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Steiner Reciprocal Degree Distance Index

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Abstract: In this paper we introduce the Steiner Reciprocal Degree Distance Index $SRDD_t(G)$ for some standard graph structures as well as some properties and bounds for it.

Keywords: Graphs, Degree, Distance, Topological indices, steiner distance

1 Introduction

Let G be a finite and simple connected graph whose vertex and edge sets are respectively denoted by $V(G)$, $E(G)$ and $|V(G)| = p$, $|E(G)| = q$ called as order and size of the graph G. The number of edges incident with a vertex x is called as degree of G which is denoted as $deg_G(x)$. If G is connected and $x,y \in V(G)$ and the distance $d_G(x,y)$ is the shortest path between x and y.
For a connected graph G, The Wiener index [11] $W(G)$ of the graph G is defined as

$$W(G) = \sum_{\{x,y\}\subseteq V(G)} d_G(x,y) \tag{1}$$

and the Reciprocal Degree Distance index $RDD(G)$ of a simple connected graph G is defined as

$$RDD(G) = \sum_{\{x,y\}\subseteq V(G)} \left(\frac{deg_G(x) + deg_G(y)}{d_G(x,y)} \right) \tag{2}$$

where $deg_G(x)$ is the degree of x, $d_G(x,y)$ is the distance between x and y. Wiener index and Reciprocal Degree Distance index $RDD(G)$ are extensively studied for many types of chemical graphs [7]. The Steiner distance in a graph, introduced by Chartrand et al.[2] in 1989 is a generalization of the concept of graph distance. For a connected graph G of order at least 2 and $S \subseteq V(G)$, the Steiner distance $d(S)$ of the vertices of S is the minimum size of a connected subgraph whose vertex set is S. In view of equation (1) Li, Mao, and Gutman generalized the notion of wiener index of a graph G as the Steiner wiener index [12] is defined as

$$SW_t(G) = \sum_{S\subseteq V(G),\ |S|=t} d_G(S) \tag{3}$$

When $S = \{u,v\}$, $|S| = 2$ then the Stiener distance diminishes to the distance between a pair of vertices which is ordinary wiener index [11] that is

$$W(G) = SW_t(G) = \sum_{S\subseteq V(G),\ |S|=2} d_G(S) \tag{4}$$

Moreover when $t = 0$, $SW_t(G) = 0$, and $t = p-1$, $SW_t(G) = p-1$.
Furtula, Gutman and Katanic introduced the concept of Steiner Harary index [4]. Later Mao studied the properties of the Steiner harary index[16] or t-center Steiner harary index $SH_t(G)$ is defined as

$$\sum_{S\subseteq G,\ |S|=t} \left(\frac{1}{d_G(S)} \right) \tag{5}$$

Gutman and B. Furtula stated an application of Steiner distance indices in [4] In this paper, we introduce Steiner reciprocal degree distance index and study some interesting properties and bounds.

2 Steiner reciprocal degree distance index of standard graph structures

The reciprocal degree distance index deals with vertex degree-weighted sum of the reciprocal distance, in analogues manner, one can generalize the notion of reciprocal degree distance with t vertices. In view of equation (2) and (3), we introduce the following definition

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Definition 1. *For a connected graph G the $SRDD_t(G)$ is defined as*

$$SRDD_t(G)) = \sum_{\substack{S\subseteq V(G)\\ |S|=t}} \frac{\sum_{u\in S} deg_G(u)}{d_G(S)} \tag{6}$$

Where $1 \leq t \leq p-1$ and when $t = 1$ then $SRDD_t(G) = 0$. One can observe that in the special case $t = 2$ of equation (6), this implies reciprocal degree distance index.

Definition 2. *For a connected graph G the Steiner reciprocal degree distance polynomial $SRDD_t(G,z)$ is defined as*

$$SRDD_t(G,z) = \sum_{\substack{S\subseteq V(G)\\ |S|=t}} \frac{\sum_{u\in S} deg_G(u)}{d_G(S)^2} z^{d_G(S)} \tag{7}$$

the first derivative of the Steiner reciprocal degree distance polynomial with $z = 1$ gives the $SRDD_t(G)$.

Theorem 1. *The Steiner reciprocal degree distance index of the Star graph is*

$$SRDD_t(S_p) = \frac{p(2t-1)-t}{(p-t)(t-1)}\binom{p-1}{t} \tag{8}$$

where $2 \leq t \leq p-2$.

Proof. Let v_1 be the center vertex of the star graph S_p. Divide the vertex set in to two partitions as follows. Let $S \subseteq V(S_p)$ and $|S| = t$, if $v_1 \notin S$, then $d_{S_p}(S) = t$ and $\sum_{v\in S} deg_{S_p}(v) = t$. If $v_1 \in S$, then $d_{S_p}(S) = t-1$ and $\sum_{u\in S} deg_{S_p}(v) = p-1$. Therefore

$$\begin{aligned} &SRDD_t(S_p) \\ &= \sum_{\substack{S\subseteq V(S_p)\\ v_1\notin S,\ |S|=t}} \frac{\sum_{v\in S} deg_{S_p}(v)}{d_{S_p}(S)} + \sum_{\substack{S\subseteq V(S_p)\\ v_1\in S,\ |S|=t}} \frac{\sum_{v\in S} deg_{S_p}(v)}{d_{S_p}(S)} \\ &= \binom{p-1}{t} + \frac{p+t-2}{t-1}\binom{p-1}{t-1} \\ &= \binom{p-1}{t} + \binom{p-1}{t}\frac{t}{p-t}\frac{p+t-2}{t-1} \\ &= \frac{p(2t-1)-t}{(p-t)(t-1)}\binom{p-1}{t} \end{aligned}$$

Proposition 1. *Let K_p be the complete graph of order p and t be an integer $2 \leq t \leq p$ then $SRDD_t(K_p) = \binom{p}{t}\frac{t(p-1)}{t-1}$*

For r regular graph of order p

$$\begin{aligned} SRDD_t(G)) &= \sum_{\substack{S\subseteq V(G)\\ |S|=t}} \frac{\sum_{s\in V(G)} deg_G(s)}{d_G(S)} \\ &= \sum_{\substack{S\subseteq V(G)\\ |S|=t}} \frac{rt}{d_G(S)} \\ &= rt[SH_t(G)] \end{aligned}$$

Where $SH_t(G)$ is the Steiner harary index of G.

Theorem 2. *The Steiner reciprocal degree distance index of path P_p of order p is*

$$SRDD_t(P_n) = 2t\,[SH_t(P_p)] + \binom{p-2}{t-2}\frac{1}{p-1} + \frac{1}{l-1}\binom{l-2}{t-2} \tag{9}$$

where $2 \leq t \leq p-2$, $t-1 \leq l-1 \leq p-1$ and $SH_t(P_p)$ is the Steiner harary index of the path P_p

Proof. Let $V = \{v_1, v_2, ..., v_p\}$ be the vertices of P_n where v_1 and v_p are pendent vertices. Let $S \subseteq V(P_p)$ and $|S| = t$ we have v_1 & $v_p \notin S$ v_1 or $v_p \in S$ and v_1 & $v_p \in S$ then

$$\sum_{v\in S, |S|=t} deg_{P_p}(v) = \begin{cases} 2t-1 & v_1 \text{ or } v_p \in S \\ 2t-2 & v_1 \ \&\ v_p \in S \\ 2t & v_1 \ \&\ v_p \notin S \end{cases}$$

$$\begin{aligned} &SRDD_t(P_p) \\ &= \sum_{\substack{S\subseteq V(P_p)\\ v_1\in S \text{ or } v_p\in S,\\ |S|=t}} \frac{\sum_{v\in S} deg_{P_p}(w)}{d_{P_p}(S)} + \sum_{\substack{S\subseteq V(P_p)\\ v_1 \ \&\ v_n\in S,\ |S|=t}} \frac{\sum_{v\in S} deg_{P_p}(w)}{d_{P_p}(S)} \\ &+ \sum_{\substack{S\subseteq V(P_p)\\ v_1\&v_2\notin S,\ |S|=t}} \frac{\sum_{v\in S} deg_{P_p}(w)}{d_{P_p}(S)} \\ &= \sum_{\substack{S\subseteq V(P_p)\\ v_1\in S \text{ or } v_p\in S\\ |S|=t}} \frac{2t-1}{d_{P_p}(S)} + \sum_{\substack{S\subseteq V(P_p)\\ v_1\&v_p\in S,\ |S|=t}} \frac{2t-2}{d_{P_p}(S)} \\ &+ \sum_{\substack{S\subseteq V(P_p)\\ v_1\&v_2\notin S,\ |S|=t}} \frac{2t}{d_{P_p}(S)} \end{aligned}$$

$$\begin{aligned} &= 2t\left[\sum_{\substack{S\subseteq V(P_p)\\ v_1\in S \text{or} v_p\in S\\ |S|=t}} \frac{1}{d_{P_p}(S)} + \sum_{\substack{S\subseteq V(P_p)\\ v_1 \ \&\ v_p\in S,\ |S|=t}} \frac{1}{d_{P_p}(S)}\right. \\ &\left. + \sum_{\substack{S\subseteq V(P_p)\\ v_1,v_2\notin S, |S|=t}} \frac{1}{d_{P_p}(S)}\right] - \sum_{\substack{S\subseteq V(P_p)\\ \{v_1,\ v_p\}\in S,\ |S|=t}} \frac{1}{d_{P_p}(S)} \\ &- \sum_{\substack{S\subseteq V(P_p)\\ v_1\in S \text{ or } v_p\in S\\ |S|=t,}} \frac{2}{d_{P_p}(S)} \\ &= 2t\sum_{s\in V(G)} \frac{1}{d(s)} - \sum_{\substack{S\subseteq V(P_p)\\ \{v_1,v_p\}\in S,\ |S|=t}} \frac{1}{d_{P_p}(S)} - \sum_{\substack{S\subseteq V(P_p)\\ v_1\in S\text{or}v_p\in S\\ |S|=t}} \frac{2}{d_{P_p}(S)} \end{aligned}$$

for symmetry of $v_1 \in S$ and $v_p \in S$ and view of equation (5)

$$2t\left[SH_t(P_p)\right]-2\left[\sum_{\substack{S\subseteq V(P_p)\\ v_1v_p\in S,\ |S|=t}}\frac{1}{d_{P_p}(S)}+\sum_{\substack{S\subseteq V(P_p)\\ v_1\in S \text{ or } v_p\in S\\ |S|=t}}\frac{1}{d_{P_p}(S)}\right]$$

Let $M=\sum_{\substack{S\subseteq V(P_p)\\ v_1\in S \text{ or } v_p\in S\\ |S|=t}}\frac{1}{d_{P_p}(S)}$ since $\{v_1\ \ v_p\}\in S$ and without loss of generality $S=\{u_1,u_{i_2},u_{i_3},...,u_{i_{k-1}},u_p\}$ where $1\le i_2\le ...\le i_{t-2}\le n$ hence $d_G(S)=p-1$ and there are $\binom{p-2}{t-2}$ ways to compute $\{u_1,u_{i2},u_{i3},...,u_{i(t-1)}\}$ hence $M=\binom{p-2}{t-2}\frac{1}{p-1}$.

Now let $N=\sum_{\substack{S\subseteq V(P_p)\\ \{v_1,\ v_p\}\in S,\ |S|=t}}\frac{1}{d_{P_p}(S)}$. Choose $S\subseteq V(G)$ and $|S|=t$. without loss of generality let $S=\{u_1,u_{i2},u_{i3},...,u_it-1,u_p\}$ where $1\le i_2\le ...\le i_{t-2}\le n$, then $t\le i_t\le p$. Let $d_G(u_1,u_{i_n})=l-1$. $t-1\le l-1\le p-1$. Since $d_G(u_1,u_{i_p})=l-1$ for each vertex u_{i_t} we have $\binom{l-2}{t-2}$ ways to choose $\{u_1,u_{i_2},u_{i_3},...,u_{i_{t-1}}\}$. Hence $N=\frac{1}{l-1}\binom{l-2}{t-2}$

Theorem 3.*Let G be the $K_{m,n}$ complete bipartite graph of order $m+n$, for an integer t such that $2\le t\le m+n-2$, $m,n\ge 2$ then*

$$SRDD_t(m,n)=\begin{cases}\frac{x(m-n)-t(1-m)-1}{(t-1)}\left[\binom{m}{t}+\binom{n}{t}\right]+\frac{(n-m)x+at}{t-1}\binom{m+n}{t} & \text{if } 1\le t\le m\\ \frac{m}{t}\binom{n}{t}+\frac{1}{t-1}\left[\sum_{x=1}^{m}\binom{m}{x}\binom{n}{t-x}nx+a(t-x)\right] & \text{if } m<t\le n\\ \frac{(mx+nt-mx)}{t-1}\binom{m+n}{t} & \text{if } m<t\le m+n.\end{cases}$$

*Proof.*Let $K_{m,n}=G$ and let $V_1=u_1,u_2,u_3,...,u_a$ and $V_2=w_1,w_2,w_3,...,w_b$ are the partition of $V(G)$.

Case I: $1\le t\le a$. For any $S\subset V(G)$ and $|S|=t$, the following subcases $S\cap V_1=\emptyset$ or $S\cap V_2=\emptyset$ or $S\cap V_1\ne\emptyset$ and $S\cap V_2\ne\emptyset$. If $S\cap V_1=\emptyset$. Then $S\subseteq V_2$ and without loss of generality let $S=\{w_1,w_2,...,w_t\}$ then the tree induced by the edges is $\{u_1w_1,u_2w_2,...,u_1w_t\}$ which is a Steiner tree containing S implying $d_G(S)\le t$. Any tree containing t vertices must use at least t edges therefore $d_G(S)=t$ and $\sum_{\substack{v\in S\\ |S|=t}}deg_S(v)=mt$. Similarly $S\cup V_2=\emptyset$ then $d(S)=t$ and $\sum_{\substack{v\in S\\ |S|=t}}deg_S(v)=mt$. Suppose $S\cap V_1\ne\emptyset$ and $S\cap V_2\ne\emptyset$ and without loss of generality suppose $S=\{u_1,u_2,...,u_x,w_1,w_2,...,w_{t-x}\}$ then the tree induced by the edges $\{w_1u_1,u_1w_2,...,u_1w_{t-x},w_1u_1,w_1u_2,...,w_1u_x\}$ is a Steiner tree containing S that must use at least $t-1$ edges and hence $d(S)\le t-1$ and $\sum_{\substack{v\in S\\ |S|=t}}deg_S(v)=nx+m(t-x)$. Thus

$$\begin{aligned}&SRDD_t(K_{m,n})\\&=\sum_{\substack{S\subseteq V(K_{m,n})\\ S\cap V_1=\emptyset}}\frac{\sum_{v\in S}deg_{K_{m,n}}(v)}{d_{K_{m,n}}(S)}+\sum_{\substack{S\subseteq V(K_{m,n})\\ S\cap V_2=\emptyset}}\frac{\sum_{v\in S}deg_{K_{m,n}}(v)}{d_{K_{m,n}}(S)}\\&\quad+\sum_{\substack{S\subseteq V(K_{m,n})\\ S\cap V_1\ne\emptyset,\ V_2\ne\emptyset}}\frac{\sum_{v\in S}deg_{K_{m,n}}(v)}{d_{K_{m,n}}(S)}\\&=\frac{1}{t}\left[\binom{m}{t}tn+\binom{n}{t}mt\right]\\&\quad+\frac{(nx+m(t-x))}{t-1}\sum_{x=1}^{m}\binom{m}{x}\binom{n}{t-x}\\&=\frac{1}{t}\left[\binom{m}{t}tn+\binom{n}{t}mt\right]\\&\quad+\frac{(m+n)x+mt}{t-1}\left[\binom{m+n}{t}-\binom{n}{t}-\binom{n}{t}\right]\\&=\frac{x(m-n)-t(1-m)-1}{(t-1)}\left[\binom{m}{t}+\binom{n}{t}\right]\\&\quad+\frac{(n-m)x+at}{t-1}\binom{m+n}{t}\end{aligned}$$

Case II: $m<t\le n$. For any $S\subseteq V(G)$ and $|S|=t$, we have $S\cap V_1=\emptyset$ and $S\cap V_1\ne\emptyset$. If $S\cap V_1\ne\emptyset$ then $S\subseteq V_2$ and without loss of generality let $S=\{w_1,w_2,...,w_t\}$ then the tree T induced by the edges $\{u_1w_1,u_1w_2,...,u_1w_t\}$ is a Steiner tree containing S implying $d_G(S)\le r$ and any tree containing S vertices must use at least r edges and hence $d(S)\ge t$ edges therefore $d(S)=t$ and $\sum_{v\in S,|S|=t}deg_S(v)=tm$. If $S\cap V_1\ne\emptyset$ and without loss of generality, suppose $S=\{u_1,\ u_2,\ ...,\ u_x,\ w_1,\ w_2,\ ...,\ w_{(r-x)}\}\ (1\le x\le a)$ therefore the tree induced by the edges $\{w_1u_1,u_1w_2,...,u_1w_(t-x)\}$ is a Steiner tree containing S must use at least $t-1$ edges that is $d_G(S)\ge k$ and hence $d(S)=t-1$ and $\sum_{v\in S,|S|=t}deg_{S(v)}=mx+n(t-x)$. Thus

$$\begin{aligned}&SG_tK_{m,n}\\&=\sum_{\substack{S\subseteq V(K_{m,n})\\ S\cap V_1=\emptyset}}\frac{\sum_{v\in S}deg_{K_{m,n}}(v)}{d_{K_{m,n}}(S)}+\sum_{\substack{S\subseteq V(K_{m,n})\\ S\cap V_2\ne\emptyset}}\frac{\sum_{v\in S}deg_{K_{m,n}}(v)}{d_{K_{m,n}}(S)}\\&=\frac{m}{t}\binom{n}{t}+\frac{nx+m(t-x)}{t-1}\left[\sum_{x=1}^{m}\binom{m}{x}\binom{b}{t-x}\right]\end{aligned}$$

Case III: we consider the remaining case $b<t\le m+n$. For any $S\subseteq V(G)$ and $|S|=t$, we have $S\cap V_1\ne\emptyset$, and $|S|=t$ and $S\cap V_2\ne\emptyset$ and without loss generality let $S=\{u_1,u_2,...,u_x,w_1,w_2,...,w_{r-x}\}$. Therefore the tree induced by the edges $\{w_1u_1,w_1u_2,...,w_1u_x,u_1w_2,u_1w_3,...,u_1w_{(r-x)}\}$ is a Steiner tree connecting S which implies $d_G(S)\le t-1$.

Since $|S| = t$, it follows that any tree connecting S must use at least $t-1$ edges and hence $d_G(S) = t-1$ Thus

$$SG_t(G) = \sum_{S\subseteq V(G)} \frac{\sum_{v\in S} deg_{K_{m,n}}(v)}{d_{K_{m,n}}} + \frac{(nx+mt-nx)}{t-1}\binom{m+n}{t}$$

Corollary 1.*Let G be the connected graph of order p and size q.*
Then $SRDD_p(G) = \frac{2q}{p-1}$

Theorem 4.*Let G be a connected graph with p vertices and q edges if*
$\kappa(G) \geq 2$ then

$$SRDD_{p-1}(G) = \frac{2q(p-1)}{p-2}$$

and if $\kappa(G) = 1$ then,

$$SRDD_{p-1}(G) = \frac{2pq}{p-2} - \frac{1}{p^2-3p+2}\left[2lq - 2q(p+1) + \sum_{i=1}^{l} deg_u(x_i)\right]$$

where x_i $(1 \leq i \leq l)$ are the cut vertices of G

*Proof.*Since $\kappa(G) \geq 2$, therefore $d_G(S)$ which consists of $p-1$ vertices is $p-2$, hence

$$SRDD_{p-1}(G) = \frac{1}{p-2}\sum_{\substack{S\subseteq V(G)\\|S|=p-1}}\sum_{v\in S} deg(v)$$

For each $v \in V(G)$ and $|S| = p-1$ we have $p-1$ subsets of $V(G)$ such that each edge of v contributes exactly by $p-1deg_G(v)$ to $SRDD_{p-1}(G)$. Hence
$SRDD_{p-1}(G) = \frac{2q(p-1)}{p-2}$
Case(ii) Suppose $\kappa(G) = 1$
since each vertices of G is one connected therefore each edge is cut edges. For any $S \subseteq V(G)$ and $|S| = p-1$ and $S = \{x_1, x_2, ..., x_l\}$ If $V(G)\backslash S = x_i$, then $d_G(S) = p-1$. If $V(G)\backslash S \neq \{x_i\}$, $1 \leq i \leq p-1$ then $d_G(S) = p-2$,

$$\begin{aligned}
&SRDD_{p-1}(G)\\
&= \frac{1}{p-2}\sum_{\substack{S\subseteq V(G),\\ V(G)\backslash S\neq\{x_i\}}}\left[\sum_{v\in S} deg_G(v)\right]\\
&\quad + \frac{1}{p-2}\sum_{\substack{S\subseteq V(G),\\ V(G)\backslash S\neq\{x_i\}}}\left[\sum_{v\in S} deg_G(v)\right]\\
&= \frac{1}{p-2}\left[2(p-l)q - \sum_{i=l+1}^{p} deg_G(w_i)\right]\\
&\quad + \frac{1}{p-1}[2(ql) - \sum_{i=1}^{l} deg_G(x_i)]\\
&= \frac{2pq}{p-2} - \frac{1}{p^2-3p+2}\left[2pq - 2q(p+1) + \sum_{i=1}^{l} deg_G(x_i)\right]
\end{aligned}$$

Theorem 5.*Let T be a tree with p vertices, possessing l pendent vertices Thus*
$SRDD_{p-1}(T) = \frac{2p^3-8p^2+2pl+10p-3l-4}{(n-2)(n-1)}$

*Proof.*Since $t = p-1$ vertices therefore each subset S contains $p-1$ vertices of T. Let v_i be a pendent vertices such that $v_i \in V(G)\backslash S$ is pendent, Therefore $v_i \in S$ produce a tree of order $p-1$. Therefore $d_T(S) = p-2$ and $\sum_{v\in S} deg_T(v) = 2q-1$. Suppose there are l subsets with $p-1$ in $V(G)\backslash S$. If $V(G)\backslash S$ is non-pendent in S and let w_i be the non-pendent vertices of the graph G then $v \in S$ alone does not produce a tree. Therefore the steiner tree must have all p vertices of T. Therefore $d_T(S) = p-1$ and $\sum_{v\in S} deg_T(v) = 2q - deg_T(w_i)$, where $w_i \in V(G)\backslash S$. There are $p-l$ such subsets. Hence

$$\begin{aligned}
&SRDD_t(T)\\
&= \sum_{\substack{S\subseteq V(T)\\ v_i\in V(G)\backslash S}} \frac{\sum_{v\in S} deg_{S_p}(v)}{d_{S_p}(S)} + \sum_{\substack{S\subseteq V(S_p)\\ v_i\notin S,|S|=t}} \frac{\sum_{v\in S} deg_{S_p}(v)}{d_{S_p}(S)}\\
&= \frac{l(2q-1)}{(p-2)} + \frac{p-l}{p-1}[2q] - \frac{1}{p-1}deg_T(w_i)\\
&= \frac{l(2p-3)}{(p-2)} + \frac{p-l}{p-1}[2(p-1)] - \frac{1}{p-1}\sum_{deg_T\geq 2} deg_T(w_i)\\
&= \frac{l(2p-3)}{(p-2)} + \frac{p-l}{p-1}[2(p-1)] - \frac{1}{p-1}[2p-1-l]\\
&= \frac{2p^3-8p^2+2pl+10p-3l-4}{(n-2)(n-1)}
\end{aligned}$$

3 Some Bounds for Steiner Reciprocal Degree Distance index

Let Γ be a connected Graph and denote $\Delta(\Gamma)$ and $\delta(\Gamma)$ to be the maximum and minimum degree of vertices of Γ.

Theorem 6. *For a connected Graph Γ with p vertices. Then*

$$t\delta(\Gamma)SH_t(\Gamma) \le SRDD_t(\Gamma)) \le t\Delta(\Gamma)SH_t(\Gamma) \quad (10)$$

holds for all r, $2 \le t \le n$ with equality holding iff G is regular graph.

Proof. From the definition of $SRDD_t(\Gamma)$ we have

$$SRDD_t(\Gamma) = \sum_{S\subseteq V(\Gamma)} \left(\frac{\sum_{u\in S} deg(S)}{d(S)}\right)$$
$$\le \sum_{S\subseteq V(\Gamma)} \frac{t\Delta(\Gamma)}{d_\Gamma(S)} = t\Delta(\Gamma)SH_t(\Gamma)$$

and

$$SRDD_t(\Gamma) = \sum_{S\subseteq V(\Gamma)} \left(\frac{\sum_{deg(S)}}{d(S)}\right)$$
$$\ge \sum_{S\subseteq V(\Gamma)} \frac{r\delta(\Gamma)}{d_\Gamma(S)} = t\delta(\Gamma)SH_\Gamma(G)$$

For r regular graph $\Delta(\Gamma) = \delta(\Gamma) = r$ and $SRDD_t(\Gamma) = ktSW_k(\Gamma) = rtSH_t(\Gamma)$

Theorem 7. *For a connected graph Γ of order p with size q then*

$$\frac{2q}{p-1}\binom{p-1}{t-1} \le SRDD_t(G) \le \frac{2q}{r-1}\binom{p-1}{t-1}$$

Proof. Let $S \subset V(G)$ and $|S| = t$, then Steiner distance ranges $t-1 \le d_\Gamma(S) \le p-1$ therefore

$$\frac{1}{p-1}\sum_{\substack{S\subseteq V(T)\\|S|=t}}\sum_{v\in S} deg_\Gamma(v) \le SRDD_t(\Gamma)$$
$$\le \frac{1}{t-1}\sum_{\substack{S\subseteq V(T)\\|S|=t}}\sum_{v\in S} deg_\Gamma(v)$$

for any vertex $v \in V(\Gamma)$ we have $\binom{p-1}{t-1}$ subsets S of $V(G)$ with $|S| = t$ which contains the vertex v. Hence

$$\sum_{\substack{S\subseteq V(\Gamma)\\|S|=t}}\sum_{v\in S} deg_\Gamma(v) = \binom{p-1}{t-1}\sum_{v\in S} deg_\Gamma(v) = 2q\binom{p-1}{t-1}$$

4 Polynomial in Steiner distance

In [6], Haruo Hosoya introduced Hosoya polynomial to generate distance distributions for graphs which are defined as

$$H(G,z) = \sum_{t\ge 1} d(\Gamma,t)z^t$$

the first derivative of the Hosoya polynomial implies the wiener index of the graph[10]. Later Schultz polynomial was introduced by H.P. Schultz in 1989 for the molecular graph and modified Schultz index was defined by S. Klavzar and I. Gutman in 1997[5]. The Schultz polynomial of the graph G is defined as

$$Sc(G,z) = \sum_{S\subseteq V(\Gamma)} (deg_x + deg_y)x^{d_\Gamma(x,y)} \quad (11)$$

Also the modified Schultz polynomial of Γ is defined as

$$Sc^*(G,z) = \sum_{\{x,y\}\subset V(\Gamma)} (deg_x \times deg_y)z^{d_\Gamma(x,y)} \quad (12)$$

The distance generalization of the wiener polynomial defined as the stiener wiener polynomial which is denoted as

$$SW_t(\Gamma,z) = \sum_{S\subset V(\Gamma)} z^{d_\Gamma(S)} \quad (13)$$

where $d_\Gamma(S)$ is the steiner distance of the graph. Similarly one can generalize the Schultz polynomial and molecular graph and modified Schultz polynomial. The Steiner degree distance polynomial is the generalization of the Schultz polynomial which is defined by

$$SDD_t(\Gamma,z) = \sum_{S\subseteq V(\Gamma)} \left(\sum_{v\in S} deg_\Gamma(u)\right) z^{d_\Gamma(S)} \quad (14)$$

the Steiner Gutman polynomial is the generalization of the Schultz polynomial which is defined by

$$SGut_t(\Gamma,z) = \sum_{S\subseteq V(\Gamma)} \left(\prod_{v\in S} deg_\Gamma(u)\right) z^{d_\Gamma(S)} \quad (15)$$

Further it is noted that

$$\frac{\partial\left(SDD_t(\Gamma,z)\right)}{\partial z}|_{z=1} = SDD_t(\Gamma)$$

$$\frac{\partial\left(SGut_t(\Gamma,z)\right)}{\partial z}|_{z=1} = SGut_t(\Gamma)$$

and

$$\frac{\partial\left(SRDD_t(\Gamma,z)\right)}{\partial z}|_{z=1} = SRDD_t(\Gamma)$$

Example 1. Let S_n, P_n K_n be the star, path, complete graph then their Steiner Gutman polynomial is given as follows

$$SDD_t(S_p,z) = (p+t-2)\binom{p-1}{t-1}z^{t-1} + \binom{p-1}{t}z^t$$

$$SDD_t(K_p,z) = t(p-1)\binom{p}{t}z^{t-1}$$

*Example 2.*Let S_p, P_p, K_p be the star, path and complete graph then their Steiner Gutman polynomial is given as follows

$$SGut_t(S_p,z) = (p-1)\binom{p-1}{t-1}z^{t-1} + \binom{p-1}{t}z^t$$

$$SGut_t(K_p,z) = (p-1)^t\binom{p}{t}z^{t-1}$$

$$SGut_r(P_p,z) = 2^t\binom{p}{t+1}z^{t-1} + 2^t\binom{n-2}{t-2}x^{p-1}$$

*Example 3.*Let S_p, P_p K_p be the star, path, complete graph then their Steiner reciprocal degree distance polynomial is given as follows

$$SRDD_t(S_p,z) = \frac{1}{t^2}\binom{p-1}{t}z^t + \frac{1}{(t-1)^2}\frac{t-1}{t-1}z^t - 1$$

$$SRDD_t(k_p,z) = \frac{t(p-1)}{(t-1)^2}\binom{p}{t}$$

$$SRDD_t(P_p,z) = (2tqSH_t(P_p,z)) + \frac{1}{(p-1)^2}\binom{p-2}{t-2}z^{p-1} + \frac{1}{(l-1)^2}\binom{l-2}{t-2}z^{l-1}$$

Observation 1*Let G be the graph with p vertices then*

$$SW_t(\Gamma,z,i) \le SRDD_t(\Gamma,z,i) \le SDD_t(\Gamma,z,i) \le Gut_t(\Gamma,z,i) \tag{16}$$

where (Γ,z,i) *is the coefficient of the* i^{th} *power of the equation.*

5 Conclusion

The Steiner reciprocal degree distance index introduced in this paper has application in the study of QSAR and QSPR study since it is a combination of Steiner distance and reciprocal degree distance. It is easy to find the Steiner Gutman index of wheel graph, windmill graph, caterpillar, and cartesian product of standard graphs.

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