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Lomax-Rayleigh Distribution with an Application

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Abstract: We propose and study a two-parameter univariate distribution called the Lomax-Rayleigh distribution, which is shown to belong to the Lomax-Weibull family of distributions. It is generated using the Lomax-G generator class of distributions. Various structural properties including expressions for the moments, skewness and kurtosis coefficients of the distribution are derived. The model parameters are estimated by moment and maximum likelihood methods. We hope that the distribution proposed here serves as an alternative model to other models with two parameters proposed in the literature for modeling real data in many areas.

Keywords: Lomax-G generator distributions, moment, maximum likelihood estimation, lifetime data.

1 Introduction

An important distribution in modeling random phenomena, especially positive phenomena, is the Rayleigh distribution. It arises in many areas of application, including reliability, life testing and survival analysis. A Rayleigh random variable X, denoted as $X \sim R(\sigma)$, has probability density function (pdf)

$$f_X(x;\sigma) = \frac{x}{\sigma}\exp\left(-\frac{x^2}{2\sigma}\right), \qquad (1)$$

and cumulative distribution function (cdf)

$$F_X(x;\sigma) = 1-\exp\left(-\frac{x^2}{2\sigma}\right), \qquad (2)$$

where $x>0$ and $\sigma>0$ is a scale parameter.
A generalization of the Rayleigh distribution is the Weibull model. A Weibull random variable X, denoted as $X \sim W(\theta,\alpha)$, has probability density function

$$f_X(x;\theta,\alpha) = \frac{\alpha}{\theta^\alpha}x^{\alpha-1}\exp\left(-\left(\frac{x}{\theta}\right)^\alpha\right),$$

where $x>0$, $\theta>0$ is a scale parameter and $\alpha>0$ is a shape parameter. If $X \sim W(\sqrt{2\sigma},2)$, then X follows a Rayleigh distribution.
More details on the Rayleigh distribution can be found in Johnson et al. [5]. Several extensions of the Rayleigh distribution have been proposed in the literature, for example, (Vodă [11]; Balakrishnan and Kocherlakota [1]; Surles and Padgett [10]; Kundu and Raqab [6]; Manesh and Khaledi [7]; Cordeiro et al. [2]; Iriarte et al. [4]).
Another important distribution in modeling random phenomena is the Lomax distribution (also known as the Pareto distribution of the second kind). A random variable X follows a Lomax distribution, denoted as $X \sim L(\beta,\alpha)$, if its pdf and cdf are given by

$$f_X(x;\beta,\alpha) = \frac{\alpha\beta^\alpha}{(\beta+x)^{\alpha+1}}$$

and

$$F_X(x;\beta,\alpha) = 1-\left(\frac{\beta}{\beta+x}\right)^\alpha,$$

where $x>0$, $\beta>0$ is a scale parameter and $\alpha>0$ is a shape parameter.
Cordeiro et al. [4] proposed a family of univariate distributions generated by Lomax random variables. For any baseline cdf $G(x)$ ($x \in \mathbb{R}$), they defined the Lomax-G generator distribution with cdf and pdf given by

$$\begin{aligned} F_X(x) &= \alpha\beta^\alpha \int_0^{-\log[1-G(x)]} \frac{du}{(\beta+u)^{\alpha+1}} \qquad (3)\\ &= 1-\left(\frac{\beta}{\beta-\log[1-G(x)]}\right)^\alpha, \end{aligned}$$

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and

$$f_X(x) = \alpha\beta^{\alpha}\frac{g(x)}{[1-G(x)]\{\beta-\log[1-G(x)]\}^{\alpha+1}}, \quad (4)$$

respectively, where $\beta > 0$ is a scale parameter, $\alpha > 0$ is a shape parameter and $g(x) = dG(x)/dx$ is the parent density function.

This article describes our study of a two-parameter distribution called the Lomax-Rayleigh distribution which is a particular case of the Lomax-Weibull distribution. We have defined the probability density function of the distribution by inserting equations (1) and (2) into equation (4) considering the parametrization $\theta = 2\sigma\beta$. The corresponding cumulative distribution function is obtained by inserting equation (2) into equation (3) and considering the same parametrization. We propose this distribution as an alternative model to other two-parameter models proposed in the literature for modeling real data.

The rest of the paper is organized as follows. In Section 2 we present the probability density function and cumulative distribution function of the Lomax-Rayleigh model. We also provide its ordinary moments and derived asymmetry and kurtosis coefficients, and we also derive the survival function and the density function of order statistics. In Section 3 we discuss moment and maximum likelihood estimations and calculate the elements of the observed information matrix. In Section 4 we present an application to a real data set. Finally, in Section 5 we report the final conclusions.

2 Lomax-Rayleigh distribution

In this section we present the pdf and cdf of the Lomax-Rayleigh distribution and derive its distributional moments.

2.1 Density and cumulative functions

Definition 1.*A random variable X follows a Lomax-Rayleigh distribution, denoted as $X \sim LR(\theta,\alpha)$, if its cumulative distribution function (cdf) is given by*

$$F_X(x;\theta,\alpha) = 1-\left(\frac{\theta}{\theta+x^2}\right)^{\alpha}, \quad (5)$$

and the respective pdf is

$$f_X(x;\theta,\alpha) = \frac{2\alpha\theta^{\alpha}x}{(\theta+x^2)^{\alpha+1}}, \quad (6)$$

where $x > 0$, $\theta > 0$ is a scale parameter and $\alpha > 0$ is a shape parameter.

Next, we present some transformations related to LR distributions.

Proposition 1.*Let $X \sim LR(\theta,\alpha)$. Then,*

$(a)W = aX \sim LR(a^2\theta,\alpha)$ for all $a > 0$;
(b)The pdf of $W = X^{-1}$ is

$$f_W(w;\theta,\alpha) = \frac{2\alpha\theta^{\alpha}w^{1-2\alpha}}{(\theta w^2+1)^{\alpha+1}}, w > 0;$$

(c)The pdf of $W = \log(X)$ is given by

$$f_W(w;\theta,\alpha) = \frac{2\alpha\theta^{\alpha}e^{2w}}{(\theta+e^{2w})^{\alpha+1}}, w \in \mathbb{R}.$$

Proof. Parts (a)-(c) are directly obtained by the change-of-variable method. □

*Remark.*Part (a) of Proposition 1 indicates that the LR distributions belong to the scale family, Part (b) demonstrates that these distributions are not closed under reciprocation, while the result in Part (c) can be used to study regression models on the same lines as in the context of regression models for positive random variables; see McDonald and Butler [8]. In addition, Part (a) allows us to obtain a one parameter LR distribution. That is, if $X \sim LR(\theta,\alpha)$, then $X/\sqrt{\theta} \sim LR(1,\alpha)$.

Figure 1 depicts some of the shapes that the LR density can take for different parameter values.

2.2 Moments

Proposition 2.*Let $X \sim LR(\theta,\alpha)$. Then, for $r = 1,2,\ldots$ and $\alpha > r/2$, it follows that r-th moment is given by*

$$\mu_r = E(X^r) = \frac{\theta^{\frac{r}{2}}\Gamma\left(\alpha-\frac{r}{2}\right)\Gamma\left(\frac{r}{2}+1\right)}{\Gamma(\alpha)},$$

where $\Gamma(\alpha) = \int_0^{\infty} u^{\alpha-1}e^{-u}du$ is the gamma function.

Proof. Using the defining moments, the r-th moment is given by

$$\mu_r = \int_0^{\infty} x^r \frac{2\alpha\theta^{\alpha}x}{(\theta+x^2)^{\alpha+1}}dx,$$

and considering the change of variable $u = \theta\left(\theta+x^2\right)^{-1}$, the r-th moment is given by

$$\mu_r = \theta^{\frac{r}{2}}\alpha\int_0^1 u^{\alpha-\frac{r}{2}-1}(1-u)^{\frac{r}{2}}du,$$

and recognizing the beta function $B(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du$ in the above expression, and writing the beta function in terms of the gamma function $\Gamma(\alpha) = \int_0^{\infty} u^{\alpha-1}e^{-u}du$, we obtain the result. □

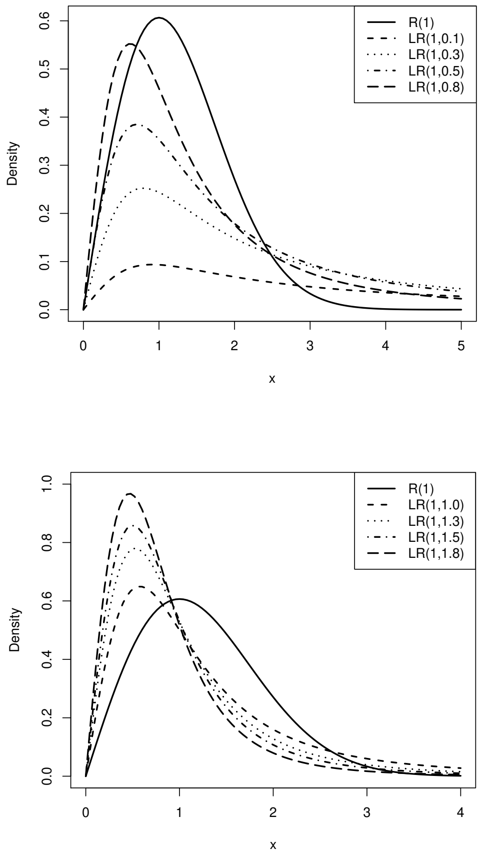

Fig. 1: Lomax-Rayleigh density for different parameter values.

Corollary 1.*Let $X \sim LR(\theta,\alpha)$, then it follows that*

$$E(X)=\frac{\sqrt{\pi\theta}\Gamma\left(\alpha-\frac{1}{2}\right)}{2\Gamma(\alpha)} \qquad \alpha>1/2,$$

and

$$Var(X)=\frac{\theta[4\Gamma(\alpha)\Gamma(\alpha-1)-\pi\Gamma^2\left(\alpha-\frac{1}{2}\right)]}{4\Gamma^2(\alpha)} \qquad \alpha>1.$$

Corollary 2.*Let $X \sim LR(\theta,\alpha)$. Then the asymmetry $(\sqrt{\beta_1})$ and kurtosis (β_2) coefficients for $\alpha>3/2$ and $\alpha>2$ are given respectively by*

$$\sqrt{\beta_1}=\frac{2\sqrt{\pi}\,[3\Gamma^2(\alpha)\Gamma(\alpha-\frac{3}{2})-6\Gamma(\alpha)\Gamma(\alpha-\frac{1}{2})\Gamma(\alpha-1)+\pi\Gamma^3(\alpha-\frac{1}{2})]}{[4\Gamma(\alpha)\Gamma(\alpha-1)-\pi\Gamma^2(\alpha-\frac{1}{2})]^{\frac{3}{2}}}$$

and

$$\beta_2=\frac{32\Gamma^3(\alpha)\Gamma(\alpha-2)-24\pi\Gamma^2(\alpha)\Gamma(\alpha-\frac{1}{2})\Gamma(\alpha-\frac{3}{2})}{[4\Gamma(\alpha)\Gamma(\alpha-1)-\pi\Gamma^2(\alpha-\frac{1}{2})]^2}+\frac{24\pi\Gamma(\alpha)\Gamma(\alpha-1)\Gamma^2(\alpha-\frac{1}{2})-3\pi^2\Gamma^4(\alpha-\frac{1}{2})}{[4\Gamma(\alpha)\Gamma(\alpha-1)-\pi\Gamma^2(\alpha-\frac{1}{2})]^2}.$$

Table 1 shows some skewness and kurtosis values for different values of the parameter α. Figure 2 depicts plots for the asymmetry and kurtosis coefficients, respectively. It can be seen that as the parameter α decreases, this increases the asymmetry and kurtosis of the model.

Table 1: Some skewness and kurtosis values for different values of parameter α.

α	3	4	6	8	10	15	20
$\sqrt{\beta_1}$	1.9086	1.4322	1.0944	0.9580	0.8839	0.7926	0.7497
β_2	12.4635	7.3558	5.1176	4.4426	4.1221	3.7685	3.6175

2.3 Survival Analysis

The survival function $S(t)$, which is the probability of an item not failing prior to a time t, is defined by $S(t)=1-F(t)$. The reliability function of a LR distribution is given by

$$S(t)=\left(\frac{\theta}{\theta+t^2}\right)^{\alpha}.$$

The other characteristic of interest for a random variable is the hazard rate function defined by $h(t)=\frac{f(t)}{S(t)}$, which is an important quantity characterizing life phenomena. It can be loosely interpreted as the conditional probability of failure, given that it has survived to time t. The hazard rate function for a LR random variable is given by

$$h(t)=\frac{2\alpha t}{(\theta+t^2)}.$$

i)$\forall t\in(0,\sqrt{\theta})$, $h(t)$ is monotonically increasing with $h(0)=0$.
ii)$\forall t\in(\sqrt{\theta},+\infty)$, $h(t)$ is monotonically decreasing with $h(\sqrt{\theta})=\frac{\alpha}{\sqrt{\theta}}$.
iii)$\forall\theta,\alpha$, $h(t)\to 0$, as $t\to\infty$.

Figure 3 depicts some of the shapes that the hazard function of the LR distribution can take for different parameter values. We can see that the hazard function is generally unimodal and is not monotonous.

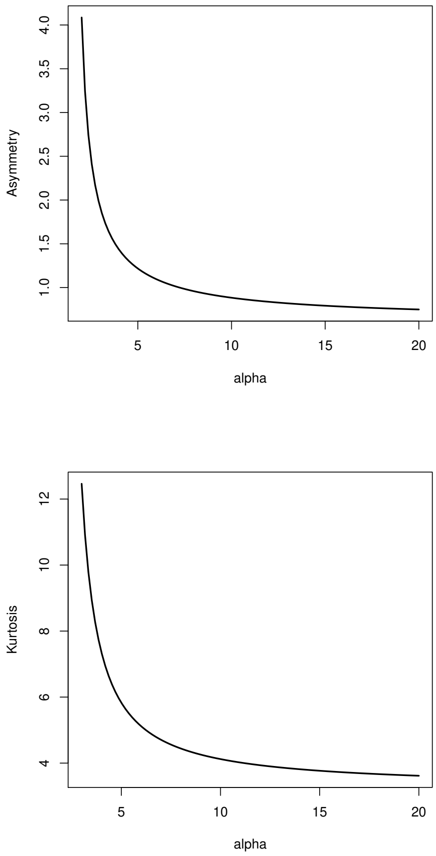

Fig. 2: Plot of the asymmetry and kurtosis coefficients for a Lomax-Rayleigh distribution.

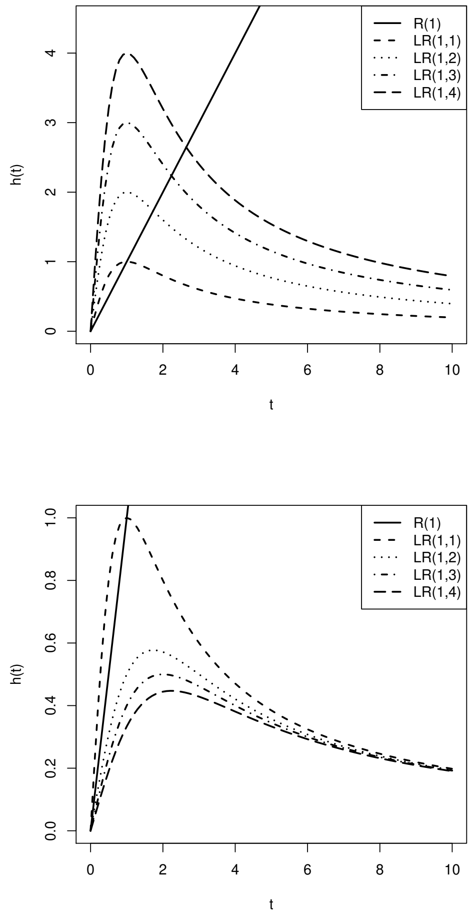

Fig. 3: Hazard function of a Lomax-Rayleigh distribution.

2.4 Order Statistics

In statistics, the k^{th} order statistical sample is equal to its k^{th}-smallest value. Together with rank statistics, order statistics are among the most fundamental tools in non-parametric statistics and inference. For a sample of size n, the n^{th} order statistic (or largest order statistic) is the maximum, that is,

$$X_{(n)} = \max\{X_1, X_2, ..., X_n\} .$$

The sample range is the difference between the largest and smallest order statistics. It is clearly a function of order statistics:

$$\text{Range}\{X_1, X_2, ..., X_n\} = X_{(n)} - X_{(1)} .$$

We know that if $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, ..., X_n$ from a continuous population with cdf $F_X(x)$ and pdf f_X then the pdf of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f_X(x) \left[F_X(x)\right]^{j-1} \left[1 - F_X(x)\right]^{n-j},$$

for $j = 1, 2, ..., n$. The pdf of the j^{th} order statistics for a LR distribution is given by

$$f_{X_{(j)}}(x) = \frac{2n!\,\alpha\theta^{\alpha} x}{(j-1)!(n-j)!(\theta+x^2)^{\alpha+1}} \left[1 - \left(\frac{\theta}{\theta+x^2}\right)^{\alpha}\right]^{j-1} \left(\frac{\theta}{\theta+x^2}\right)^{\alpha(n-j)} .$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is given by

$$f_{X_{(n)}}(x) = \frac{2n\alpha\theta^{\alpha} x}{(\theta+x^2)^{\alpha n+1}} \left[\left(\theta + x^2\right)^{\alpha} - \theta^{\alpha}\right]^{n-1},$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$f_{X_{(1)}}(x) = \frac{2n\alpha\theta^{n\alpha} x}{(\theta+x^2)^{\alpha n+1}} .$$

3 Inference

In this section we discuss moments and maximum likelihood estimations for parameters θ and α for the LR distribution.

3.1 Moment estimators

Below we discuss moment estimators for the parameter procedures.

Proposition 3. *Let $X_1,\ldots,X_n$ be a random sample from random variable $X \sim LR(\theta,\alpha)$, such that the moment estimators for $\boldsymbol{\theta}=(\theta,\alpha)$ given that $\alpha>1$, are given by*

$$\widehat{\theta}_M = \overline{X^2}(\alpha-1)$$

and

$$4\overline{X}^2(\alpha-1)\Gamma^2(\alpha-1) - \pi\overline{X^2}\Gamma^2\left(\alpha-\frac{1}{2}\right)=0,$$

where $\overline{X}$ is the sample mean and $\overline{X^2}$ is the sample mean for the squared observations.

Proof. From Proposition 2 and considering the first two equations in the moments method, we have

$$\frac{\sqrt{\pi\theta}\,\Gamma(\alpha-1/2)}{2\Gamma(\alpha)}=\overline{X} \qquad \text{and} \qquad \frac{\theta\,\Gamma(\alpha-1)}{\Gamma(\alpha)}=\overline{X^2}.$$

Solving the second equation above for θ yields $\widehat{\theta}_M$. Replacing $\widehat{\theta}_M$ in the first equation above, we obtain the required result. □

3.2 Maximum Likelihood estimation

For a random sample $X_1,\ldots,X_n$ from the distribution $LR(\theta,\alpha)$, the log likelihood function can be written as

$$l(\theta,\alpha) = n\log(2)+n\log(\alpha)+n\alpha\log(\theta)+ \sum_{i=1}^{n}\log(x_i)-(\alpha+1)\sum_{i=1}^{n}\log\left(\theta+x_i^2\right),$$

so that the maximum likelihood equations are given by

$$\frac{n\alpha}{\theta}-(\alpha+1)\sum_{i=1}^{n}(\theta+x_i^2)^{-1}=0, \tag{7}$$

$$\frac{n}{\alpha}+n\log(\theta)-\sum_{i=1}^{n}\log(\theta+x_i^2)=0. \tag{8}$$

Numerical algorithms are therefore required for solving the score equations. One possibility is to use the subroutine optim from R Core Team [9].

3.3 Observed information matrix

In this subsection we consider the observed information matrix for the LR distribution.

Let $X\sim LR(\theta,\alpha)$, so the observed information matrix is given by

$$I_n(\theta,\alpha)=\begin{pmatrix} -\frac{n\alpha}{\theta^2}+(\alpha+1)\sum_{i=1}^{n}(\theta+x_i^2)^{-1} & \frac{n}{\theta}-\sum_{i=1}^{n}\left(\theta+x_i^2\right)^{-1} \\ \frac{n}{\theta}-\sum_{i=1}^{n}\left(\theta+x_i^2\right)^{-1} & -\frac{n}{\alpha^2}\end{pmatrix}.$$

3.4 Fisher information matrix

The log-likelihood for $\boldsymbol{\theta}=(\theta,\alpha)$ based on a single observation X, that is,

$$l(\boldsymbol{\theta};x) = \log(2)+\log(\alpha)+\alpha\log(\theta)+\log(x) -(\alpha+1)\log(\theta+x^2).$$

3.4.1 Score function:

$$\frac{\partial l(\boldsymbol{\theta};x)}{\partial\theta} = \frac{\alpha}{\theta}-\frac{\alpha+1}{\theta+x^2},$$
$$\frac{\partial l(\boldsymbol{\theta};x)}{\partial\alpha} = \frac{1}{\alpha}-\log(\theta+x^2)+\log(\theta).$$

The second derivatives of $l(\boldsymbol{\theta};x)$ are:

$$\frac{\partial^2 l(\boldsymbol{\theta};x)}{\partial\theta^2} = -\frac{\alpha}{\theta^2}+\frac{\alpha+1}{(\theta+x^2)^2},$$

$$\frac{\partial^2 l(\boldsymbol{\theta};x)}{\partial\alpha^2} = -\frac{1}{\alpha^2},$$

$$\frac{\partial^2 l(\boldsymbol{\theta};x)}{\partial\theta\partial\alpha} = \frac{1}{\theta}-\frac{1}{(\theta+x^2)}.$$

After extensive algebraic manipulations, it follows that the Fisher information matrix is given by

$$I_F(\theta,\alpha)=\begin{pmatrix} \frac{\alpha}{\theta^2(\alpha+2)} & -\frac{1}{\theta(\alpha+1)} \\ -\frac{1}{\theta(\alpha+1)} & \frac{1}{\alpha^2}\end{pmatrix}.$$

3.5 Simulation study

In this section, we conduct a small scale simulation study illustrating the Maximum Likelihood (ML) estimates behavior for parameters θ and α in small and moderate sample sizes. One thousand random samples of sizes $n=50$, 100 and 200 has been generated from the

$LR(\theta,\alpha)$ model for fixed parameter values. Random numbers $X \sim LR(\theta,\alpha)$ can be generated as

$$X = \sqrt{\theta\left[(1-U)^{-1/\alpha}-1\right]},$$

where $U \sim U(0,1)$, $\theta > 0$ and $\alpha > 0$.

ML estimates can be obtained as described above using R Core Team [9]. Empirical means and standard deviations (SD) are reported in Table 2 and Table 3 indicating good performances.

Table 2: ML estimates for parameters θ and α for the LR distribution.

		$n=50$		$n=100$	
θ	α	$\widehat{\theta}$ (SD)	$\widehat{\alpha}$ (SD)	$\widehat{\theta}$ (SD)	$\widehat{\alpha}$ (SD)
1	1	1.380 (1.103)	1.194 (0.566)	1.149 (0.487)	1.083 (0.281)
	2	1.585 (1.320)	2.627 (1.371)	1.260 (0.692)	2.364 (0.954)
	3	1.601 (1.539)	3.690 (2.536)	1.399 (1.121)	3.542 (1.586)
2	1	2.482 (1.309)	1.176 (0.544)	2.259 (0.819)	1.071 (0.248)
	2	2.814 (1.709)	2.729 (1.522)	2.430 (1.236)	2.328 (0.950)
	3	2.761 (1.892)	3.668 (1.805)	2.619 (1.531)	3.639 (1.613)
3	1	3.607 (1.712)	1.166 (0.517)	3.332 (1.197)	1.065 (0.254)
	2	3.758 (1.899)	2.708 (1.475)	3.617 (1.647)	2.416 (1.093)
	3	3.642 (1.977)	3.713 (1.842)	3.633 (1.703)	3.669 (1.625)

Table 3: ML estimates for parameters θ and α for the LR distribution.

		$n=200$	
θ	α	$\widehat{\theta}$ (SD)	$\widehat{\alpha}$ (SD)
1	1	1.030 (0.249)	1.019 (0.150)
	2	1.104 (0.417)	2.151 (0.613)
	3	1.255 (0.571)	3.388 (1.185)
2	1	2.151 (0.581)	1.043 (0.168)
	2	2.258 (0.876)	2.183 (0.620)
	3	2.210 (1.113)	3.265 (1.202)
3	1	3.173 (0.773)	1.028 (0.148)
	2	3.412 (1.270)	2.185 (0.673)
	3	3.526 (1.350)	3.257 (1.296)

4 Real Data Illustration

In this Section we analyze a real data set using the LR and Weibull (W) distributions. In addition, we incorporate to the analysis the following models of two parameters:

1. Inverse gaussean (IG) distribution

$$f_X(x;\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right),$$

where $x > 0$ and $\mu,\lambda > 0$.

2. Birnbaum-Saunders (BS) distribution

$$f_X(x;\alpha,\beta) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{x}\right)^{1/2}+\left(\frac{\beta}{x}\right)^{3/2}\right]\exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta}+\frac{\beta}{x}-2\right)\right],$$

where $x > 0$ and $\alpha,\beta > 0$.

3. Lognormal (LN) distribution

$$f_X(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2 x}}\exp\left(-\frac{(\log(x)-\mu)^2}{2\sigma^2}\right),$$

where $x > 0$ and $\mu,\sigma > 0$.

4.1 Lifetime data

Yang et al. [12] reported the following observations on drill lifetime (number of hole that a drill machines before it breaks) when holes new driller in a certain brass alloy: 11, 14, 20, 23, 31, 36, 39, 44, 47, 50, 59, 61, 65, 67, 68, 71, 74, 76, 78, 79, 81, 84, 85, 89, 91, 93, 96, 99, 101, 104, 105, 105, 112, 118, 123, 136, 139, 141, 148, 158, 161, 168, 184, 206, 248, 263, 289, 322, 388, 513. The authors suggest that a reasonable probability model for the analysis of the lifetime of the drill is the lognormal model. Table 4 presents summary statistics for the lifetime data, with b_1 and b_2 standing for asymmetry and kurtosis coefficients, respectively.

Table 4: Summary statistics for lifetime data set.

n	$\overline{X}$	s^2	b_1	b_2
50	119.260	9503.013	2.030	7.680

Using results from Subsection 3.1, the moment estimates are computed and are given by: $\widehat{\theta}_M$=21573.620 and $\widehat{\alpha}_M$=1.916, which are used as starting values for computing the ML estimates.

Where the estimated information matrix is

$$I_F(\hat{\theta},\hat{\alpha}) = \begin{pmatrix} 3.760\times10^{-9} & -4.534\times10^{-5} \\ -4.534\times10^{-5} & 6.883\times10^{-1}\end{pmatrix}.$$

Therefore,

$$\hat{\boldsymbol{\theta}} = \begin{pmatrix}\hat{\theta}\\ \hat{\alpha}\end{pmatrix} \to N_2\left(\begin{pmatrix}10000.000\\ 1.205\end{pmatrix}, \hat{\Sigma} = \begin{pmatrix}2.586\times10^{7} & 1.703\times10^{3}\\ 1.703\times10^{3} & 1.412\times10^{-1}\end{pmatrix}\right),$$

as $n\to\infty$, where $\hat{\Sigma} = \frac{1}{50}I_F^{-1}(\hat{\theta},\hat{\alpha})$. Using Section 3.4 we have computed asymptotic confidence intervals of 95% for θ and α, with result $(31.654, 19968.350)$ and $(0.468, 1.942)$ respectively.

Table 5 depicts parameter estimates for the W, IG, BS, LN and LR models, using the ML approach and the corresponding Akaike information criterion (AIC) and Bayesian criterion (BIC). For these data, AIC and BIC show that the LR model gives a better fit. Standard Errors (SE) are computed using the inverse of the Hessian matrix. Figure 4 presents the histogram for the data with the fitted densities and we can observe that there are atypical observations and the right tail of the LR model is slightly heavier. Figure 5 presents the empirical and fitted cumulative distribution functions and we can see that the fitted cdf of the LR model is closer to the empirical distribution.

Table 5: Parameter estimates, AIC and BIC values for the W, IG, BS, LN and LR models for the lifetimes data set.

	W (SE)	IG (SE)	BS (SE)	LN (SE)	LR (SE)
θ	1.370	119.260	86.988	0.780	10000.000
	(0.140)	(15.640)	(9.569)	(0.078)	(5085.890)
α	131.364	138.677	0.853	4.499	1.205
	(14.360)	(27.735)	(0.085)	(0.110)	(0.375)
AIC	574.3086	574.1852	572.8716	571.0002	570.0852
BIC	578.1326	578.0092	576.6956	574.8242	573.9092

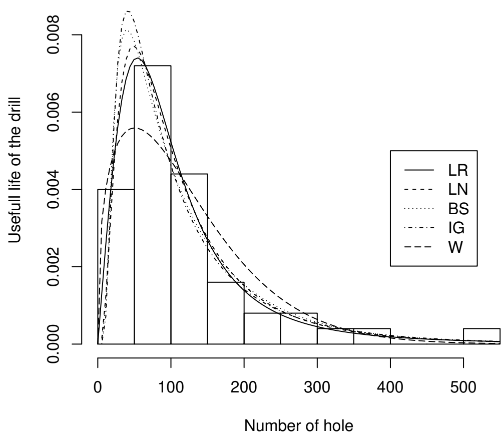

Fig. 4: The histogram for the data with the fitted densities.

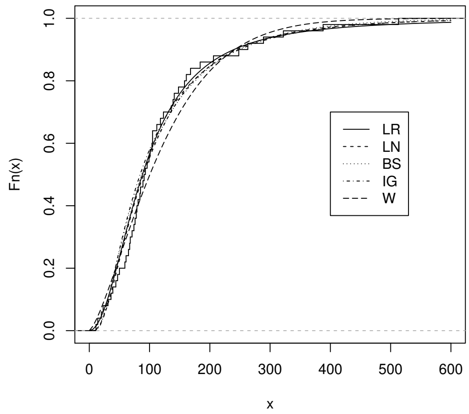

Fig. 5: The empirical and fitted cumulative distribution functions.

5 Concluding Remarks

In this paper we study a two-parameter univariate distribution called LR distribution, which is a sub-model of the Lomax-Weibull distribution. This distribution is generated using the family of distributions Lomax-G generator proposed by Cordeiro et al. [3]. LR distribution is an alternative model to other positive support distributions with two parameters proposed in the statistics literature.

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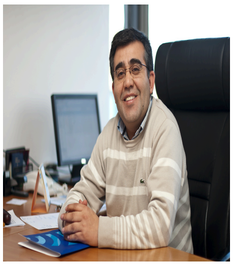

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