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# A Lower Bound for Edge-congestion of an Embedding

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**Abstract:** In network theory, the problem of simulating one architecture into another architecture is converted into a graph embedding problem. In this paper, we have extended our work in [1] and give algorithms to compute optimal edge-congestion of embedding hypercubes, folded hypercubes, crossed cubes and circulant networks into hypertrees thereby proving that the edge-congestion bound obtained in [1] is sharp.

**Keywords:** Embedding, edge-congestion, hypercube, folded hypercube, crossed cube, circulant network, hypertree.

## 1 Introduction

A electronic network, or just a network, may be a assortment of computers and alternative hardware parts interconnected by communication channels that enable sharing of resources and knowledge. The interconnection network is a key element of a tightly coupled multiprocessor system. In the implementation of any algorithm, it is necessary that the code should be compilable and executable on any machine. However, it is far complicated in the case of parallel algorithms and machines. This is due to the fact that the properties of parallel machines highly depend on their interconnection structure [2]. Small degree, small diameter, efficient routing and embedding are desirable properties of an interconnection network. [3].

In the study of interconnection networks, the simulation of one architecture by another is important. This problem is modeled as a graph embedding problem. Graph embedding problems can be used to model parallel processing systems. If a process can be subdivided into subprocesses that can be executed parallely with communications between certain subprocesses, then the model depicting the process is a graph with vertices represents subprocesses and edges representing communications between them. Graph embedding problems find applications in optimal storage representations for data structures, implementation of parallel algorithms and in architecture simulation of various interconnection networks [4]. A tasks interaction graph whose vertices represents tasks and edges represents direct communication between tasks is a model for a parallel algorithm.

The quality of an embedding can be measured by certain cost criteria. One of these criteria which is considered very often is the *edge-congestion*. The edge-congestion of an embedding is the maximum number of edges of the guest graph that are embedded on any single edge of the host graph. An embedding with a large edge-congestion faces many problems, such as long communication delay, circuit switching and the existence of different types of uncontrolled noise. In data networking, network congestion occurs when a link or node is carrying so much data that its quality of service deteriorates. Typical effects include packet loss or the blocking of new connections. Therefore, a minimum edge-congestion is a most desirable feature in network embedding [5]. Edge-congestion of an embedding has been well studied for a number of networks [6,7,8,9,10].

Hypercubes are structured interconnection networks with a giant capacity for parallel computation and a high degree of fault tolerance. The Cosmic Cube from Caltech, the iPSC/2 from Intel are machines based on hyper cubes that have been implemented commercially [12]. Mapping

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a communication graph into a hypercube multiprocessor is known as the hypercube embedding problem. Hypercubes are also known to simulate other interconnection networks [13,14].

The circulant network is a derivation of the double loop network [15]. Its optimal fault-tolerance and routing potentials have induced the design of telecommunication networks [16]. It is also utilized in VLSI design and distributed computation [17,18,19]. Binary codes have also been designed using circulant graphs [20]. Properties of circulant graphs have been studied vastly and have been surveyed by Bermond et al. [17]. Every circulant graph is a vertex transitive graph and a Cayley graph [2].

An interconnection network called hypertree is devised combining the best features of a binary tree and the hypercube for multicomputer systems which allows a considerable amount of memory to fit on a single VLSI chip [21]. In this paper, we embed hypercubes, folded hypercubes, crossed cubes and circulant networks into hypertrees with minimum edge-congestion.

## 2 Basic Concepts

In this section we begin with the basic definitions and preliminaries required for our subsequent work.

**Definition 2.1.** [22] Let $G$ and $H$ be finite graphs. An *embedding* of $G$ into $H$ is an injective mapping $f$: $V(G) \to V(H)$ which induces an injective mapping $P_f$: $E(G) \to X$ where $X = \{P_f(u,v) : P_f(u,v)$ is a path in $H$ between $f(u)$ and $f(v)$ for $(u,v) \in E(G)\}$.

The ratio of the number of vertices of $H$ to the number of vertices of $G$ is addressed as the "expansion" of an embedding $f$. In this paper, we consider embeddings with expansion one.

**Definition 2.2.** [22] Let $f : G \to H$ be an embedding. For $e \in E(H)$, let $C_f(e)$ denote the number of edges $(u,v)$ of $G$ such that $e$ is in the path $P_f(u,v)$ between $f(u)$ and $f(v)$ in $H$. In other words,

$$C_f(e) = \left|\{(u,v) \in E(G) : e \in P_f(u,v)\}\right|.$$

Then the edge-congestion of $f : G \to H$ is $C_f(G,H) = \max C_f(e)$, where the maximum is taken over all edges $e$ of $H$.

The edge-congestion of $G$ into $H$ is defined as $C(G,H) = \min C_f(G,H)$, where the minimum is taken over all embeddings $f : G \to H$. Further, if $S$ is any subset of $E(H)$, then we define $C_f(S) = \sum_{e \in S} C_f(e)$. Illustration of edge-congestion of an embedding from cylinder into a path is given in Figure 1.

**Definition 2.3.** [2,24] For $r \geq 1$, let $Q^r$ denote the $r$-dimensional hypercube. The vertex set of $Q^r$ is formed by the collection of all $r$-dimensional binary strings. Two vertices $x, y \in V(Q^r)$ are adjacent if and only if the corresponding binary strings differ exactly in one bit. The vertices of $Q^r$ can also be identified with integers

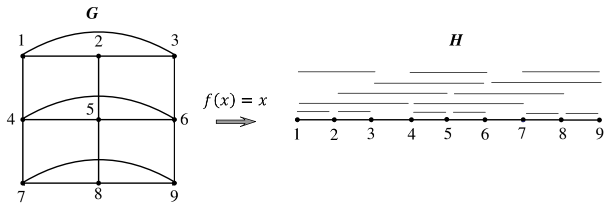


**Fig. 1:** Wiring diagram of a cylinder $G$ into a path $H$ with $C_f(G,H) = 5$

$0,1,\ldots,2^r-1$. For convenience we use the symbol $x+1$ instead of $x$, and therefore, the set of labels of the vertices is $1,2,\ldots,2^r$.

**Definition 2.4.** [25] An incomplete hypercube on $i$ vertices of $Q^r$ is the subcube induced by $\{1,2,\ldots,i\}$ and is denoted by $L_i$, $1 \leq i \leq 2^r$.

**Definition 2.5.** [2] For two vertices $x = x_1x_2\cdots x_r$ and $y = y_1y_2\cdots y_r$ of $Q^r$, $(x,y)$ is a complementary edge if and only if the bits of $x$ and $y$ are complements of each other, that is, $y_i = \overline{x}_i$ for each $i = 1,2,\ldots,r$. The $r$-dimensional folded hypercube, denoted by $FQ^r$ is an undirected graph obtained from $Q^r$ by adding all complementary edges.

**Definition 2.6.** [26] Two 2-digit binary strings $\mathbf{x} = x_1x_0$ and $\mathbf{y} = y_1y_0$ are pair-related, denoted by $\mathbf{x} \sim \mathbf{y}$, if and only if $(\mathbf{x},\mathbf{y}) \in \{(00,00),(10,10),(01,11),(11,01)\}$.

**Definition 2.7.** [26] The $r$-dimensional crossed cube $CQ^r$ is recursively constructed as follows: $CQ^1$ is a complete graph with two vertices labeled by 0 and 1. $CQ^r$ consists of two identical $(r-1)$-dimension crossed cubes, $CQ_0^{r-1}$ and $CQ_1^{r-1}$. The vertex $\mathbf{u} = 0u_{r-2}u_{r-3}\cdots u_0 \in V(CQ_0^{r-1})$ and vertex $\mathbf{v} = 1v_{r-2}v_{r-3}\cdots v_0 \in V(CQ_1^{r-1})$ are adjacent in $CQ^r$ if and only if

1. $u_{r-2} = v_{r-2}$, if $r$ is even; and
2. For $0 \leq i < \lfloor \frac{r-1}{2} \rfloor$, $u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$

**Definition 2.8.** [17,24] The undirected circulant graph $G(n;\pm S)$, $S \subseteq \{1,2,\ldots,j\}$, $1 \leq j \leq \lfloor n/2 \rfloor$ is a graph with vertex set $V = \{0,1,\ldots,n-1\}$ and the edge set $E = \{(i,k) : |k-i| \equiv s (mod\ n), s \in S\}$.

It is clear that $G(n;\pm\{1\})$ is the undirected cycle $C_n$ and $G(n;\pm\{1,2,\ldots,\lfloor n/2 \rfloor\})$ is the complete graph $K_n$. The cycle $G(n;\pm\{1\}) \simeq C_n$ contained in $G(n;\pm\{1,2,\ldots,j\})$, $1 \leq j \leq \lfloor n/2 \rfloor$ is sometimes referred to as the outer cycle $C$ of $G$.

**Definition 2.9.** [21] The basic drawing of a hypertree is a complete binary tree $T_r$. The nodes of the tree are numbered as follows: The root node has label 1. The root is said to be at level 1. Labels of left and right children are formed by appending a 0 and 1, respectively, to the label of the parent node. See Figure 2$(a)$. The decimal labels of the hypertree are depicted in Figure 2$(b)$. Here the children of the node $x$ are labeled as $2x$ and $2x+1$.

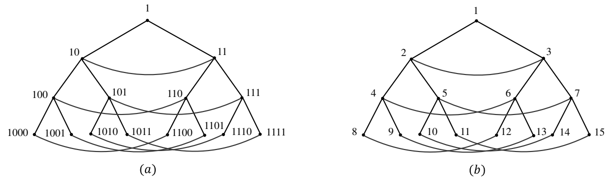


**Fig. 2:** $(a)$ $HT(4)$ with binary labels $(b)$ $HT(4)$ with decimal labels

Additional links in a hypertree are horizontal and two nodes in the same level $i$ of the tree are joined if their label difference is $2^{i-2}$. We denote an $r$ level hypertree as $HT(r)$. The rooted hypertree $RHT(r)$ is obtained from the hypertree $HT(r)$ by attaching to its root a pendant edge. The new vertex is called the root of $RHT(r)$, $r \geq 2$.

**Remark 2.10.** $HT(r)$ has $2^r - 1$ vertices and $3\ (2^{r-1} - 1)$ edges. The diameter and connectivity of $HT(r)$ are $2r-3$ and 2 respectively and it is a planar graph [27].

## 3 A Lower Bound for Edge-congestion

For an embedding $f$ of $G$ into $H$, the dilation of an embedding is outlined because the most distance between pairs of vertices of host graph that are pictures of adjacent vertices of the guest graph. The minimum taken all the embeddings $f$ is termed the dilation of embedding $G$ into $H$. Additional the add of the dilations in $H$ of edges in $G$ is termed the wirelength of $f$ . The minimum taken all the embeddings $f$ is termed the wirelength of embedding $G$ into $H$. In 1979, Garey *et al.* verified that embedding issues are NP-complete [28].

The dilation, edge-congestion and the wirelength parameters are completely different within the sense that an embedding provides the minimum dilation needn't give the minimum edge-congestion or the minimum wirelength and vice-versa. Even though there numerous outputs and comparison on the edge-congestion problem, there's no economical methodology to work out precise edge-congestion of graph embeddings [6,7,23]. In recent years, Manuel and his team obtained a lower range for dilation of an embedding exploitation minimum wirelength associated developed the result as IPS Lemma [29]. Further, the similar authors improved the range of the dilation of an embedding without knowing wirelength in the year 2014 and call it as dilation lemma [30]. Now, we propose and prove the Edge-congestion Lemma to get a tight range for edge-congestion of an embedding. We also prove that the range is sharp by embedding the hypercubes, crossed cubes, folded hypercubes and circulant networks into hypertrees.

The following problem has been considered in the literature [23,31], and is $NP$-complete [28].

**Discrete Isoperimetric Problem :** Let $G = (V,E)$ be a graph and $A \subseteq V$. Denote

$$\theta_G(A) = \{(u,v) \in E \mid u \in A, v \notin A\}$$

and

$$\theta_G(m) = \min_{A \subseteq V, |A|=m} |\theta_G(A)|.$$

For a given $m$, where $m = 1,2,\ldots,n$, we consider the problem of finding a subset $A$ of vertices of $G$ such that $|A| = m$ and $|\theta_G(A)| = \theta_G(m)$. Such subsets are called optimal [31,32].

**Theorem 3.1.** [32,33,34] Let $Q^r$ be an $r$-dimensional hypercube. For $1 \leq i \leq 2^r$, $L_i$ is an optimal set on $i$ vertices.

**Lemma 3.2.** (Modified Congestion Lemma) [35] Let $f$ be an embedding of an arbitrary graph $G$ into $H$. Let $S$ be an edge cut of $H$ such that the removal of edges of $S$ separates $H$ into 2 components $H_1$ and $H_2$ and let $G_1$ and $G_2$ be subgraphs of $G$ induced by $f^{-1}(V(H_1))$ *and* $f^{-1}(V(H_2))$ respectively. Also, $S$ satisfies the following conditions:

(i). For every edge $(a,b) \in E(G_i)$, $i = 1,2, P_f(a,b)$ has no edges in $S$.

(ii). For every edge $(a,b) \in E(G)$ with $a \in V(G_1)$ and $b \in V(G_2)$, $P_f(a,b)$ has exactly one edge in $S$.

(iii). $V(G_1)$ and $V(G_2)$ are optimal sets.

Then $C_f(S)$ is minimum over all $f$ and $C_f(S) = \sum\limits_{v \in V(G_1)} \deg_G(v) - 2|E(G_1)| = \sum\limits_{v \in V(G_2)} \deg_G(v) - 2|E(G_2)|$.

**Remark 3.3.** When the guest graph $G$ is regular, it is enough to check whether $V(G_1)$ is an optimal set [13].

The following lemma is the main result of this paper and is formulated as Edge-congestion Lemma which gives a tight bound for the embedding parameter 'edge-congestion'.

**Lemma 3.4.** (Edge-congestion Lemma) Let $G$ and $H$ be graphs of the same order and let $f : G \rightarrow H$ be an embedding. Let $S$ be an edge cut of $H$ satisfying the conditions of the modified congestion lemma. Then

$$C(G,H) \geq \frac{C_f(S)}{|S|}.$$

*Proof.* By the Modified Congestion Lemma, the inverse images of the sets $H_1$ and $H_2$ with respect to the embedding $f$ are maximum subgraphs of $G$. Therefore $C_f(S)$ is minimum.

There is at least one edge in $S$ with edge-congestion at least $\frac{C_f(S)}{|S|}$. Further, for any embedding $g$ of $G$ into $H$,

$$C_g(G,H) \geq \frac{C_g(S)}{|S|} \geq \frac{C_f(S)}{|S|}, \text{ and hence}$$

$$C(G,H) \geq \min_g C_g(G,H) \geq \min_g \frac{C_f(S)}{|S|} \geq \frac{C_f(S)}{|S|}. \square$$

## 4 Results and Discussions

Minimum edge-congestion has been obtained for embedding hypercubes into rectangular grids [23] and $n$-dimensional grids [13]. In this section, we obtain the edge-congestion of embedding hypercubes, folded hypercubes, crossed cubes and circulant networks into hypertrees and show that the bound obtained from Edge-congestion Lemma is tight.

### *4.1 Hypercubes into Hypertrees*

For proving the main result, we need the following results.

**Lemma 4.1.** For $i = 1,2,\dots,r-1$, $NcutS_i^{2^i} = \{2^i+1, 2^i+2,\dots,2^{i+1}-2\}$ is an optimal set in $Q^r$.

*Proof.*Let $L_{2^i}$ denote the incomplete hypercube on $2^i$ vertices. Define $\varphi : NcutS_i^{2^i} \to L_{2^i}$ by $\varphi(2^i+k) = k$. If the binary representation of $2^i+k$ is $\alpha_1\alpha_2\cdots\alpha_r$ then the binary representation of $k$ is $\underbrace{00\cdots00}_{r-i \text{ times}}\alpha_{r-i+1}\alpha_{r-i+2}\cdots\alpha_r$. Thus the binary representation of two numbers $x$ and $y$ differ in exactly one bit $\Leftrightarrow$ the binary representation of $\varphi(x)$ and $\varphi(y)$ differ in exactly one bit. Therefore $(x,y)$ is an edge in $NcutS_i^{2^i}$ $\Leftrightarrow$ $(\varphi(x),\varphi(y))$ is an edge in $L_{2^i}$. Hence $NcutS_i^{2^i}$ and $L_{2^i}$ are isomorphic. By Theorem 3.1, $NcutS_i^{2^i}$ is an optimal set in $Q^r$.

**Theorem 4.2.** Let $G$ be an $r$-dimensional hypercube $Q^r$ and $H$ be the rooted hypertree $RHT(r)$, $r \geq 3$. Then the edge-congestion of embedding $G$ into $H$ satisfies

$$C(G,H) \geq 2^{r-2}+r-3.$$

*Proof.* Let $f$ be an embedding from $G$ into $H$. Label the vertices of $Q^r$ by lexicographic order [22] from 0 to $2^r-1$ and use the symbol $x+1$ instead of $x$. Therefore the labeling of $Q^r$ is from 1 to $2^r$. Removal of the horizontal edges in a rooted hypertree $RHT(r)$ yields a rooted complete binary tree. Label its vertices by inorder labeling [36] using the sequence of numbers $1,3,5,\dots,2^r-1,2,4,6,\dots,2^r$. See Figure 3. We assume that the labels of hypertree vertices represent the hypercube vertices that are assigned to them.

Let $S$ be the edge cut in $RHT(r)$ given by $S = \{(2^{r-2}-1, 2^{r-1}-1),(2^{r-2}, 2^{r-1})\}$. See Figure 3. For $r \geq 3$, $E(RHT(r))\backslash S$ has two components $H_1$ and $H_2$, where $V(H_1) = \{1,2,\dots,2^{r-1}-2\}$. Let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. By Theorem 3.1 and Lemma 4.1, $G_1$ is an optimal set and $S$ satisfies conditions (i), (ii) and (iii) of the Modified Congestion Lemma. Therefore $C_f(S) = 2^{r-1}+2r-6$ is minimum. By Edge-congestion Lemma,

$$C(G,H) \geq \frac{2^{r-1}+2r-6}{2} = 2^{r-2}+r-3. \square$$

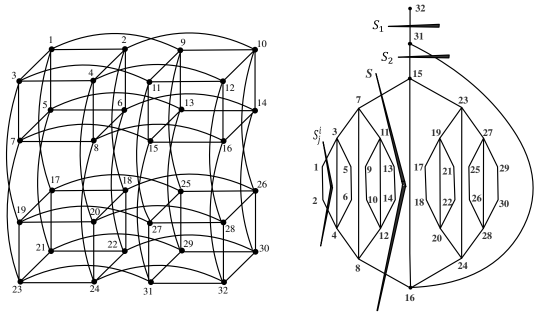


**Fig. 3:** Labeling of $Q^5$ and $RHT(5)$

**Congestion Algorithm A**

**Input:** The $r$-dimensional hypercube $Q^r$ and the rooted hypertree $RHT(r)$, $r \geq 3$.

**Algorithm:** Label the vertices of $Q^r$ by lexicographic order [22] from 0 to $2^r-1$ and use the symbol $x+1$ instead of $x$. Therefore the labeling of $Q^r$ is from 1 to $2^r$. Removal of the horizontal edges in a rooted hypertree $RHT(r)$ yields a rooted complete binary tree. Label its vertices by inorder labeling [36] using the sequence of numbers $1,3,5,\dots,2^r-1,2,4,6,\dots,2^r$. Let $f(x) = x$ for all $x \in V(Q^r)$ and for $(a,b) \in E(Q^r)$, let $P_f(a,b)$ be a shortest path between $f(a)$ and $f(b)$ in $RHT(r)$.

**Output:** An embedding $f$ of $Q^r$ into $RHT(r)$ with edge-congestion $2^{r-2}+r-3$.

**Proof of correctness:** Label the vertices of $Q^r$ and $RHT(r)$ using Congestion Algorithm A. We assume that the labels represent the vertices to which they are assigned.

For $1 \leq i \leq r-2$, $1 \leq j \leq 2^{r-(i+1)}$ and $j$ odd, let $S_j^i$ be edge cuts in $RHT(r)$ given by $S_j^i = \{(j\,2^{i+1}-2^i-1, j\,2^{i+1}-1),(j\,2^{i+1}-2^i, j\,2^{i+1})\}$.

For $1 \leq i \leq r-2$, $1 \leq j \leq 2^{r-(i+1)}$ and $j$ even, let $S_j^i$ be edge cuts in $RHT(r)$ given by $S_j^i = \{(j\,2^{i+1}-2^i-1, j\,2^{i+1}-2^{i+1}-1),(j\,2^{i+1}-2^i, j\,2^{i+1}-2^{i+1})\}$.

Let $S_1$ and $S_2$ be the edge cuts in $RHT(r)$ given by $S_1 = \{(2^r, 2^r-1)\}$ and $S_2 = \{(2^r-1, 2^{r-1}),(2^r-1, 2^{r-1}-1)\}$. Let $S_3$ be the set of all horizontal edges in $RHT(r)$.

Then $\{S_j^i : 1 \leq i \leq r-2, 1 \leq j \leq 2^{r-(i+1)}\} \cup \{S_i : i = 1,2,3\}$ is a partition of $E(RHT(r))$. See Figure 3. Note that, for $i = r-2$ and $j = 1$, $S_j^i = S = \{(2^{r-2}-1, 2^{r-1}-1),(2^{r-2}, 2^{r-1})\}$.

For each $i,j$, $1 \leq i \leq r-2$, $1 \leq j \leq 2^{r-(i+1)}$, $E(RHT(r))\backslash S_j^i$ has two components $H_{j1}^i$ and $H_{j2}^i$, where $V(H_{j1}^i) = \{(j-1)2^{i+1}+1, (j-1)2^{i+1}+2,\dots, j\,2^{i+1}-2\}$. Let $G_{j1}^i = f^{-1}(H_{j1}^i)$ and $G_{j2}^i = f^{-1}(H_{j2}^i)$. By Theorem 3.1 and Lemma 4.1, $G_{j1}^i$ is an optimal set and each $S_j^i$ satisfies conditions (i), (ii) and (iii) of the

Modified Congestion Lemma. Therefore $C_f(S_j^i) = 2^{i+1}(r-i-1) - 2r + 4i + 2$ is minimum.

Let $e_1$ and $e_2$ be cut edges in $S_j^i$, $1 \le i \le r-2$, $1 \le j \le 2^{r-(i+1)}$. Then by Congestion Algorithm A, there exists a unique shortest path between any pair of vertices $f(x), f(y)$, where $(x,y) \in E(Q^r)$. By symmetry property, the number of paths crossing the edge $e_1$ is equal to $2^i(r-i-1) - r + 2i + 1 = e_2 \le 2^{r-2} + r - 3$.

For $i = 1$, $E(RHT(r)) \backslash S^i$ has two components $H_{i1}$ and $H_{i2}$, where $V(H_{i1}) = \{2^r\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. $G_{i1}$ is an optimal set and $S^i$ satisfies conditions (i), (ii) and (iii) of the Modified Congestion Lemma. Therefore $C_f(S^i) = r$ is minimum and is less than $2^{r-2} + r - 3$.

For $i = 2$, $E(RHT(r)) \backslash S^i$ has two components $H_{i1}$ and $H_{i2}$, where $V(H_{i1}) = \{2^r - 1, 2^r\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. $G_{i1}$ is an optimal set and $S^i$ satisfies conditions (i), (ii) and (iii) of the Modified Congestion Lemma. Therefore $C_f(S^i) = 2r - 2$ is minimum and is less than $2^{r-2} + r - 3$. It is easy to see that for $e, e \in S_3$, the edge-congestion of the edge 1. Hence the result. □

Proceeding along the same lines, we prove the following results.

**Theorem 4.3.** Let $G$ be an $r$-dimensional folded hypercube $FQ^r$ and $H$ be the rooted hypertree $RHT(r)$, $r \ge 3$. Then the edge-congestion of embedding $G$ into $H$ is given by

$$C(G,H) = 2^{r-1} + r - 4.$$

**Theorem 4.4.** Let $G$ be an $r$-dimensional crossed cube $CQ^r$ and $H$ be the rooted hypertree $RHT(r)$, $r \ge 3$. Then the edge-congestion of embedding $G$ into $H$ is given by

$$C(G,H) = 2^{r-2} + r - 3.$$

### *4.2 Circulant Networks into Hypertrees*

For proving the main result, we need the following result.

**Theorem 4.5.** [37] A set of $k$ consecutive vertices of $G(n;\pm 1)$, $1 \le k \le n$ induces a maximum subgraph of $G(n;\pm S)$, where $S = \{1,2,\ldots,j\}$, $1 \le j \le \lfloor n/2 \rfloor$, $n \ge 3$. □

**Theorem 4.6.** Let $G$ be the circulant graph $G(2^r - 1; \pm S)$, $S \subseteq \{1,2,\ldots,j\}$, $1 \le j < \lfloor n/2 \rfloor$ and $H$ be the hypertree $HT(r)$, $r \ge 3$. Then the edge-congestion of embedding $G$ into $H$ satisfies

$$C(G,H) \ge \frac{j(j+1)}{2}.$$

*Proof.* Let $f$ be an embedding from $G$ into $H$. Label the consecutive vertices of $G(2^r - 1; \pm\{1\})$ in $G(2^r - 1; \pm S)$, $S \subseteq \{1,2,\ldots,j\}$, $1 \le j < \lfloor n/2 \rfloor$ as $1,2,\ldots,2^r - 1$ in the clockwise sense. Removal of the horizontal edges in a hypertree $HT(r)$ yields a complete binary tree. Label its vertices by inorder labeling [36] using the sequence of numbers $1,3,5,\ldots,2^r - 1, 2,4,6,\ldots,2^r - 2$. We assume that the labels of the hypertree vertices represent the circulant graph vertices that are assigned to them.

Let $S$ be the edge cut in $HT(r)$ given by $S = \{(2^{r-2} - 1, 2^{r-1} - 1), (2^{r-2}, 2^{r-1})\}$. For $r \ge 3$, $E(HT(r)) \backslash S$ has two components $H_1$ and $H_2$, where $V(H_1) = \{1,2,\ldots,2^{r-1} - 2\}$. Let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. By Theorem 4.5, $G_1$ is an optimal set and $S$ satisfies conditions (i), (ii) and (iii) of the Modified Congestion Lemma. Therefore $C_f(S)$ is minimum and $C_f(S) \ge j(j+1)$, where $1 \le j < \lfloor n/2 \rfloor$. Then by Edge-congestion Lemma,

$$C(G,H) \ge \frac{j(j+1)}{2}. \square$$

**Congestion Algorithm B**

**Input:** The circulant graph $G(2^r - 1; \pm S)$, $S \subseteq \{1,2,\ldots,j\}$, $1 \le j < \lfloor n/2 \rfloor$ and the hypertree $HT(r)$, $r \ge 3$.

**Algorithm:** Label the consecutive vertices of $G(2^r - 1; \pm\{1\})$ in $G(2^r - 1; \pm S)$, $S \subseteq \{1,2,\ldots,j\}$, $1 \le j < \lfloor n/2 \rfloor$ as $0,1,\ldots,2^r - 1$ in the clockwise sense. Removal of the horizontal edges in a hypertree $HT(r)$ yields a complete binary tree. Label its vertices by inorder labeling [36] using the sequence of numbers $1,3,5,\ldots,2^r - 1, 2,4,6,\ldots,2^r - 2$. Let $f(x) = x$ for all $x \in V(G(2^r - 1; \pm S))$ and for $(a,b) \in E(G(2^r - 1; \pm S))$, let $P_f(a,b)$ be a shortest path between $f(a)$ and $f(b)$ in $HT(r)$.

**Output:** An embedding $f$ of $G(2^r - 1; \pm S)$ into $HT(r)$ with edge-congestion $\frac{j(j+1)}{2}$. □

Proof of correctness of Congestion Algorithm B follows the same lines as that of Congestion Algorithm A.

**Theorem 4.7.** Let $G$ be the circulant graph $G(2^r - 1; \pm S)$, $S \subseteq \{1,2,\ldots,j\}$, $1 \le j < \lfloor n/2 \rfloor$ and $H$ be the hypertree $HT(r)$, $r \ge 3$. Then the edge-congestion of embedding $G$ into $H$ is given by

$$C(G,H) = \frac{j(j+1)}{2}. \square$$

## 5 Concluding Remarks

In this paper, we obtain a strategy to compute edge-congestion of an embedding. Further, we compute the edge-congestion of embedding hypercubes, folded hypercubes, crossed cubes and circulant networks into hypertrees. Using the techniques of Section 3, we have the following result.

**Theorem 5.1.** Let $G$ be an $r$-dimensional augmented hypercube $AQ^r$ [29] and $H$ be the rooted hypertree $RHT(r)$ $r \ge 3$. Then the edge-congestion of embedding $G$ into $H$ is minimum.

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