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Game Chromatic Number and Game Chromatic Index of the Mycielski Graphs of Some Families of Graphs

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Abstract: In this paper, we determine the game chromatic number and game chromatic index of the Mycielski graphs of some families of graphs.

Keywords: Graphs, Game chromatic number, Game chromatic index, Mycielski graph

1 Introduction

Let G be a finite graph and X be a set of colors. The vertex coloring game on G is defined to be a game played by two people Alice and Bob with Alice start playing the game first. They take turns to color a vertex of G from X so that no two adjacent vertices are colored with the same color. Alice wins the game if it is possible to color all the vertices of G with colors in X. Bob wins the game if at any point of the game, there is a vertex which cannot be colored with colors from X. The *game chromatic number* of G, denoted by $\chi_g(G)$, is the minimum number of colors needed in the color set X for which Alice has a strategy to win. If $\chi(G)$ is the chromatic number of G and $\triangle(G)$ is the maximum degree of G then $\chi(G) \leq \chi_g(G) \leq \triangle(G)+1$. Bodlaender [1] introduced the above game chromatic number,$\chi_g(G)$, of a graph G. Later many people have tried to determine the game chromatic number or its bounds for various classes of graphs [2,3,5,9,10,14,15].

The game chromatic index is based on coloring the edges of a graph instead of coloring the vertices and this was introduced by Lam et al [4], Cai and Zhu [6]. Alice and Bob color an uncolored edge of a graph G alternately with a color from X with Alice start playing first. They color the edges in such a way that adjacent edges do not get the same color. Alice wins the game if it is possible to color all the edges of G with colors in X. Otherwise Bob wins. The *game chromatic index* of G, denoted by ${\chi_g}'(G)$, is the minimum number of colors needed in the color set X for which Alice has a strategy to win. There are a few results on game chromatic index of graphs which are given in [6,7,8,12].

Let G be a graph with vertices $v_1, v_2, ..., v_n$. The Mycielski graph of G, denoted by $\mu(G)$, has G as an induced subgraph together with $n+1$ additional vertices $u_1, u_2, ..., u_n, w$ such that each vertex u_i is adjacent to w and to the neighbors of v_i $(1 \leq i \leq n)$. Hence if there are n vertices and m edges in G then $\mu(G)$ will have $2n+1$ vertices and $3m+n$ edges.

In this paper, we find the game chromatic number and game chromatic index of the Mycielski graphs of some families of graphs. While playing the vertex coloring game, a vertex v which is not colored is said to be color i-critical[11] or critical, if the following conditions are satisfied.

(i).color i is the only color available for v
(ii).v has an uncolored neighbor v' such that color i is one of the available colors for v'.

Observation 1: At any point of the game, if a vertex v is color i-critical and if

(i).it is Bob's turn to play the game then he will win.
(ii).it is Alice's turn to play the game, then she has to save the vertex v either by coloring it with color i or by making color i unavailable for all the neighbors of v.

Observation 2: If at any point of the game, two vertices are made critical and if Alice is not able to save both the vertices in her next move, then Bob wins.

Notations:

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– If G is a graph, $V(G)$ denotes its vertex set and $E(G)$ its edge set.
– P_m denotes the path of length $m-1$.
– C_m denotes the cycle of length m.
– $K_{a,b}$ denotes the complete bipartite graph with a vertices in one partition and b vertices in other partition.
– F_m denotes the friendship graph with a set of m triangles having a common centre vertex.
– In $\mu(G)$, for $1 \leq i, j \leq n$, let us call the edge joining v_i and u_j as *cross edge* denoted by c_{ij}, the edge joining w and u_i as *star edge* denoted by s_i and the edge joining v_i and v_j as *graph edge*.

2 Game chromatic number of the Mycielski graphs of some families of graphs

Theorem 2.1. For any integer $n \geq 5$, $\chi_g(\mu(P_n)) = 4$.

Proof.
First we show that for $n \geq 5$, $\chi_g(\mu(P_n)) \geq 4$. To prove this, we give a strategy for Bob to win using three colors. This strategy is as given below.

The vertex in which Alice colors in her first move	The vertex with which Bob responds
w	v_i, $3 \leq i \leq n-2$, with a new color. Now u_{i-1} and u_{i+1} become critical. Hence Bob wins.
v_i, $3 \leq i \leq n-2$	Either v_{i-2} or v_{i+2} with a new color. This makes (v_{i-1} and u_{i-1}) or (v_{i+1} and u_{i+1}) critical respectively. Hence Bob wins.
u_i, $3 \leq i \leq n-2$	Either u_{i-2} or u_{i+2} with a new color. This makes (v_{i-1} and w) or (v_{i+1} and w) critical respectively. Hence Bob wins.
v_1, v_2, v_{n-1}, v_n, u_1, u_2, u_{n-1} and u_n	$v_3, v_4, v_{n-3}, v_{n-2}, u_3, u_4$, u_{n-3} and u_{n-2} respectively with a new color and each one of these moves makes two vertices critical. Hence Bob wins.

Thus $\chi_g(\mu(P_n)) \geq 4$. Now to show that $\chi_g(\mu(P_n)) = 4$, we give a strategy for Alice to win the game using four colors.

Initial Step:

Alice colors the vertex w.

Recursive Step:

When Bob colors any of u_i (v_i), $1 \leq i \leq n$, then Alice colors the corresponding v_i (u_i) with the same color, if it is available. Otherwise she uses some other color.

Observe that, using this strategy, at any stage of the game, any vertex which is not yet colored is adjacent to at most three distinctly colored vertices. Hence at least one color is always available for any vertex. Thus $\chi_g(\mu(P_n)) = 4$, for $n \geq 5$.□

Note:

It can be easily checked that for $2 \leq n \leq 4$, $\chi_g(\mu(P_n)) = 3$.

Theorem 2.2. For any integer $n \geq 5$, $\chi_g(\mu(C_n)) = 4$.

Proof. We can see that $V(\mu(P_n)) = V(\mu(C_n))$ and $\mu(C_n)$ contains all the edges of $\mu(P_n)$ and three more edges namely v_1v_n, v_1u_n and v_nu_1. It can be easily verified that for $n \geq 5$, $\chi_g(\mu(C_n)) = 4$ by using a similar argument as given in the previous theorem.□

Note:

It can be easily checked that, $\chi_g(\mu(C_3)) = 4$ and $\chi_g(\mu(C_4)) = 3$.

Theorem 2.3. For any integer $m \geq 2$, $\chi_g(\mu(K_{m,m})) = 3$.

Proof. It can be easily checked that $\chi_g(\mu(K_{m,m})) \geq 3$. Now we show that Alice has a strategy to win the game using three colors.

Let us partition the vertices of $\mu(K_{m,m})$ into five sets. Denote them by A_1,A_2,A_3,A_4 and A_5 such that $A_1 = \{v_1, v_2, v_3, ..., v_m\}$, $A_2 = \{v_{m+1}, v_{m+2}, v_{m+3}, ..., v_{2m}\}$, $A_3 = \{u_1, u_2, u_3, ..., u_m\}$, $A_4 = \{u_{m+1}, u_{m+2}, u_{m+3}, ..., u_{2m}\}$ and $A_5 = \{w\}$. Observe that each vertex in A_1 is adjacent to each vertex in A_2 and each vertex in A_4; each vertex in A_2 is adjacent to each vertex in A_3; the vertex in A_5 is adjacent to each vertex in A_4 and each vertex in A_3. The strategy for Alice to win the game with three colors is given below.

Alice starts the game by coloring the vertex w. Bob can color any of the remaining vertices in the second move and the course of the game is explained as discussed below.

Case 1: When Bob colors a vertex of A_1

Alice replies with a vertex of A_4. In the following moves, whenever Bob
(i) colors a vertex of A_1 with the color which is already used in A_1 or colors a vertex of A_4 with any color then Alice remains playing in these two sets A_1, A_4 with colors already used in A_1 and A_4.
This is always possible for Alice since $|A_1 \cup A_4|$ is even.
(ii) colors a vertex of A_1 with a color which is not already used in A_1, then Alice colors a vertex of A_2.
(iii) colors a vertex of A_2 or A_3 then Alice colors a vertex of A_3 or A_2 respectively.

Case 2: When Bob colors a vertex of A_2

Alice replies with a vertex of A_3. In the following moves, whenever Bob
(i) colors a vertex of A_2 with the color which is already used in A_2 or colors a vertex of A_3 with any color then Alice remains playing in these two sets A_2, A_3 with colors already used in A_2 and A_3.
This is always possible for Alice since $|A_2 \cup A_3|$ is even.
(ii) colors a vertex of A_2 with a color which is not already used in A_2, then Alice colors a vertex of A_1.
(iii) colors a vertex of A_1 or A_4 then Alice colors a vertex of A_4 or A_1 respectively.

Case 3: When Bob colors a vertex of A_4 or A_3

Alice replies with a vertex of A_1 or A_2 respectively. In the following moves, wherever Bob plays, Alice will play in the same way as discussed above in Case 1 and Case 2.

Observe that, using this strategy, at any stage, any uncolored vertex is adjacent with at most two distinctly colored vertices. As the set X has three colors, for any uncolored vertex, at least one color is always available.□

Theorem 2.4. For any integer $m \geq 3$, $\chi_g(\mu(F_m)) = 4$.

Proof. F_m has $2m+1$ vertices. Let $n = 2m+1$ and v_1 be the centre vertex of F_m, where $m \geq 3$. First we show that $\chi_g(\mu(F_m)) \geq 4$. To prove this, we give a strategy for Bob to win the game using three colors. This strategy is as given below.

The vertex in which Alice colors in her first move	The vertex with which Bob responds
$v_1(u_1)$	$u_1(v_1)$ respectively. This makes v_i, $2 \leq i \leq n$, critical. Hence Bob wins.
w	u_1. Now Alice can color any of the remaining vertices with any available color. But this makes at least two vertices critical. Hence Bob wins.
$v_i(u_i)$, $2 \leq i \leq n$	$v_j(u_j)$, $i \neq j$, $2 \leq j \leq n$. This makes (v_1 and u_1) or (v_1 and w) critical respectively. Hence Bob wins.

Thus $\chi_g(\mu(F_m)) \geq 4$. Now to show that $\chi_g(\mu(F_m)) = 4$ we give a strategy for Alice to win the game using four colors. The strategy is as follows.

Alice starts by coloring v_1. Now depending on Bob's move the game is decided. We have the following two cases.

Case 1: Bob colors u_1 or w

Alice replies with w or u_1 respectively with the color of v_1 if it is available. If not, she will choose any of the available colors. In the following moves, whenever Bob colors $u_i(v_i)$ then Alice replies with $v_i(u_i)$, $2 \leq i \leq n$, with the same color.

Case 2: Bob colors u_i or v_i, $2 \leq i \leq n$

Alice replies with u_1 with the color of v_1. Now Bob colors any one of the remaining vertices. Now Alice responds with w, if it is uncolored. Otherwise she colors any uncolored vertex.

Alice wins in both the above situations. Thus $\chi_g(\mu(F_m)) = 4$.□

3 Game chromatic index of the Mycielski graphs of some families of graphs

For any simple graph G, $\triangle(G) \leq \chi_g'(G) \leq 2\triangle(G) - 1$ where $\triangle(G)$ is the maximum degree of G. Game chromatic index is known for paths, cycles, wheels and some small graphs[6,7,8,12].

Definition Two edges e_1 and e_2 of a graph G are said to be adjacent if they both are incident at a common vertex v. For an edge e of a graph G, the neighbors of e are the edges of G which are adjacent to it.

Theorem 3.1. For any integer $n \geq 7$, $\chi_g'(\mu(P_n)) = n$.

Proof. As the maximum degree of $\mu(P_n)$ is n, $\chi_g'(\mu(P_n)) \geq n$. Now we show that $\chi_g'(\mu(P_n)) = n$. Let $X = \{1,2,3,\ldots,n\}$ be the set of n colors. Observe that the maximum number of neighbours of a cross edge is five and that of graph edge is six. As we have n colors and $n \geq 7$, at least one color is always available for these edges. Therefore, it is enough to show that Alice can color the star edges using n colors, however Bob plays. The strategy of Alice is as follows.

Initial Step:

Alice colors any one of the star edges $s_2, s_3, s_4, \ldots, s_{n-1}$.

Recursive Step:

Bob can color any edge in each of his turns that follows. Irrespective of Bob's moves, Alice keeps coloring the star edges until all the star edges are colored or Situation I or Situation II discussed below arises. Also she prefers to use the color which Bob has used in the previous move, if it is available. Otherwise she will use a new color.

Each time when Alice colors a star edge, her first preference would be a star edge which is adjacent to two uncolored cross edges and her second preference would be a star edge which is adjacent to one uncolored cross edge and her last preference would be a star edge which is adjacent to two colored cross edges.

Situation I: *All but two star edges say s_i and s_j are colored and exactly one cross edge adjacent to each of s_i and s_j are colored, say c_{pi} and c_{qj}, $1 \leq i,j,p,q \leq n$.*

Now, Alice will wait for Bob to color among these four edges. Till then she colors edges other than these four edges. Note that this is always possible for Alice because the total number of edges in the graph is odd.

In this situation, depending on Bob's move the game is decided. We have the following three cases. Let the two uncolored cross edges at s_i and s_j be c_{ki} and c_{lj}, $1 \leq k,l \leq n$.

If Bob colors c_{ki} or c_{lj} with a new color then Alice colors s_j or s_i respectively with the same color.

If Bob colors c_{ki} or c_{lj} with an already used color then Alice colors c_{lj} or c_{ki} respectively with any available color which is already used in the game.

If Bob colors s_i or s_j then Alice colors s_j or s_i respectively with any available color.

Situation II: *All but two star edges say s_i and s_j are colored and both the cross edges adjacent to exactly one of s_i and s_j (say s_i) are uncolored (let these cross edges be c_{ki} and c_{li}) and exactly one cross edge adjacent to s_j is uncolored, say c_{pj}, $1 \leq i,j,k,l,p \leq n$.*

In this situation,

(i).if Alice has to play, then she chooses c_{ki} or c_{li} and colors it with an already used color. Now the situation is same as Situation I.

(ii).if Bob has to play, then he can choose either a star edge or a cross edge to color. If Bob colors a star edge s_i or s_j then Alice colors s_j or s_i respectively. Suppose Bob colors c_{ki} or c_{li} then the play comes down to Situation I. So Alice will play accordingly. If Bob colors a cross edge c_{pj} then Alice colors s_i with the same color.

Observe that using this strategy, at any stage, any uncolored edge is adjacent to at most $n-1$ distinctly colored edges and hence at least one color is always available.□

Theorem 3.2. For any integer $n \geq 6$, $\chi_g^{'}(\mu(C_n)) = n+1$.

Proof. As the maximum degree of $\mu(C_n)$ is n, $\chi_g^{'}(\mu(C_n)) \geq n$. Now we show that Bob has a strategy to win the game using n colors. Let $X = \{1,2,3,...,n-1,n\}$ be the set of n colors.

Initial Step:

In the first move, Alice colors any edge of $\mu(C_n)$.

Recursive Step:

Bob's reply would be in coloring a star edge. Now irrespective of Alice's moves, Bob keeps coloring the star edges until all but two star edges, say s_i and s_j are colored. While choosing the star edge, Bob's first preference would be a star edge which is adjacent to the cross edge which Alice colored in the previous move if it is uncolored. Otherwise, Bob colors some other star edge.

Now consider the stage, where only two star edges are uncolored. As there are totally n star edges in the graph and we have n colors in X, let the remaining two unused colors be $n-1$ and n. In this stage, Bob will color some edge other than $s_i, s_j, c_{i(i-1)}, c_{i(i+1)}, c_{j(j-1)}$ and $c_{j(j+1)}$ and wait for Alice to color first among these six edges. Note that this is always possible for Bob since the total number of edges in the graph is even. Now depending on Alice's move the game is decided. We have the following two cases.

Case 1: Alice colors one of the uncolored star edges say s_i

In this case, $n-1$ star edges are colored with $n-1$ colors. Now Bob replies with a cross edge adjacent to s_j with the color n and hence no color is available for s_j.

Case 2: Alice colors a cross edge adjacent to one of the uncolored star edges, say s_i

Bob replies with a cross edge which is adjacent to s_j with the color $n-1$. In the next move,

–if Alice colors a star edge then Bob replies with a cross edge as in Case 1.

–if Alice colors a cross edge adjacent to s_i then Bob colors a cross edge adjacent to s_j with color n and hence no color is available for s_j.

–if Alice colors a cross edge adjacent to s_j then Bob colors a cross edge adjacent to s_i with color $n-1$ and there is no color available for one of s_i or s_j.

Hence Bob wins in all the above situations with n colors. Also observe that, in the stage (where only two star edges are yet to color), if Alice uses a color $n-1$ or n for a graph edge which is adjacent to a cross edge, say c_{pq} such that c_{pq} is at least one of $\{c_{i(i-1)}, c_{i(i+1)}, c_{j(j-1)}, c_{j(j+1)}\}$ then Bob colors one of the cross edges $c_{i(i-1)}, c_{i(i+1)}, c_{j(j-1)}$ and $c_{j(j+1)}$ which is adjacent to c_{pq} with the color which Alice has used in the previous move. This makes the game to be in Bob's hand and hence Bob wins.

To show that, $\chi_g^{'}(\mu(C_n)) = n+1$, we give a strategy for Alice to win the game using $n+1$ colors.

In the first move, Alice will color a star edge. In the following moves, if Bob colors a cross edge then Alice will reply with the star edge which is adjacent to it, if it is uncolored. Otherwise she will color some other star edge. Note that for every edge in the graph other than the star edges, maximum number of neighbours is six. Hence at least one color is always available for these edges as $n+1 \geq 7$. Thus Alice can win with $n+1$ colors.□

Theorem 3.3. For any integer $n \geq 3$, $\chi_g^{'}(\mu(K_{1,n})) = 2n$.

Proof. Let $m = n+1$. Let v_1 be the centre vertex of $K_{1,n}$ and the remaining vertices be $v_2, v_3, v_4, ..., v_m$. As the maximum degree of $\mu(K_{1,n})$ is $2n$, $\chi_g^{'}(\mu(K_{1,n})) \geq 2n$. To show that $\chi_g^{'}(\mu(K_{1,n})) = 2n$, we give a strategy for Alice to win the game using $2n$ colors. The strategy is as follows.

Initial Step:

Alice colors the edge wu_1.

Recursive Step:

Now depending on Bob's move the game is decided. We have the following two cases.

Case 1: Bob colors u_1v_i or wu_i $(2 \leq i \leq m)$

Now Alice replies with v_1v_j or v_1u_j respectively, $2 \leq j \leq m$ and $i \neq j$, preferably with the same color.

Case 2: Bob colors v_1v_i or v_1u_i $(2 \leq i \leq m)$

Now Alice replies with any edge incident to v_1 if it is uncolored. Otherwise she colors any other uncolored edge.

Observe that, using this strategy, Alice can win with $2n$ colors. Thus $\chi_g^{'}(\mu(K_{1,n})) = 2n$.□

Conclusion

We have determined the game chromatic number of the Mycielski graphs of paths, cycles, complete bipartite graphs and friendship graphs and the game chromatic index of the Mycielski graphs of paths, cycles and star graphs.

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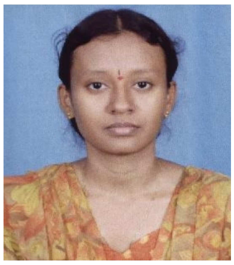

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