ideal of R, and let M be an R-module. In [2], the i-th local cohomology module $\mathrm{H}^i_I(M)$ of M with respect to I is defined by

$$\mathrm{H}^i_I(M) = \varinjlim_{t\in\mathbb{N}} \mathrm{Ext}^i_R\left(R/I^t, M\right),$$

for all $0\leq i\in\mathbb{Z}$.

Now, for $\mathfrak{a}$ other ideal of R, we consider that $(R,\mathfrak{m})$ is a local ring, and we consider the family of local cohomology modules given by $\left\{\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)\right\}_{n\in\mathbb{N}}$. According to [5], for every $n\in\mathbb{N}$, we have that there exists a natural homomorphism

$$\phi_{n+1,n} : \mathrm{H}^i_{\mathfrak{m}}\left(M/\mathfrak{a}^{n+1}M\right)\to \mathrm{H}^i_{\mathfrak{m}}\left(M/\mathfrak{a}^n M\right).$$

These families form an inverse system. Their inverse limit that, for all $0\leq i\in\mathbb{Z}$, is given by

$$\varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}\left(M/\mathfrak{a}^n M\right),$$

is called, according to [5], the i-th formal local cohomology module of M with respect to $\mathfrak{m}$, and will be denoted by $\mathfrak{F}^i_{\mathfrak{a},\mathfrak{m}}(M)$.

In the next section, we introduce the edge ideal of a graph. And, with this definition, we put the object we dealt with in the result of this article.

Moreover, for a Noetherian local ring $(R,\mathfrak{m})$ and M an R-module we have the Matlis dual module $\mathrm{D}(M) = \mathrm{Hom}_R(M,E)$ of M, where $E = \mathrm{E}(R/\mathfrak{m})$ is the injective envelope of the residue field $R/\mathfrak{m}$.

The next definition will be used in the sequence of the paper.

Definition 21 Let $(R,\mathfrak{m},k)$ and $(S,\mathfrak{n},l)$ be two local rings. A ring homomorphism $(R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ is a *local homomorphism* if $\mathfrak{m}S\subset\mathfrak{n}$.

In the next section, the following remark will be used.

Remark 22([5, Remark 4.6]) Note that, for $(R,\mathfrak{m})$ a local ring, the short exact sequence

$$0\to\mathfrak{a}^n M/\mathfrak{a}^{n+1}M\to M/\mathfrak{a}^{n+1}M\to M/\mathfrak{a}^n M\to 0$$

induces an epimorphism

$$\mathrm{H}^i_{\mathfrak{m}}\left(M/\mathfrak{a}^{n+1}M\right)\to \mathrm{H}^i_{\mathfrak{m}}\left(M/\mathfrak{a}^n M\right)\to 0,$$

of non-zero R-modules for all $n\in\mathbb{N}$. Hence, the inverse limit

$$\varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}\left(M/\mathfrak{a}^n M\right)$$

is not zero.

The following definition will be used in the next section, and concerns the dual definition of the local cohomology module.

Definition 23([3, Definition 3.1]) Let R be a Noetherian ring. Let I be an ideal of R and let M be an R-module. The i-th *local homology module* $\mathrm{H}^I_i(M)$ of M with respect to I is defined by,

$$\mathrm{H}^I_i(M) := \varprojlim_{t\in\mathbb{N}} \mathrm{Tor}^R_i\left(R/I^t, M\right),$$

for all $i\geq 0$, an integer number.

3 The edge ideal of a graph

This section is in accordance with [1] and [6].

Let $R = K[v_1,\ldots,v_s]$ be a polynomial ring over a field K, and let $Z=\{z_1,\ldots,z_q\}$ be a finite set of monomials in R. The *monomial subring* spanned by Z is the K-subalgebra,

$$K[Z] = K[z_1,\ldots,z_q]\subset R.$$

In general, it is very difficult to certify whether $K[Z]$ has a given algebraic property - e.g., Cohen-Macaulay, normal - or to obtain a measure of its numerical invariants - e.g., Hilbert function. This arises because the number q of monomials is usually large.

Thus, we consider any graph G with vertex set $V(G) = \{v_1,\ldots,v_s\}$.

Let Z be the set of all monomials $v_{i-1}v_i$ in $R = K[v_1,\ldots,v_s]$, such that $\{v_{i-1}v_i\}$ is an edge of G, i.e., the graph finite and simple G, with no isolated vertices, is such that the squarefree monomials of degree two are defining the edges of the graph G.

Definition 31A *walk* of length s in G is an alternating sequence of vertices and edges $w = \{v_1, z_1, v_1, \ldots, v_{s-1}, z_h, v_s\}$, where $z_i = \{v_{i-1}v_i\}$ is the edge joining v_{i-1} and v_i.

Definition 32A walk is closed if $v_1 = v_s$. A walk may also be denoted by $\{v_1, \ldots, v_s\}$, the edges being evident by context. A *cycle* of length s is a closed walk, in which the points $v_1, \ldots, v_s$ are distinct.

A *path* is a walk with all the points distinct. A *tree* is a connected graph without cycles and a graph is *bipartite* if all its cycles are even. A vertex of degree one will be called an *end point*.

Definition 33A subgraph $G^{'} \subseteq G$ is called **induced** if $v_{i-1}v_i$ is an edge of $G^{'}$ whenever v_{i-1} and v_i are vertices of $G^{'}$ and $v_{i-1}v_i$ is an edge of G.

The **complement** of a graph G, for which we write G^c, is the graph on the same vertex set in which $v_{i-1}v_i$ is an edge of G^c if and only if it is not an edge of G. Finally, let C_k be denote the cycle on k vertices; a **chord** is an edge which is not in the edge set of C_k. A cycle is called **minimal** if it has no chord.

If G is a graph without isolated vertices then let R be denote the polynomial ring on the vertices of G over some fixed field K.

Definition 34([1]) According to the previous context, the **edge ideal** of a finite simple graph G, with no isolated vertices, is defined by

$$I(G) = (v_{i-1}v_i : v_{i-1}v_i \text{ is an edge of } G).$$

We denote by $\mathfrak{m} = (v_1, \ldots, v_s)$ the homogeneous maximal ideal of $R = K[v_1, \ldots, v_s]$, where we have $I(G)$ a monomial ideal of R which is finitely generated.

Now, we put the definition that will be dealt with in the next section.

Definition 35 Let $R = K[v_1, \ldots, v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Since that $I(G)$ is an R-module, in the same context of the definition of i-th formal local cohomology module, we define the i-th **edge formal local cohomology module** of $I(G)$ with respect to $\mathfrak{m} = (v_1, \ldots, v_s)$ of the following form:

$$\varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}(I(G)/\mathfrak{a}^n I(G)),$$

and will be denoted by $\mathfrak{F}^i_{\mathfrak{a},\mathfrak{m}}(I(G))$.

According to Definition 21, we can to define in the context of the edge ideal the following concept.

Definition 36 Let $(R, \mathfrak{m}, k)$ and $(S, \mathfrak{n}, \mathfrak{l})$ be two polynomial rings. And let $I(G)$ be an edge ideal in R, and let $I(G^{'})$ be an edge ideal in S. A ring homomorphism $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \mathfrak{l})$ is an edge homomorphism if $\mathfrak{m}S \subset \mathfrak{n}$.

Now, according to Remark 22, we can to remark in the context of the edge ideal the following.

Remark 37 Let $R = K[v_1, \ldots, v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Note that for $\mathfrak{m} = (v_1, \ldots, v_s)$, the short exact sequence

$$0 \to \mathfrak{a}^n I(G)/\mathfrak{a}^{n+1} I(G) \to I(G)/\mathfrak{a}^{n+1} I(G) \to I(G)/\mathfrak{a}^n I(G) \to 0$$

induces an epimorphism

$$\mathrm{H}^i_{\mathfrak{m}}\left(I(G)/\mathfrak{a}^{n+1} I(G)\right) \to \mathrm{H}^i_{\mathfrak{m}}\left(I(G)/\mathfrak{a}^n I(G)\right) \to 0,$$

of non-zero R-modules for all $n \in \mathbb{N}$.

Hence, the inverse limit

$$\varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}\left(I(G)/\mathfrak{a}^n I(G)\right)$$

is not zero.

4 The result of application

In this section, we presented a result about the edge formal local cohomology module which involve the theory of graphs together with the edge ideal of a graph G.

Here, we take K a fixed field and we consider $K[v_1, \ldots, v_s]$ the polynomial ring over the field K. Since K is a field, we have that K is a Noetherian ring and then $K[v_1, \ldots, v_s]$ is also a Noetherian ring (Theorem of the Hilbert Basis). Moreover, $K[v_1, \ldots, v_s]$ is not a local ring because K is a field.

We presented the result in the form of theorem.

Theorem 41*Let $(R, \mathfrak{m}, k)$ be a Noetherian ring, where we have that the ring $R = K[v_1, v_2, \ldots, v_s]$, and $k = R/\mathfrak{m}$. Let $I(G)$ be an edge ideal, of a graph G, in the polynomial ring R. Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be an edge homomorphism of polynomial rings, with S a Noetherian ring, where $S = K[v_1, v_2, \ldots, v_r]$ with $r \leq s$, and let J be an ideal of R. We observe that $I(G)$, which is an R-module, is a finitely generated S-module. If $\mathfrak{F}^i_{J,\mathfrak{m}}(I(G)) = 0$, for each $i \geq 1$, then $\mathrm{D}(I(G)/J^n I(G))$ is a flat R-module, for some $n \in \mathbb{N}$.*

*Proof.*By the hypothesis, for all $i \geq 1$, we have that:

$$\mathfrak{F}^i_{J,\mathfrak{m}}(I(G)) := \varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}\left(I(G)/J^n I(G)\right) = 0.$$

By the Remark 37, we have that there exists $n \in \mathbb{N}$ such that the local cohomology module $\mathrm{H}^i_{\mathfrak{m}}\left(I(G)/J^n I(G)\right) = 0$, where $\mathfrak{m} = (v_1, \ldots, v_s)$.

Therefore, it follows, as given in prerequisites, that we have:

$$\varinjlim_{t\in\mathbb{N}} \mathrm{Ext}^i_R\left(R/\mathfrak{m}^t, I(G)/J^n I(G)\right) = 0. \quad (*)$$

Thus, applying the Matlis dual module $\mathrm{D}(\bullet)$ (see prerequisites) to $(*)$ we obtain that $\mathrm{D}\left(\varinjlim_{t\in\mathbb{N}} \mathrm{Ext}^i_R\left(R/\mathfrak{m}^t, I(G)/J^nI(G)\right)\right) = 0$. Now, by [4, Theorem 2.27], it follows that $\mathrm{D}\left(\varinjlim_{t\in\mathbb{N}} \mathrm{Ext}^i_R\left(R/\mathfrak{m}^t, I(G)/J^nI(G)\right)\right)$, which is equal to

$$\mathrm{Hom}_R\left(\varinjlim_{t\in\mathbb{N}} \mathrm{Ext}^i_R\left(R/\mathfrak{m}^t, I(G)/J^nI(G)\right), \mathrm{E}(R/\mathfrak{m})\right),$$

is isomorphic to $\varprojlim_{t\in\mathbb{N}} \mathrm{Hom}_R\left(\mathrm{Ext}^i_R\left(R/\mathfrak{m}^t, I(G)/J^nI(G)\right), \mathrm{E}(R/\mathfrak{m})\right)$, which in turn is equal to $\varprojlim_{t\in\mathbb{N}} \mathrm{D}\left(\mathrm{Ext}^i_R\left(R/\mathfrak{m}^t, I(G)/J^nI(G)\right)\right)$.

By [7, Proposition 3.4.14 (ii)], we have that:

$$\mathrm{D}\left(\mathrm{Ext}^i_R\left(R/\mathfrak{m}^t, I(G)/J^nI(G)\right)\right) \cong \mathrm{Tor}^R_i\left(R/\mathfrak{m}^t, \mathrm{D}\left(I(G)/J^nI(G)\right)\right).$$

Therefore, by the Definition 23, we have that

$$\mathrm{H}^{\mathfrak{m}}_i\left(\mathrm{D}\left(I(G)/J^nI(G)\right)\right) = \varprojlim_{t\in\mathbb{N}} \mathrm{Tor}^R_i\left(R/\mathfrak{m}^t, \mathrm{D}\left(I(G)/J^nI(G)\right)\right) = 0.$$

By Remark 37, it follows that there exists $t \in \mathbb{N}$ such that

$$\mathrm{Tor}^R_i\left(R/\mathfrak{m}^t, \mathrm{D}\left(I(G)/J^nI(G)\right)\right) = 0, \text{ for all } i \geq 1.$$

Thus, also we have

$$\mathrm{Tor}^R_i\left(R/\mathfrak{m}, \mathrm{D}\left(I(G)/J^nI(G)\right)\right) = 0, \quad (**)$$

for all $i \geq 1$.

To end the theorem, it suffices to prove that

$$\mathrm{Tor}^R_i\left(N, \mathrm{D}\left(I(G)/J^nI(G)\right)\right) = 0$$

for each finitely generated R-module N, and for all $i \geq 1$. This we achieve by an induction on $\dim(N)$.

When $\dim(N) = 0$, let's induce on the length of N. If $\mathfrak{l}_R(N) = 1$, then $N \cong R/\mathfrak{m}$, so the desired result is the mentioned in $(**)$. When $\mathfrak{l}_R(N) \geq 2$, one can get an exact sequence of R-modules $0 \to R/\mathfrak{m} \to N \to N' \to 0$. Applying $\bullet \otimes_R \mathrm{D}\left(I(G)/J^nI(G)\right)$ we obtain an exact sequence

$$\ldots \to \mathrm{Tor}^R_i\left(N, \mathrm{D}\left(I(G)/J^nI(G)\right)\right) \to \mathrm{Tor}^R_i\left(N', \mathrm{D}\left(I(G)/J^nI(G)\right)\right).$$

Since $\mathfrak{l}_R\left(N'\right) = \mathfrak{l}_R(N) - 1$, the induction hypothesis provides us the vanishing.

Let $d \geq 1$ be an integer such that for $i \geq 1$ we have that the functor $\mathrm{Tor}^R_i\left(\bullet, \mathrm{D}\left(I(G)/J^nI(G)\right)\right)$ whether vanishes on finitely generated R-modules of dimension up to $d-1$. Let N be a finitely generated R-module of dimension d. We consider the exact sequence of R-modules

$$0 \to \Gamma_{\mathfrak{m}}(N) \to N \to N' \to 0,$$

and the induced exact sequence on $\mathrm{Tor}^R_i\left(\bullet, \mathrm{D}\left(I(G)/J^nI(G)\right)\right)$.

Since $\mathfrak{l}_R\left(\Gamma_{\mathfrak{m}}(N)\right)$ is finite, it suffices to verify the vanishing for N'. Thus, replacing N by N', one may assume that $\operatorname{depth}(N) \geq 1$. Let x in R be an N-regular element; then $\dim\left(N/(x)N\right) = \dim(N) - 1$. In view of the induction hypothesis, the exact sequence $0 \to N \xrightarrow{x\cdot} N \to N/(x)N \to 0$ induces an exact sequence

$$\mathrm{Tor}^R_i\left(N, \mathrm{D}\left(I(\tilde{G})\right)\right) \xrightarrow{x\cdot} \mathrm{Tor}^R_i\left(N, \mathrm{D}\left(I(\tilde{G})\right)\right) \to 0,$$

for $i \geq 1$, with

$$\mathrm{Tor}^R_i\left(N/(x)N, \mathrm{D}\left(I(\tilde{G})\right)\right) = 0,$$

and where $I(\tilde{G}) = I(G)/J^nI(G)$.

As an S-module, $\mathrm{Tor}^R_i\left(N, \mathrm{D}\left(I(G)/J^nI(G)\right)\right)$ is finitely generated: we compute it using a resolution of N by finitely generated free R-modules.

Since, by Definition 36, xS is in the maximal ideal of S, the exact sequence above implies

$$\mathrm{Tor}^R_i\left(N, \mathrm{D}\left(I(G)/J^nI(G)\right)\right) = 0,$$

by Nakayama's lemma, for all $i \geq 1$.

This completes the induction step and hence the proof, as required.

5 Conclusion

The principal idea of this work was to do a more specific study about the formal local cohomology module.

Let $\mathfrak{a}$ denote an ideal of a local ring $(R, \mathfrak{m})$. Let M be a finitely generated R-module. There exists a systematic study of the formal local cohomology modules, which, as we have already seen in the text, are given by

$$\varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}\left(\frac{M}{\mathfrak{a}^n M}\right), \text{ for } i \in \mathbb{Z}, \text{ with } i \geq 0.$$

These cohomology modules occur in relation to the formal completion of the punctured spectrum $\mathrm{Spec}(R) \setminus \mathrm{V}(\mathfrak{m})$.

We can not fail to mention in our conclusion that we use many things from the commutative algebra theory to make the results of the theory of local cohomology and formal local cohomology.

We can then observe that the formal local cohomology theory can be further explored in order that we can obtain more results, which relate more definitions and concepts. For example, for future studies on the subject we can establish conditions for that the formal local cohomology module satisfies the conditions for to be an co-Cohen-Macaulay R-module.

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