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A Generalized Transformation Function that Transforms Multiplicative Preference Relation into Fuzzy Preference Relation in MPDM

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Abstract: This paper proposes a generalized transformation function that transforms multiplicative preference relation (MPR) into fuzzy preference relation (FPR) in multi person decision making (MPDM). In analytic hierarchy process (AHP), reciprocal multiplicative preference relation is considered to be the preference representation and in fuzzy majority based selection, FPR acts as uniform representation element. Though, the effectiveness of AHP is to find incompatible judgments, it does not take into account the uncertainty to a number. In decision making problems (DMP) , the lack of consistency leads to get the inconsistent solutions. By exploiting this proposed transformation, it is possible to find a preference of alternatives with strongly consistent solutions in decision making processes. The proposed work is an improvement to the existing work of Herrera et., al.,.

Keywords: Multiperson decision making, Analytic hierarchy process, Multiplicative preference relation, fuzzy preference relation, Consistency ratio.

1 Introduction

In real world situations, making a decision with multi person or multi criteria is a vital area in decision theory. A MPDM problem can be interpreted as the solution alternatives which have been selected, based on the data given by different experts [7]. These type of problems, taken into account of two classical methods that are available in literature, AHP [7] and fuzzy majority based selection scheme [8]. Even though AHP is used in pairwise comparison matrices to find consistency, it will lead to inconsistent solutions many times [9]. It is very tough to ensure a consistent in pair wise comparison matrices. In decision making problems, pairwise comparison matrices are rarely consistent, and the inconsistent judgments may lead to unreasonable decisions. Using some necessary definitions such as transitivity property the consistency is characterized. Since the accuracy is improved by using fuzzy majority scheme, FPRs are mostly used in decision making problems [10] and some of the required properties are to be verified to check the consistency. The existing transformation [8] satisfies the required properties of consistency. But it lacks behind with respect to consistency ratio(CR). By [7], an exact consistency ratio is evaluated by dividing the consistency index (CI) for the collection of judgments may be too consistent. The lacking of consistency will be overcome by this proposed new generalized transformation function in FPR.

In the proposed paper, in section II, some literature review is enhanced. In section III, AHP technique in multiplicative preference relation is discussed. In section IV, fuzzy preference relation is defined and the new generalized transformation function is proposed to be consistent, a collection of the required criteria are being convinced as in [1,2,3]. In section V, the collective preference relation is obtained in fuzzy majority based selection scheme [7]. In section VI, aggregation operation is used to obtain consistency level of MPDM shown by a numerical example. In section VII, the conclusion is drawn.

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2 Literature Review

Cengiz Kahraman et al., [11] classified the recent status of fuzzy multi person/criteria decision-making techniques into two different parts: One is, fuzzy multiattribute decision-making (MADM) and the other one is, fuzzy multi objective decision-making (MODM). In fuzzy model, weak transitivity is considered as the minimum condition and strong stochastic transitivity is considered as the stronger condition for consistency [1,3]. Chiclana et al. [1] presented a fuzzy MPDM which combines distinctive preference relations and Saaty's MPRs [7]. The author transforms the MPR into FPR in [1]. Though the transformation satisfies the required properties of consistency it lacks behind with respect to consistency ratio. To rectify the complication, a new generalized transformation function is proposed in this paper. F.Herrera et.al., [8] defined a transformation function which relates both the kinds of preference relations, reciprocal MPR, $A = (a_{ij}), a_{ij} \in [1/9, 9]$ and reciprocal FPR, $P = (p_{ij}), p_{ij} \in [0, 1]$. In [1], the author obtained that the consistency for FPRs is based on the additive property. The author [1] designed a technique to built compatible FPRs from a collection of (n-1) choices provided by the experts. The existing transformation fucntion satisfies the required properties of consistency. But it lacks behind with respect to consistency ratio. Later, Sridevi [2] determined a transformation fucntion which transforms Saaty's reciprocal preference relation into fuzzy preference relation. For this transformation, various consistency properties were satisfied. Then the preference values were aggregated using two different techniques OWA and GMO [6]. By extending the existing transformation function of [1], here a new preference structure is generalized in fuzzy majority to relate MPR and FPR using fuzzy majority and consistency is analyzed and compared with the existing system.

3 Analytic hierarchy process [7]

A procedure which includes and integrates judgments and measurements in a hierarchical way is widely known as AHP, recommended by L. Saaty. It is acceptable for complex situations which involve comparison of decision elements that are difficult to quantify. In AHP evaluations, information about the alternatives is provided in multiplicative preference relations.

3.1 Multiplicative preference relation[8]

By Saaty's ratio scale [7], in a MPDM problem, a professional prioritizes a collection of alternatives $X = \{x_1, x_2, \cdots, x_n\}$ as a multiplicative preference relation $A^k = [a^k_{ij}]$ where $a^k_{ij} \in [1/9, 9]$, of $k^t h$ and $a^k_{ij}.a^k_{ji} = 1 \ \ \forall i, j$, where $i, j = 1, 2, \cdots, n$.

Table 1: A positive reciprocal matrix

	E	U	R	P
E	1	1/3	1/9	1/5
U	3	1	1	1
R	9	1	1	3
P	5	1	1/3	1

Saaty suggests estimating a^k_{ij} with the correlation scale [1/9,9] and exactly the (1-9) scale , as
$a^k_{ij} = 1$ represents indifference in x_i and x_j,
$a^k_{ij} = 19$ represents that x_i is said to be precisely opt for x_j,
$a^k_{ij} \in 2, 3, \cdots, 8$ represents the transitional computations.

The main advantage of the AHP is its ability to rank the choices [7]. On the other hand, AHP is limited to work only if the matrices are all of positive reciprocal matrices such as, in table 1 [7] considering E,U,R,P as different set of objectives..

The other limitation is that, if the scale is changed from 1 to 9 to any other number the end result also account to change [7]. To overcome this drawback , F. Herrera et.al.,[8] used a fuzzy preference relation, adopted fuzzy majority based choice scheme to convert the Saaty's MPR into FPR. Also, the merging of AHP and fuzzy set tends to give more adaptability in discernment and decision making.

4 Fuzzy majority based Scheme[8]

Fuzzy majority allows more flexibility in the decision making method since every decision is attained by utilizing a soft majority of expert's choice. The proposed method is developed using the fuzzy majority guided aggregated operator,as Ordered Weighted Average (OWA) operator, Quantifier Guided Dominance Degree (QGDD) and Quantifier Guided Non Dominance Degree (QGNDD).

4.1 OWA operator [6]

The OWA operator is an important technique for aggregation of data given by the experts. It provides connection with the concepts of fuzzy majority scheme in DM. According to R.R.Yager, an OWA operator is mapped as $F \subset R^n \rightarrow R$, associated with n vectors$n$ vectors $w = (w_1, w_2, \cdots w_n)^T$ such as $w_i \in [0, 1], 1 \leq i \leq n$, and $\sum_{i=1}^n w_i = 1$ such that, $\text{OWA}(a_1, a_2, \cdots a_n) = w_1 b_1 + w_2 b_2 + \cdots + w_n b_n$ where b_j is the j^{th} largest value of a_i[6]. OWA operator plays a vital role in selection process to calculate collective MPR as well as the QGDD and QGNDD from MPR.

4.2 A fuzzy preference relation[8]

A FPR, $P^k \subset X \times X$ having the membership $\mu_{P}k : X \times X \rightarrow [0,1]$ where $\mu_{P}k(x_i, x_j) = p_{ij}^k$ specifies the degree of choices or strength of alternatives x_i over x_j.

$p_{ij}^k = \frac{1}{2}$ represents that there is no significant difference between x_i and x_j
$p_{ij}^k = 1$ represents that x_i is said precisely opt for x_j
$p_{ij}^k > \frac{1}{2}$ represents x_i is better than x_j

Here P^k is considered as additive reciprocal if $p_{ij}^k + p_{ji}^k = 1$ and $p_{ii}^k = \frac{1}{2}$. The aim of this proposed work is to define a continuous function f from [1/9, 9] to [0, 1] such that $f(A^k) = P^k$ for all k, where $A^k = [a_{ij}^k]$ in multiplicative preference relation.

The proposed transformation function must satisfy the condition,

$$f(a_{ij}^k) + f\left(\frac{1}{a_{ij}^k}\right) = 1, \forall i, j$$

$$\text{i.e., } p_{ij}^k + p_{ji}^k = 1.$$

4.3 A New Generalized Transformation function that transforms MPR into FPR using fuzzy majority

If $A^k = [a_{ij}^k]$ is a MPR for the set X then the corresponding FPR, $P^k = [p_{ij}^k]$ is obtained by the generalized transformation function as,

$$p_{ij}^k = f\left(a_{ij}^k\right) = \frac{1}{\alpha}\left\{\beta\left(\frac{1}{1+a_{ji}^k}\right) - 1\right\} \tag{1}$$

$$\textit{where } \beta = \alpha + 2, 8 \leq \alpha \leq \infty, \alpha \text{ is an integer.} \tag{2}$$

Various properties that are required for a consistent preference relation [1] are verified here.

*Property 1.*Additive reciprocal [1]
The pairwise comparison matrix(PCM) is said as additive reciprocal if $p_{ij}^k + p_{ji}^k = 1$ and $p_{ii}^{=}0.5$. Reciprocity is the necessary condition for preference relation and it verifies that indifference between any alternatives and itself holds [1].

Proof.

$$p_{ij}^k + p_{ji}^k = \frac{1}{\alpha}\left\{\beta\left(\frac{1}{1+a_{ji}^k} + \frac{1}{1+a_{ij}^k}\right) - 2\right\}$$

$$= \frac{1}{\alpha}\left\{\beta\left(\frac{1+a_{ij}^k}{1+a_{ji}^k}\right) - 2\right\} = \frac{1}{\alpha}(\beta - 2) = 1 \text{ where } \beta - 2 = \alpha.$$

$$\text{Hence } p_{ij}^k + p_{ji}^k = 1$$

$$p_{ii}^k = f(a_{ii}^k) = \frac{1}{\alpha}\left\{\beta\left(\frac{1}{2}\right) - 1\right\}.$$

$$\text{Since } a_{ii}^k = 1$$

$$p_{ii}^k = \frac{1}{\alpha}\left\{\beta\left(\frac{1}{2}\right) - 1\right\} = 0.5. \tag{3}$$

Lemma 1.*Whenever $p_{ij}^k \geq 0.5$ then $a_{ij}^k < 1$ and vice versa*

*Proof.*For,

$$p_{ij}^k \geq 0.5 \Rightarrow \frac{1}{\alpha}\left\{\beta\left(\frac{1}{1+a_{ji}^k}\right) - 1\right\} \geq 0.5.$$

$$\Rightarrow \beta\left(\frac{1}{1+a_{ji}^k}\right) - 1 \geq 0.5\alpha \Rightarrow 1 + a_{ji}^k < \frac{\beta}{0.5\alpha + 1}$$

$$\Rightarrow a_{ji}^k < \frac{\beta - 0.5.\alpha - 1}{0.5\alpha + 1}$$

Since $\beta - 2 = \alpha$

$$\Rightarrow a_{ji}^k < \frac{\alpha + 2 - 0.5\alpha - 1}{0.5\alpha + 1} \Rightarrow a_{ji}^k < 1$$

$$\text{Conversely, if } a_{ji}^k < 1 \text{ then } 1 + a_{ji}^k < 2$$

$$\frac{1}{1+a_{ji}^k} \geq 0.5 \Rightarrow p_{ji}^k \geq 0.5$$

*Property 2.*Weak Transitivity [1] A fuzzy relation is weakly transitive if for $p_{ij}^k \geq 0.5, p_{jl}^k \geq 0.5 \Rightarrow p_{il}^k \geq 0.5 \forall i, j, l.$
If x_i is preferred to x_j and x_j is preferred to x_l, then x_i should be preferred to x_l. Such a condition is used by a consistent person to make the decisions. Therefore it is considered as the minimum requirement to verify the consistency of FPR[1,2].

*Proof.*By Lemma 3.1,

$$p_{ij}^k \geq 0.5 \Rightarrow a_{ji}^k < 1 \quad \text{and } p_{jl}^k \geq 0.5 \Rightarrow a_{lj}^k < 1$$

$$a_{lj}^k a_{ji}^k < 1 \Rightarrow a_{li}^k < 1$$

So $p_{il}^k \geq 0.5 \forall i, j, l.$

*Property 3.*Moderate stochastic transitivity (Restricted max-min transitivity) [1] It is proved $p_{il} \geq \min(p_{ij}, p_{jl}) \forall i, j, k,$ provided $p_{ij} \geq 0.5$ and $p_{jl} \geq 0.5$. The preference p_{ij} is valued when the alternative x_i is preferred to x_j with a value p_{ij} and x_j is preferred to x_l is with a value p_{jl}, then x_i should be preferred to x_l with at least an intensity of preference p_{il} is the same as the minimum of the above values[1]. This condition is required to verify a step addition to weak transitivity [2].

Proof. By Lemma 3.1,

$$p_{ij}^k \geq 0.5 \text{ and } p_{jl}^k \geq 0.5$$

$$\text{Let } \min(p_{ij}, p_{jl}) = p_{ij}$$

$$\text{Now to prove } \; p_{il} \geq \min(p_{ij}, p_{jl}) \text{ i.e., } p_{il} \geq p_{ij}$$

$$\text{Suppose } p_{il} \leq p_{ij}$$

$$\frac{1}{\alpha}\left\{\beta\left(\frac{1}{1+a_{li}^k}\right)-1\right\} \leq \frac{1}{\alpha}\left\{\beta\left(\frac{1}{1+a_{ji}^k}\right)-1\right\}$$

$$\frac{1}{1+a_{li}^k} \leq \frac{1}{1+a_{li}^k}$$

$$\therefore a_{ji} \leq a_{li}, \text{ Since } a_{ji} \leq 1, a_{lj} \leq 1 \Rightarrow a_{li} \leq 1$$

by multiplicative reciprocity

$$a_{lj}a_{ji} < a_{lj}a_{li}$$

$$a_{li} < a_{lj}a_{li} \Rightarrow\Leftarrow$$

$\therefore p_{il} \leq p_{ij}$ is not possible. Hence $p_{il} \geq \min(p_{ij}, p_{jl})$

Property 4. Strong Stochastic transitivity[1] (Restricted max-max transitivity) If $p_{ij}^k \geq 0.5$ and $p_{jl}^k \geq 0.5 \Rightarrow p_{il}^k \geq \max(p_{ij}^k, p_{jl}^k) \forall i,j,k$. The equality condition holds good provided there exists indifference between two or more alternatives. So, in the proposed transformation, restricted max-max transitivity property and restricted max-min transitivity property coincides. This is a stronger condition comparing with the restricted max-min transitivity and milder than max-max transitivity. And it is also a necessary condition by a consistent FPR [1] to be verified.

Proof. By weak transitivity property,

$p_{ij}^k \geq 0.5, p_{jl}^k \geq 0.5 \Rightarrow p_{il}^k \geq 0.5$.

We know that $p_{ij}^k \geq 0.5 \Rightarrow a_{ji}^k < 1$.

To prove $p_{il}^k \geq \max(p_{ij}^k, p_{jl}^k)$. Let $\max(p_{ij}^k, p_{jl}^k) = p_{ij}^k$

now $p_{il}^k \geq p_{ij}^k$

$$\frac{1}{\alpha}\left\{\left(\frac{\beta}{1+a_{li}^k}\right)-1\right\} < \frac{1}{\alpha}\left\{\left(\frac{\beta}{1+a_{ji}^k}\right)-1\right\}$$

$$\frac{1}{1+a_{li}^k} < \frac{1}{1+a_{ji}^k}$$

$a_{ji} \leq a_{li}$, since $a_{ji}^k \leq 1, a_{ij}^k \leq 1 \Rightarrow a_{li}^k \leq 1$

by multiplicative reciprocity

$a_{lj}^k a_{ji}^k < a_{lj}^k a_{li}^k$

$a_{li}^k < a_{lj}^k a_{li}^k \Rightarrow\Leftarrow$

$\therefore p_{il}^k \leq p_{ij}^k$ is not possible. Hence $p_{il}^k \geq \max(p_{ij}^k, p_{jl}^k)$.

5 Making the uniform information and the fuzzy majority based selection process

After the transformation is uniformed, the individual FPRs in the set of choices X, is set and a selection process with two important phases is applied according to [5] as,

5.1 Aggregation phase [8]

Using the fuzzy majority using an OWA operator, $P = [p_{ij}]$ represents the priority in the ordered pair of choices, acquired by using average of discrete FPRs.

5.2 Exploitation phase [8]

These choices are used in finding the collective choices of alternatives. In fuzzy majority selection model, there are two degrees QGDD (Quantifier Guided Dominance Degree) and QGNDD (Quantifier Guided Non-Dominance Degree) to be computed for obtaining collective fuzzy preference relation P^C [5].

5.3 Quantifier Guided Dominance Degree and Quantifier Guided non-Dominance Degree [5]

For the alternative x_i, the QGDD is used to measure the influence of a choice in a fuzzy majority sense as follows:
Quantifier Guided Dominance Degree is
$QGDD_i^k = \sum w_t q_{it}^k$
and the Quantifier Guided non Dominance Degree is
$QGNDD_i^k = \phi_G(1 - p_{li}^{k,s}, l = 1,2,\dots n)$ indicates the degree to which x_i is greater than x_j.

5.4 Propositions of QGDD and QGNDD for MPRs

Using them in the consistency demonstration it has to be defined for any preference relation. The consistency condition is explained in the following proposition.

Proposition 1. *If $\alpha_i^k \leq \alpha_j^k$, for a consistent multiplicative preference relation A^k, the quantifier guided dominance degree obtained from the fuzzy preference relation $P^k = f(A^k)$ which satisfies the relationship, $QGDD_j^k \geq QGDD_i^k$.*

Proof:- If $\alpha_i^k \leq \alpha_j^k \Rightarrow QGDD_j^k \geq QGDD_i^k$. A fuzzy linguistic quantifier Q is chosen to calculate the weighting vector $w = [w_1, w_2, \cdots w_n]$ then

$$QGDD_i^k = \sum w_t q_{it}^k \; .$$

where q_{it}^k is the t^{th} largest value which is the collection $p_{i1}^k, p_{i2}^k, \cdots p_{in}^k$. Since multiplicative preference relation is consistent

$$a_{ij}^k = \frac{\alpha_i^k}{\alpha_j^k}$$
$$QGDD_i^k = \sum w_t q_{it}^k$$
$$= \sum_{t=1}^{n} w_t \frac{1}{\alpha}\left\{\left(\beta\frac{1}{1+a_{ti}^k}\right)-1\right\} = \frac{\beta}{\alpha}\sum_{t=1}^{n} w_t \frac{1}{1+a_{ti}^k} - \frac{1}{\alpha}\sum_{t=1}^{n} w_t$$
$$= \frac{\beta}{\alpha}\sum_{t=1}^{n} w_t \frac{\alpha_i^k}{\alpha_i^k+\alpha_t^k} - \frac{1}{\alpha} = \sum_{t=1}^{n} w_t \frac{1}{\alpha}\left\{\left(\beta\frac{\alpha_i^k}{\alpha_i^k+\alpha_t^k}\right)-1\right\}$$
$$= \frac{\beta}{\alpha}\alpha_i^k\sum_{t=1}^{n} w_t \frac{1}{\alpha_i^k+\alpha_t^k} - \frac{1}{\alpha}$$

Similarly $QGDD_i^k = \sum_{t=1}^{n} w_t \frac{1}{\alpha}\left\{\left(\beta\frac{\alpha_i^k}{\alpha_i^k+\alpha_t^k}\right)-1\right\}$

$$= \frac{\beta}{\alpha}\alpha_i^k\sum_{t=1}^{n} w_t \frac{1}{\alpha_i^k+\alpha_t^k} - \frac{1}{\alpha}$$

For $\alpha_i^k \le \alpha_j^k$

$$\alpha_i^k + \alpha_t^k \le \alpha_j^k + \alpha_t^k$$
$$\frac{1}{\alpha_i^k+\alpha_t^k} \ge \frac{1}{\alpha_j^k+\alpha_t^k} \Rightarrow QGDD_j^k \ge QGDD_i^k.$$

Proposition 2.*If $\alpha_i^k \le \alpha_j^k$, for a consistent multiplicative preference relation A^k, the quantifier guided dominance degree obtained from the fuzzy preference relation $P^k = f(A^k)$ satisfies $QGNDD_j^k \ge QGNDD_i^k$.*

Proof: If Q is the fuzzy linguistic quantifier chosen to obtain the weights of a fuzzy majority based aggregation then,

$$QGNDD_i^k = \phi_G(1-p_{li}^{k,s}, l=1,2,\dots n)\ .$$

Where the strict preference value

$$p_{li}^{k,s} = \max p_{li}^k - p_{il}^k, 0$$

For σ being the permutation over the set $\{p_{li}^{k,s}, \forall l\}$, such that for $r \le s, p_{\sigma(r)i}^{k,s} \le p_{\sigma(s)i}^{k,s}, r,s \in \{1,2,\dots n\}$.
Suppose the vector associated with $[p_{\sigma(1)i}^{k,s},\ dots p_{\sigma(n)i}^{k,s}]$ is $[q_{1i}^{k,s},\ dots q_{ni}^{k,s}]$,

$$QGNDD_i^k = \phi_G(1-p_{li}^{k,s}, l=1,2,\dots n) = \textstyle\sum_{t=1}^{n} w_t(1-q_{ti}^{k,s})$$

where,

$$(1-q_{ti}^{k,s}) = \begin{cases} 1 t=1,2,\dots n_i-1 \\ 2p_{i\sigma(t)}^k t=n_i,\dots n \end{cases} \quad 2 \le n_i \le n$$

$$QGNDD_i^k = \sum_{t=1}^{n} w_t(1-q_{ti}^{k,s})$$
$$= \sum_{t=1}^{n_i-1} w_t(1-q_{ti}^{k,s} + \sum_{t=n_i}^{n} w_t(1-q_{ti}^{k,s})$$
$$= \sum_{t=1}^{n_i-1} w_t + \sum_{t=n_i}^{n} w_t(2p_{i\sigma(t)}^k)$$
$$= \sum_{t=1}^{n_i-1} w_t + \sum_{t=n_i}^{n} w_t - \sum_{t=n_i}^{n} w_t + \sum_{t=n_i}^{n} w_t(2p_{i\sigma(t)}^k)$$
$$= 1 + \sum_{t=n_i}^{n} w_t(2p_{i\sigma(t)}^k - 1)$$
$$= 1 + \sum_{t=n_i}^{n} w_t\left(2\left\{\frac{1}{\alpha}\left(\beta\left(\frac{1}{1+a_{\sigma(t)i}^k}\right)-1\right)\right\}-1\right)$$
$$= 1 + \sum_{t=n_i}^{n} w_t\left(2\left\{\frac{1}{\alpha}\left(\beta\left(\frac{1}{1+a_{\sigma(t)i}^k}\right)-1\right)\right\}\right) - \sum_{t=n_i}^{n} w_t$$
$$= \sum_{t=1}^{n_i-1} w_t + \sum_{t=n_i}^{n} w_t\left\{\frac{2\beta}{\alpha}\left(\frac{1}{1+a_{\sigma(t)i}^k}\right) - \frac{\beta}{\alpha}\right\}$$
$$QGNDD_i^k = \sum_{t=1}^{n_i-1} w_t + \frac{\beta}{\alpha}\sum_{t=n_i}^{n} w_t\left(2\left(\frac{\alpha_i^k}{\alpha_{\sigma(t)}^k+\alpha_i^k}\right)-1\right).$$

similarly,

$$QGNDD_j^k = \textstyle\sum_{t=1}^{n_j-1} w_t + \frac{\beta}{\alpha}\sum_{t=n_j}^{n} w_t\left(2\left(\frac{\alpha_j^k}{\alpha_{\sigma(t)}^k+\alpha_j^k}\right)-1\right)$$

For $n_i \le n_j, \alpha_i \le \alpha_j, \sum_{t=1}^{n_j-1} w_t \ge \sum_{t=1}^{n_i-1} w_t$

$$\frac{1}{\alpha_{\sigma(t)}^k+\alpha_i^k} \ge \frac{1}{\alpha_{\sigma(t)}^k+\alpha_j^k} \Rightarrow \frac{\alpha_i^k}{\alpha_{\sigma(t)}^k+\alpha_i^k} \le \frac{\alpha_j^k}{\alpha_{\sigma(t)}^k+\alpha_j^k}$$

consequently $\sum_{t=n_j}^{n} w_t \frac{\alpha_j^k}{\alpha_{\sigma(t)}^k+\alpha_j^k} \ge, \sum_{t=n_i}^{n} w_t \frac{\alpha_i^k}{\alpha_{\sigma(t)}^k+\alpha_i^k}$,

Hence $QGNDD_j^k \ge QGNDD_i^k$ for $\alpha_i \le \alpha_j$.

6 Aggregation operation to obtain MCDM

This section explains about applying the aggregation operator in MCDM problems under FPRs according to the preferences of the expert[4].For the proposed fuzzy preference relation,

$$p_{ij}^k = f(a_{ij}^k) = \frac{1}{\alpha}\left\{\beta\left(\frac{1}{1+a_{ji}^k}\right)-1\right\}$$

The collective preference relations or the judgment matrix is found by using $u_{ij}^k = \sum w_k q^k$ where q^k is the k^{th} largest value of p_{ij}^k.
Similarly taking the preference relations $p_{ij}^l, p_{ij}^m, p_{ij}^n$, for the linguistic variables, “most of”, “atleast half” etc., with

the pair values (a,b) and the corresponding OWA operators with weight $w=(w_1,w_2,w_3,w_4)$, we get,

$$u_{ij}^C = P^C = [X_i] \text{ where } X_i = \frac{X_i'}{\sum X_i'}, i,j=1,2,\ldots,n$$

using the OWA operator calculating the alternative index [4],
$X_i' = \sum_{j=1} n(w_j)(v_j)$ where v_j is the j^{th} largest value of the $\{u_{i1}^C, u_{i2}^C, \cdots u_{in}^C\}$.
Here X_i, are found using fuzzy majority, and are equivalent to the eigenvector which is the term used by Saaty in AHP method [7].

6.1 Numerical example with ordered weighting average (OWA) operators [6]

OWA operators introduced by Yager [6] are used as aggregation procedure to combine the fuzzy preference relations. For the randomized pairwise comparison matrices,

$$P^1 = \begin{pmatrix} 1 & 1/7 & 1/7 & 1/5 \\ 7 & 1 & 1/2 & 1/3 \\ 7 & 2 & 1 & 1/9 \\ 5 & 3 & 9 & 1 \end{pmatrix} P^2 = \begin{pmatrix} 1 & 1/5 & 1/3 & 1/9 \\ 5 & 1 & 4 & 1/8 \\ 3 & 1/4 & 1 & 1/9 \\ 9 & 8 & 9 & 1 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 1 & 1/3 & 1/7 & 1/9 \\ 3 & 1 & 1/2 & 1/5 \\ 7 & 2 & 1 & 1/7 \\ 9 & 5 & 7 & 1 \end{pmatrix} P^4 = \begin{pmatrix} 1 & 1/3 & 7 & 1/9 \\ 3 & 1 & 1/2 & 1/5 \\ 7 & 2 & 1 & 1/5 \\ 9 & 5 & 5 & 1 \end{pmatrix}$$

the corresponding fuzzy preference matrices (FP^k) using the proposed transformation are,

$$FP^1 = \begin{pmatrix} 0.5 & 0.9688 & 0.9688 & 0.0833 \\ 0.0312 & 0.5 & 0.2917 & 0.1875 \\ 0.0312 & 0.7083 & 0.5 & 0 \\ 0.9167 & 0.8125 & 1 & 0.5 \end{pmatrix}$$

$$FP^2 = \begin{pmatrix} 0.5 & 0.0833 & 0.1875 & 0 \\ 0.9167 & 0.5 & 0.9750 & 0.0139 \\ 0.8125 & 0.125 & 0.5 & 0 \\ 1 & 0.9861 & 1 & 0.5 \end{pmatrix}$$

$$FP^3 = \begin{pmatrix} 0.5 & 0.1875 & 0.9688 & 0 \\ 0.8125 & 0.5 & 0.2917 & 0.0833 \\ 0.0312 & 0.7083 & 0.5 & 0.9688 \\ 1 & 0.9167 & 0.0312 & 0.5 \end{pmatrix}$$

$$FP^4 = \begin{pmatrix} 0.5 & 0.1875 & 0.9688 & 0 \\ 0.8125 & 0.5 & 0.2917 & 0.0833 \\ 0.0312 & 0.7083 & 0.5 & 0.0833 \\ 1 & 0.9167 & 0.9167 & 0.5 \end{pmatrix}$$

If the proposed transformation function is applied in the above relation, to obtain the equivalent collective FPR (FP^c), we get,

$$u_{ij}^C = FP^C = \begin{pmatrix} 0.5 & 0.1875 & 0.5782 & 0.0416 \\ 0.8125 & 0.5 & 0.2917 & 0.1353 \\ 0.0312 & 0.7083 & 0.5 & 0.0417 \\ 0.9584 & 0.9167 & 0.9583 & 0.5 \end{pmatrix}$$

where $\alpha = 8, \beta = 10$

The collective preference relations or the judgment matrix is calculated using $u_{ij}^C = \sum_{k=1}^m w_k q^k$ where q^k is the k^{th} largest value of the p_{ij}^k.Applying the linguistic operator "most of" with pair value (0.25, 0.75) and the corresponding OWA operators with weight w = (0,0.5, 0.5,0) we get, the alternative index, $X_i' = \sum_{j=1}^n (w_j)(v_j)$ where v_j is the j^{th} largest value of the $\{u_{i1}^C, u_{i2}^C, \cdots u_{in}^C\}$. Using the linguistic quantifier "most of" with pair value $X_1' = 0.1246, X_2' = 0.4212, X_3' = 0.5789, X_4' = 0.9188, \sum X_i' = 2.0435$.

To calculate the normalized vector, if $X_i = \frac{X_i'}{\sum X_i'}$ we get the weights as, $X_1 = 0.0610, X_2 = 0.2061, X_3 = 0.2833, X_4 = 0.4496$, Total=1.0000, and this value is equivalent to that of the sum of eigenvectors of relative importance, derived by Saaty [7].

Also Saaty[7] suggested that if the consistency ratio is more than 0.1, the collection of decisions may be too incompatible to be reliable. And the CR is equal to 0 then the judgments are perfectly consistent.

To evaluate the consistency level of the choices, we find the consistency index(CI) and CR. Here, table 2 represents a 3x3 additive reciprocal matrix and its corresponding CI [9]. For such types of matrices, λ_{max} cannot be used to find the inconsistency and hence we find CI using the method of [9]. Pelaez and Lamata [9] presented an alternative measure which is based on minimal element of consistency to test the consistency of entries in a PCM.

Table 2: 3×3 additive matrix

C	A_i	A_j	A_k
A_i	0.5	0.6	0.8
A_j	0.4	0.5	0.8
A_k	0.2	0.2	0.5

Consistency index is 0.05.

The matrix given below is consistent according to the CI criterion and consistency ratio.

$$FP^C = \begin{pmatrix} 0.5 & 0.1875 & 0.5782 & 0.0416 \\ 0.8125 & 0.5 & 0.2917 & 0.1353 \\ 0.0312 & 0.7083 & 0.5 & 0.0417 \\ 0.9584 & 0.9167 & 0.9583 & 0.5 \end{pmatrix}$$

By Lamata [9], Number of transitivities, NT = $\frac{n!}{(n-3)!3!}$ if $n \geq 3$, here NT=4,

Table 3: A comparison matrix

Method	n	CI	CR	Comment
Existing	4	0.0339	0.04	Strongly Consistent
Proposed	4	0.0469	0.05	Strongly consistent

By definition [9], the consistency index CI^* of an $M_{n\times n}$ matrix is given by the average of the consistency index of the matrix transitivities.

$$\frac{1}{4}\sum_{i=1}^{4} CI^*(\Gamma_i) = \frac{0.18754}{4} \Rightarrow CI^* = 0.0469.$$

Consistency ratio, CR $= \dfrac{CI^*}{0.9} = 0.05(<0.10)$

which is Strongly consistent[7].

According to L. Saaty [7], if the consistency ratio is exactly equal to zero then the pair wise comparison matrix of alternatives is perfectly consistent. By the proposed method, CR is found very close to 0. Therefore the new generalized transformation function is strongly consistent. The result of the proposed transformation using FPR is compared with the existing transformation and is given in Table 3. It is seen that the existing transformation is lacking behind in the value of consistency ratio.

7 Conclusion

The paper proposes new generalized transformation function that satisfies various properties of consistency of preference relation. Furthermore, to design it, an aggregation OWA operator guided by fuzzy majority is introduced to define QGDD and QGNDD degrees in a fuzzy majority for FPRs. Using fuzzy majority scheme in AHP technique, the consistency ratio is measured. It is significant that, the consistency Ratio valued by the proposed generalized transformation function is strongly consistent comparing with the existing transformation function. Therefore, by applying this new transformation function in a pairwise comparison matrix, it is possible to find a preference of alternatives with strongly consistent solutions in decision making processes.

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