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Analytical Solutions of Fuzzy Fractional Boundary Value Problem of Order 2α by Using RKHS Algorithm

Mohammad Abdel Aal[1], *Nisrein Abu-Darwish*[2], *Omar Abu Arqub*[3], *Mohammed Al-Smadi* [3,*] *and Shaher Momani*[3,4,]

[1] Department of Basic Sciences, Faculty of Art and Science, Middle East University, Amman, Jordan
[2] Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan
[3] Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan
[4] Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah 21589, Kingdom of Saudi Arabia

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Abstract: In this article, an effective numerical solution for fractional fuzzy differential equations of order 2α subject to appropriate fuzzy boundary conditions has been provided by using the Reproducing Kernel (RK) algorithm in Caputo sense. The reproducing kernel functions are built, in which the constraint conditions are satisfied, to yield a fast and accurate RK algorithm for handling these BVPs. The solution methodology is based on constructing the fractional series solution based on the reproducing-kernel theory in the form of a rapidly convergent series with a minimum size of calculations using symbolic computation software. The analytical solution is formulated in the form of a finite series, while, the n-term numerical solution is obtained and proved to converge uniformly to the analytical solution in the space of interest. Simulations, as well as the computational algorithm, are provided to guarantee the RK procedure, to show potentiality, generality, and superiority of RK algorithm and to illustrate the theoretical statements of the present algorithm.

Keywords: Computational algorithm, Fuzzy logic, Fractional derivative, Numerical simulations, Reproducing kernel algorithm

1 Introduction

The study of Fuzzy Differential Equations (FDEs) of fractional order is rapidly expanding as a novel branch of fuzzy mathematics. Both theory and applications have been actively discussed over the last few years. The fuzzy DEs of fractional order are the essential part of the fuzzy analysis and have the vital values for theory and application in control theory, measure theory, and radiation transfer in a semi-infinite atmosphere. Anyhow, crisp DEs of fractional order are indispensable for modeling real-world phenomena in time domain. Unfortunately, every time uncertainty can intervene with real-world problems. The uncertainty can arise from deficient data, measurement errors, or when determining initial conditions. However, fuzzy set theory is a powerful tool to overcome these problems. Thus, it is immensely important to develop appropriate and applicable algorithm to accomplish the mathematical construction that would appropriately treat fuzzy DEs of fractional order and solve them. Anyhow, the authors in [1] considered a generalization of the H-differentiability for the fractional sense. Further, numerous papers have been dedicated to study and solve the FFDEs of order $0 < \alpha \leq 1$, (see [2,3,4,5]).

A generalized concept of higher order H-derivative for fuzzy functions for integer order has been introduced by [6]. This article aims to investigate the exact and the numerical solutions for fuzzy fractional differential equations of order 2α with appropriate fuzzy boundary conditions based on fuzzy Caputo H-derivative by utilizing the Fractional Reproducing Kernel (FRK) algorithm. The proposed algorithm possesses robust features to handle numerous complex problems in the natural sciences that avoid the linearization and perturbation of its nature components for getting the best results with the least effort, time and money. To see some applications of the RK algorithm for solving differential equations of different types, the reader is asked to refer to [7]-[13]. For numerical solutions about RK method, we

* Corresponding author e-mail: mhm.smadi@bau.edu.jo

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refer to [14]-[20], as well as for numerical methods about fractional differentiation, we refer to [21]-[29].

The current article is organized as follows: In the next section, we recall some basic concepts, preliminaries and necessary notations about the fuzzy fractional analysis. In Section 3, formulation of fuzzy fractional differential equation of order 2α is constructed subject to given fuzzy boundary conditions. Solutions of fuzzy fractional boundary value problems by FRK algorithm are presented in Section 4. In Section 5, numerical applications are performed to test the validity and reliability of the present algorithm. Finally, some conclusions are drawn in the last section.

2 Preliminaries

In this section, we present certain essential definitions and properties about fuzzy fractional concept, which is used throughout this article.

Definition 2.1. A fuzzy set v in $\mathbb{R}$ is characterized by its membership function $v:\mathbb{R}\to[0,1]$, while $v(\zeta)$ is interpreted as the degree of membership of an element ζ in the fuzzy set v for each $\zeta\in\mathbb{R}$.

A fuzzy set v on $\mathbb{R}$ is called convex, if for each ζ, $\eta\in\mathbb{R}$ and $\lambda\in[0,1]$, we have $v(\lambda\zeta+(1-\lambda)\eta)\geq min\{v(\zeta),v(\eta)\}$. While a fuzzy set v on $\mathbb{R}$ is called upper semi-continuous, if the set $\{\zeta\in\mathbb{R}\mid v(\zeta)\geq r\}$ is closed for each $r\in[0,1]$. Also, a fuzzy set v on $\mathbb{R}$ is said to be normal, if there is $\zeta\in\mathbb{R}$ such that $v(\zeta)=1$. Moreover, the support of v, which is denoted by $sup(v)$, is defined by the set $sup(v)=Cl\{\zeta\in\mathbb{R}:v(\zeta)>0\}$.

Definition 2.2. [30] A fuzzy number v is a fuzzy subset of the real line with normal, convex, and upper semi-continuous membership function of bounded support.

For each $r\in(0,1]$, put $[v]^r=\{\zeta\in\mathbb{R}:v(\zeta)\geq qr\}$ and $[v]^0=Cl\{\zeta\in\mathbb{R}:v(\zeta)>0\}$, where the symbol $[v]^r$ is called the r-cut representation or parametric form of a fuzzy number and the symbol $Cl\{\cdot\}$ is the closure of the set $\{\cdot\}$. Here, it should be mentioned that v is a fuzzy number if and only if $[v]^r$ is compact and convex subset of $\mathbb{R}$ for each $r\in[0,1]$ and $[v]^1\neq\phi$. Thus, if v is a fuzzy number, then $[v]^r$ can be represented as $[v]^r=[\underline{v}(r),\overline{v}(r)]$, where $\underline{v}(r)=min\{\zeta:\zeta\in[v]^r\}$ and $\overline{v}(r)=max\{\zeta:\zeta\in[v]^r\}$ for each $r\in[0,1]$. Therefore, the r-level set $[v]^r$ is the non-empty and compact interval for each $r\in[0,1]$ and for each $v\in\mathbb{R}_F$, where $\mathbb{R}_F$ denotes the set of fuzzy numbers on $\mathbb{R}$.
Theorem 2.1 is a basic rule in fuzzy numbers theory and its applications.

Theorem 2.1. [31] Let $\underline{v}:[0,1]\to$ and $\overline{v}:[0,1]\to\mathbb{R}$ satisfy the following conditions:

(i) The function $\underline{v}$ is a bounded increasing and the function $\overline{v}$ is a bounded decreasing such that $\underline{v}(1)\leq\overline{v}(1)$.
(ii) The functions $\underline{v}$ and $\overline{v}$ are left-hand continuous functions at $r=\lambda$, for each $\lambda\in(0,1]$.
(iii) The functions $\underline{v}$ and $\overline{v}$ are right-hand continuous functions at $r=0$.

Then, the function $v:\mathbb{R}\to[0,1]$ that is defined by $v(\zeta)=sup\{r:\underline{v}(r)\leq\zeta\leq\overline{v}\}$ is a fuzzy number with parameterization $[\underline{v}(r),\overline{v}(r)]$. However, if $v:\mathbb{R}\to[0,1]$ is a fuzzy number with parameterization $[\underline{v}(r),\overline{v}(r)]$, then the functions $\underline{v}$ and $\overline{v}$ satisfy the conditions mentioned in Theorem 2.1.

Consequently, the arbitrary fuzzy number v can be generally represented as an order pair of functions $(\underline{v}(r),\overline{v}(r))$. On the other hand, let $v=(\underline{v}(r),\overline{v}(r))$, $(\omega=\underline{\omega}(r),\overline{\omega}(r))$ be arbitrary fuzzy numbers and $\lambda\in\mathbb{R}$, then the addition and scalar multiplication can be defined, respectively, by $[v+\omega]^r=[v]^r+[\omega]^r=[\underline{v}(r)+\underline{\omega}(r),\overline{v}(r)+\overline{\omega}(r)]$ $[\lambda v]^r=\lambda[v]^r=[min\{\lambda\underline{v}(r),\lambda\overline{v}(r)\},max\{\lambda\underline{v}(r),\lambda\overline{v}(r)\}]$. For equality property, the fuzzy numbers v and ω are equal if $[v]^r=[\omega]^r$ for each $r\in[0,1]$, that is, $\underline{v}(r)=\underline{\omega}(r)$ and $\overline{v}(r)=\overline{\omega}(r)$.

Definition 2.3. [32] Let $D:\mathbb{R}_F\times\mathbb{R}_F\to\mathbb{R}^+\cup\{0\}$ be the Hausdorff distance, then the metric structure on $\mathbb{R}_F$ is given by D such that
$D(v,\omega)=\sup_{0\leq r\leq1} max\{|\underline{v}(r)-\underline{\omega}(r)|,|\overline{v}(r)-\overline{\omega}(r)|,\}$
for arbitrary fuzzy numbers v and ω. It should be mentioned that $(\mathbb{R}_F,D)$ is a complete metric space.

Definition 2.4. [32] Let $c,\omega\in\mathbb{R}_F$. If there exists $\rho\in\mathbb{R}_F$ such that $c=\omega+\rho$, then ρ is called the Hukuhara difference (H-difference) of c and ρ, and is denoted by $c\ominus\omega$.

Remark 2.1. The sign "$\ominus$" stands always for Hukuhara difference. Thus, it should be noted that $v\ominus\omega\neq v+(-1)\omega$. Normally, $v+(-1)\omega$ is denoted by $v-\omega$. If the H-difference $v\ominus\omega$ exists, then $[v\ominus\omega]^r=[\underline{v}(r)-\underline{\omega}(r),\overline{v}(r)-\overline{\omega}(r)]$.

Definition 2.5. [33] Let $\xi:[a,b]\to\mathbb{R}_F$. Then, the function $\xi(\tau)$ is continuous at $\tau_0\in[a,b]$ if for every $\varepsilon>0$, there exists $\delta=\delta(\tau_0,\varepsilon)>0$ such that $D(\xi(\tau),\xi(\tau_0))<\varepsilon$, for each $\tau\in[a,b]$, whenever $|\tau-\tau_0|<\delta$.

Definition 2.6. [34] Let $\xi:[a,b]\to\mathbb{R}_F$ and $\tau_0\in[a,b]$. We say that ξ is strongly generalized differentiable at τ_0, if there exists an element $\xi'(\tau_0)\in\mathbb{R}_F$ such that either

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1.(i) The H-differences $\xi(\tau_0+\vartheta)\ominus\xi(\tau_0)$, $\xi(\tau_0)\ominus\xi(\tau_0-\varepsilon)$ exist, for each $\vartheta>0$ sufficiently close to 0, and

$$\lim_{\vartheta\to 0^+}\frac{\xi(t_0+\vartheta)\ominus\xi(t_0)}{-\vartheta}=\xi'(\tau_0)$$
$$=\lim_{\vartheta\to 0^+}\frac{\xi(t_0)\ominus\xi(t_0-\varepsilon)}{\vartheta}$$

2.(ii) The H-differences $\xi(\tau_0)\ominus\xi(\tau_0+\vartheta)$, $\xi(\tau_0-\vartheta)\ominus\xi(\tau_0)$ exist, for each $\vartheta>0$ sufficiently close to 0, and

$$\lim_{\vartheta\to 0^+}\frac{\xi(t_0)\ominus\xi(t_0+\varepsilon)}{-\vartheta}=\xi'(\tau_0)$$
$$=\lim_{\vartheta\to 0^+}\frac{\xi(t_0-\vartheta)\ominus\xi(t_0)}{-\vartheta}$$

The limit within Definition 2.6 is taken in the complete metric space $(\mathbb{R}_F,D)$ and at the endpoints of $[a,b]$ by considering only one-sided derivatives.

Theorem 2.2. [35] Let $\xi:[a,b]\to\mathbb{R}_F$ be a fuzzy function and put $[\xi(\tau)]^r=[\underline{\xi_r}(\tau),\overline{\xi_r}(\tau)]$ for each $r\in[0,1]$.

(i) If $\xi(\tau)$ is (1)-differentiable, then $\underline{\xi_r}(\tau)$ and $\overline{\xi_r}(r)$ are differentiable functions and $[D_1^1\xi(\tau)]^r=[\underline{\xi_r}(\tau),\overline{\xi_r}(\tau)]$.

(ii) If ξ is (2)-differentiable, then $\underline{\xi_r}(\tau)$ and $\overline{\xi_r}(r)$ are differentiable functions and $[D_2^1\xi(\tau)]^r=[\overline{\xi_r'}(\tau),\underline{\xi_r'}(\tau)]$.

Definition 2.7. [36] Let $\xi:[a,b]\to\mathbb{R}_F$ be a fuzzy function and $n,m\in\{1,2\}$. We say that ξ is (n,m)-differentiable on $[a,b]$ if $D_n^1\xi$ exists and its (m)-differentiable. The second derivatives of ξ are denoted by $D_{n,m}^2\xi$.

Theorem 2.3. [36] Let $D_1^1\xi:(a,b)\to\mathbb{R}_{\mathscr{F}}$, and $D_2^1\xi:(a,b)\to\mathbb{R}_{\mathscr{F}}$ be fuzzy functions with $[\xi(t)]^r=[\underline{\xi_r}(t),\overline{\xi_r}(t)],r\in[0,1]$. Then

a) If $D_1^1\xi$ is (1)-differentiable, then $\underline{\xi_r'}(\tau)$ and $\overline{\xi_r'}(\tau)$ are differentiable functions and $[D_{1,1}^2\xi(t)]^r=[\underline{\xi_r''}(t),\overline{\xi_r''}(t)]$.

b) If $D_1^1\xi$ is (2)-differentiable, then $\underline{\xi_r'}(\tau)$ and $\overline{\xi_r'}(\tau)$ are differentiable functions and $[D_{1,2}^2\xi(t)]^r=[\overline{\xi_r''}(t),\underline{\xi_r''}(t)]$.

c) If $D_2^1\xi$ is (1)-differentiable, then $\underline{\xi_r'}(\tau)$ and $\overline{\xi_r'}(\tau)$ are differentiable functions and $[D_{2,1}^2\xi(t)]^r=[\overline{\xi_r''}(t),\underline{\xi_r''}(t)]$.

d) If $D_2^1\xi$ is (2)-differentiable, then $\underline{\xi_r'}(\tau)$ and $\overline{\xi_r'}(\tau)$ are differentiable functions and $[D_{2,2}^2\xi(t)]^r=[\underline{\xi_r''}(t),\overline{\xi_r''}(t)]$.

As for the concept of integration for a fuzzy valued function, we will consider the next definition.

Definition 2.8. [35] Suppose that $\xi:[a,b]\to\mathbb{R}_{\mathscr{F}}$, for each partition $\wp=\{\tau_0^*,\tau_1^*,\cdots,\tau_n^*\}$ of $[a,b]$ and for arbitrary points $\varepsilon_i\in[\tau_{i-1}^*,\tau_i^*]$, $1\le i\le n$, let $\Re_\wp=\sum_{i=1}^n\xi(\varepsilon_i)(\tau_i^*-\tau_{i-1}^*)$ and $\Delta=\max\limits_{1\le i\le n}|\tau_i^*-\tau_{i-1}^*|$. Then the definite integral of ξ_τ over $[a,b]$ is defined by $\int_a^b\xi(\tau)d\tau=\lim_{\Delta\to 0}\Re_\wp$ provided the limit exists in the metric space $(\mathbb{R}_{\mathscr{F}},D)$.

Theorem 2.4. [35] Let $\xi:[a,b]\to\mathbb{R}_{\mathscr{F}}$ be continuous fuzzy-valued function and put $[\xi(\tau)]^r=[\underline{\xi_r}(\tau),\overline{\xi_r}(\tau)]$ for each $r\in[0,1]$. Then $\int_a^b\xi(\tau)d\tau$ exists, belonging to $\mathbb{R}_{\mathscr{F}}$, where $\underline{\xi_r}$ and $\overline{\xi_r}$ are integrable functions on $[a,b]$, and $\left[\int_a^b\xi(\tau)d\tau\right]^r=\int_a^b[\xi(\tau)]^rd\tau=\left[\int_a^b\underline{\xi_r}(\tau)d\tau,\int_a^b\overline{\xi_r}(\tau)d\tau\right]$.

The generalized H-differentiability is used to extend the definitions of fuzzy fractional derivatives.

Definition 2.9. [36] Let $\xi:[a,b]\to\mathbb{R}_{\mathscr{F}}$ and $\xi\in C^F[a,b]\bigcap L^F[a,b]$. The fuzzy Riemann-Liouville fractional integral of order α is defined as $(J_{\alpha^+}^\alpha\xi(\tau))=\frac{1}{\Gamma(\alpha)}=\int_\alpha^\tau\frac{\xi(s)}{(\tau-s)^{1-\beta}}ds,\tau>\alpha$, where $0<\alpha\le 1$.

If we put ξ in its r-cut form, that is, $[\xi(\tau)]^r=[\underline{\xi_r}(\tau),\overline{\xi_r}(\tau)],r\in[0,1]$, then the fuzzy Riemann-Liouville fractional integral can be formed in parametric form as $[J_{\alpha^+}^\alpha]^r=\left[\frac{1}{\Gamma(\alpha)}\int_\alpha^\tau\frac{\underline{\xi_r}(s)}{(\tau-s)^{1-\beta}}ds,\frac{1}{\Gamma(\alpha)}\int_\alpha^\tau\frac{\overline{\xi_r}(s)}{(\tau-s)^{1-\beta}}ds\right]$.

Definition 2.10. [31] Assume that $\alpha\in(0,1]$ and $\xi\in C^F[a,b]\bigcap L^F[a,b]$ where $\xi[a,b]\to\mathbb{R}_{\mathscr{F}}$. Then ξ is Caputo's H-differentiable of order α at τ when $(D_{*\alpha^+}^\alpha\xi)(\tau)=\frac{1}{\Gamma(1-\alpha)}\int_\alpha^\tau\frac{\xi'(s)}{(\tau-s)^\beta}ds$ exists. Also, ξ is called Caputo $[(1)-\alpha]$-differentiable if ξ is (1)-differentiable, and ξ is Caputo $[(2)-\alpha]$-differentiable if ξ is (2)-differentiable.

Theorem 2.5. [31] Let $0<\alpha\le 1$ and $\xi\in AC^F[a,b]$. Then, the fuzzy Caputo fractional derivative exists almost everywhere on (a,b) and for all $r\in[0,1]$, we have $[(D_{*\alpha^+}^\alpha\xi)(\tau)]^r=\left[\frac{1}{\Gamma(1-\alpha)}\int_\alpha^\tau\frac{\underline{\xi_r'}(s)}{(\tau-s)^\alpha}ds,\frac{1}{\Gamma(1-\alpha)}\int_\alpha^\tau\frac{\overline{\xi_r'}(s)}{(\tau-s)^\alpha}ds\right]$ if ξ is (1)-differentiable, and $[(D_{*\alpha^+}^\alpha\xi)(\tau)]^r=\left[\frac{1}{\Gamma(1-\alpha)}\int_\alpha^\tau\frac{\overline{\xi_r'}(s)}{(\tau-s)^\alpha}ds,\frac{1}{\Gamma(1-\alpha)}\int_\alpha^\tau\frac{\underline{\xi_r'}(s)}{(\tau-s)^\alpha}ds\right]$ if ξ is (2)-differentiable.

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3 Fuzzy Fractional DEs of Order 2α

The goal of this section is to implement a relatively new analytical numerical technique, called the Fractional Reproducing Kernel Method (FRPSM), to handle the fuzzy differential equation of fractional order with fuzzy boundary conditions.

Consider the fractional fuzzy differential equation in following form

$$D^{2\alpha}_{*\alpha^+}\varphi(\tau) = \mathscr{F}(\tau, \varphi(\tau), D^{\alpha}_{*\alpha^+}\varphi(\tau)),\ \tau \in [a,b], \tag{1}$$

with the fuzzy boundary conditions

$$\varphi(a) = \delta,\ \varphi(b) = \gamma, \tag{2}$$

where $0 < a \leq 1$, $D^{2\alpha}_{*\alpha^+}$ is the fuzzy Caputo fractional derivative of order 2α, $\mathscr{F} : [a,b] \times \mathbb{R}_{\mathscr{F}} \times \mathbb{R}_{\mathscr{F}} \to \mathbb{R}_{\mathscr{F}}$ is continuous and compact fuzzy-valued function to be linear or nonlinear, $\delta, \gamma \in \mathbb{R}_{\mathscr{F}}$ and $\varphi(\tau)$ is unknown analytical function to be determined. Consequently, if $\mathscr{F}(\cdot)$ is a crisp function, then the solution $\varphi(\tau)$ of Equation (1) is a crisp. Otherwise, if $\mathscr{F}(\cdot)$ is a fuzzy function, then Equation (1) may possess only fuzzy solution $\varphi(\tau)$. Anyhow, we assume that $\mathscr{F}(\cdot)$ is a fuzzy function.

Let $\left[D^{2\alpha}_{*\alpha^+}\underline{\varphi_r}(\tau),\ D^{2\alpha}_{*\alpha^+}\overline{\varphi_r}(\tau)\right]$, $\left[\underline{\varphi_r}(\tau),\ \overline{\varphi_r}(\tau)\right]$, $\left[D^{\alpha}_{*\alpha^+}\underline{\varphi_r}(\tau),\ D^{\alpha}_{*\alpha^+}\overline{\varphi_r}(\tau)\right]$ and $\left[\underline{\mathscr{F}_r}(\cdot),\ \overline{\mathscr{F}_{2r}}(\cdot)\right]$ be the parametric forms of $D^{\alpha}_{*\alpha^+}\varphi(\tau)$, $\varphi(\tau)$, $D^{\alpha}_{*\alpha^+}\varphi(\tau)$ and $\mathscr{F}(\cdot)$, respectively. Let r-cut representation of $\mathscr{F}$ be $[\mathscr{F}(\tau, \varphi(\tau), D^{\alpha}_{*\alpha^+}\varphi(\tau))]^r = [\underline{\mathscr{F}_r}(\tau, [\varphi(\tau)]^r, [D^{\alpha}_{*\alpha}]^r),\ \overline{\mathscr{F}_r}(\tau, [\varphi(\tau)]^r, [D^{\alpha}_{*\alpha}]^r)]$, then the parametric form of Eq. (1) is given by

$$\begin{aligned}\left[D^{2\alpha}_{*\alpha^+}\varphi(\tau)\right]^r = &[\mathscr{F}(\tau, \varphi(\tau), D^{\alpha}_{*\alpha^+}\varphi(\tau))]^r,\\ &\tau \in [a,b],\ r \in [0,1],\end{aligned} \tag{3}$$

subject to fuzzy boundary conditions

$$[\varphi(a)]^r = [\underline{\delta_r},\ \overline{\delta_r}],\ [\varphi(b)]^r = [\underline{\gamma_r},\ \overline{\gamma_r}] \tag{4}$$

Consequently, if $\varphi(\tau)$ is Caputo $[(n,m)-\alpha]$-differentiable (for short ${}^{C}_{\alpha}(n,m)$ -differentiable) function for $\tau \in [a,b]$ and $n,m \in [0,1]$, then Eqs. (3) and (4) can be converted into a crisp system of second-order fractional differential equations, which is called the corresponding ${}^{C}_{\alpha}(n,m)$-system. Thereby, the four possible systems are given as follows.

- If $\varphi(\tau)$ is ${}^{C}_{\alpha}(1,1)$ -differentiable function, then

$$\begin{cases} D^{2\alpha}_{(1,1)}\underline{\varphi_r}(\tau) = \underline{\mathscr{F}_r}(\tau, [\varphi(\tau)]^r, [D^{\alpha}_{(1,1)}\varphi(\tau)]^r) \\ D^{2\alpha}_{(1,1)}\overline{\varphi_r}(\tau) = \overline{\mathscr{F}_r}(\tau, [\varphi(\tau)]^r, [D^{\alpha}_{(1,1)}\varphi(\tau)]^r) \end{cases} \tag{5}$$

- If $\varphi(\tau)$ is ${}^{C}_{\alpha}(1,2)$ -differentiable function, then

$$\begin{cases} D^{2\alpha}_{(1,2)}\underline{\varphi_r}(\tau) = \overline{\mathscr{F}_r}(\tau, [\varphi(\tau)]^r, [D^{\alpha}_{(1,2)}\varphi(\tau)]^r) \\ D^{2\alpha}_{(1,2)}\overline{\varphi_r}(\tau) = \underline{\mathscr{F}_r}(\tau, [\varphi(\tau)]^r, [D^{\alpha}_{(1,2)}\varphi(\tau)]^r) \end{cases} \tag{6}$$

- If $\varphi(\tau)$ is ${}^{C}_{\alpha}(2,1)$ -differentiable function, then

$$\begin{cases} D^{2\alpha}_{(2,1)}\underline{\varphi_r}(\tau) = \overline{\mathscr{F}_r}(\tau, [\varphi(\tau)]^r, [D^{\alpha}_{(2,1)}\varphi(\tau)]^r) \\ D^{2\alpha}_{(2,1)}\overline{\varphi_r}(\tau) = \underline{\mathscr{F}_r}(\tau, [\varphi(\tau)]^r, [D^{\alpha}_{(2,1)}\varphi(\tau)]^r) \end{cases} \tag{7}$$

- If $\varphi(\tau)$ is ${}^{C}_{\alpha}(2,2)$ -differentiable function, then

$$\begin{cases} D^{2\alpha}_{(2,2)}\underline{\varphi_r}(\tau) = \underline{\mathscr{F}_r}(\tau, [\varphi(\tau)]^r, [D^{\alpha}_{(2,2)}\varphi(\tau)]^r) \\ D^{2\alpha}_{(2,2)}\overline{\varphi_r}(\tau) = \overline{\mathscr{F}_r}(\tau, [\varphi(\tau)]^r, [D^{\alpha}_{(2,2)}\varphi(\tau)]^r) \end{cases} \tag{8}$$

with boundary conditions

$$\begin{cases} \underline{\varphi_r}(a) = \underline{\delta_r},\ \underline{\varphi_r}(b) = \underline{\gamma_r}, \\ \overline{\varphi_r}(b) = \overline{\delta_r},\ \overline{\varphi_r}(b) = \overline{\gamma_r}, \end{cases} \tag{9}$$

Remark 3.1. If $\varphi(\tau)$ is ${}^{C}_{\alpha}(n,m)$ -differentiable such that $[\varphi(\tau)]^r = [\underline{\varphi_r}(\tau),\ \overline{\varphi_r}(\tau)]$ is the (n,m)-solution of BVPs (3) and (4), in which $\underline{\varphi_r}(\tau)$ and $\overline{\varphi_r}(\tau)$ are solutions for $\tau \in [a,b]$ of the corresponding (n,m)-system for $n,m \in \{1,2\}$ and $r \in [0,1]$, then $\varphi(\tau)$ is the (n,m)-solution of fuzzy fractional BVPs (1) and (2).

The main goal of the next algorithm is to construct and approximate the possible fuzzy solutions of fuzzy fractional BVPs (1) and (2). To do so, let $\varphi(\tau)$ be (n,m)-differentiable and follow the procedure to illustrate our strategy in solving such proposed model.

Algorithm 3.1. A generalized (n,m)-solution of fuzzy fractional BVPs (1) and (2).
Input: The interval $[a,b]$, and the fuzzy numbers δ, γ.
Output: Possible (n,m)-solution over $[a,b]$.

To construct the (n,m)-solution of Eqs. (1) and (2), do the following main steps:

Step 1: Choose a case of differentiability $D^{2\alpha}_{(n,m)}\varphi(\tau)$ and $D^{\alpha}_{(n,m)}\varphi(\tau)$ as follows:

Case 1: If $\varphi(\tau)$ is ${}^{C}_{\alpha}(1,1)$ -differentiable function, then utilize $[D^{2\alpha}_{(1,1)}\varphi(\tau)]^r$ and $[D^{\alpha}_{(1,1)}\varphi(\tau)]^r = [D^{\alpha}_{(1,1)}\underline{\varphi}(\tau),\ D^{\alpha}_{(1,1)}\overline{\varphi}(\tau)]$ to convert the original fuzzy fractional problem (3) into its corresponding $(1,1)$-system of fractional order described

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in Eq. (5).

Case 2: If $\varphi(\tau)$ is ${}_{\alpha}^{C}(1,2)$ -differentiable function, then utilize $[D^{2\alpha}_{(1,2)}\varphi(\tau)]^r$ and $[D^{\alpha}_{(1,2)}\varphi(\tau)]^r = [D^{\alpha}_{(1,2)}\underline{\varphi}(\tau),\ D^{\alpha}_{(1,2)}\overline{\varphi}(\tau)]$ to convert the original fuzzy fractional problem (3) into its corresponding $(1,2)$-system of fractional order described in Eq. (6).
Case 3: If $\varphi(\tau)$ is ${}_{\alpha}^{C}(2,1)$-differentiable function, then utilize $[D^{2\alpha}_{(2,1)}\varphi(\tau)]^r$ and $[D^{\alpha}_{(2,1)}\varphi(\tau)]^r = [D^{\alpha}_{(2,1)}\underline{\varphi}(\tau),\ D^{\alpha}_{(2,1)}\overline{\varphi}(\tau)]$ to convert the original fuzzy fractional problem (3) into its corresponding $(2,1)$-system of fractional order described in Eq. (7).
Case 4: If $\varphi(\tau)$ is ${}_{\alpha}^{C}(2,2)$ -differentiable function, then utilize $[D^{2\alpha}_{(2,2)}\varphi(\tau)]^r$ and $[D^{\alpha}_{(2,2)}\varphi(\tau)]^r = [D^{\alpha}_{(2,2)}\underline{\varphi}(\tau),\ D^{\alpha}_{(2,2)}\overline{\varphi}(\tau)]$ to convert the original fuzzy fractional problem (3) into its corresponding $(2,2)$-system of fractional order described in Eq. (8).
Step 2: Solve the obtained (n,m)-system of fractional order subject to boundary conditions (9) for $\underline{\varphi_r}(\tau)$ and $\overline{\varphi_r}(\tau)$.
Step 3: Ensure that $\varphi(\tau)$ is a valid solution of the (n,m)-fractional system on $[a,b]$.
Step 4: Construct the (n,m)-solution, $\varphi(\tau)$, in which $[\varphi(\tau)]^r = [\underline{\varphi_r}(\tau),\ \overline{\varphi_r}(\tau)]$.

4 The Fractional RK Algorithm for Solving FFBVPs

To illustrate the implementation method in solving the obtained (n,m)-fractional system, we have firstly to define a linear invertible differential operator of fractional order from the space $W[a,b]$ to the space $\hat{W}[a,b]$ and then construct a complete orthogonal system to obtain a series expansion form of the approximate (n,m)-solution of the proposed model. To do this, one can define the linear operator $P_{ir} : W_2^3[a,b] \to W_2^1[a,b]$ such that $P_{ir}\varphi_{i,r}(\tau) = D^{2\alpha}\varphi_{ir}(\tau),\ i=1,2$. Consequently, if one set $\mathscr{F}_r = (\underline{F_r},\ \overline{F_r})^T,\ \varphi_r = (\underline{\varphi_r},\ \overline{\varphi_r})^T,\ D^{\alpha\varphi} = (D^{\alpha}\underline{\varphi_r},\ D^{\alpha}\overline{\varphi_r})^T$ and $\delta_r = (\underline{\delta_r},\ \overline{\delta_r})^T,\ \gamma_r = (\underline{\gamma_r},\ \overline{\gamma_r})$ as well as we define the operator $P_r : \underline{W}[a,b] \to \hat{W}[a,b]$ such that $P_r\varphi_r = diag(\underline{P_r}\varphi_r,\ \overline{P_r}\varphi_r)$, then the fuzzy fractional BVPs (3) and (4) can be equivalently converted into the following form:

$$P_r\varphi_r = \mathscr{F}_r(t,\varphi(\tau),D^{\alpha}\varphi(\tau)), \tag{10}$$

with the boundary conditions

$$\varphi_r(a) = \delta_r,\ \varphi_r(b) = \gamma_r, \tag{11}$$

where as $\varphi_r \in W[a,b]$ and $\mathscr{F}_r \in \hat{W}[a,b]$.

Remark 4.1. The linear operator $P_{ir} : W_2^3[a,b] \to W_2^1[a,b],\ i=1,2$ is a bounded. Thereby, $P_r : W[a,b] \to \hat{W}[a,b]$ is also bounded linear operator as follows

$$\begin{aligned}|P_{ir}\varphi_{ir}(\tau)| &= |D^{2\alpha}\varphi_{ir}(\tau)| = |\langle\varphi_{ir}(s),D^{2\alpha}\mathscr{T}_t^3(s)\rangle_{W_2^3}|\\ &\le \|D^{2\alpha\mathscr{T}_t^3}\|_{W_2^3}\|\varphi_{ir}\|_{W_2^3}\\ &= A_i^2\|\varphi_{ir}\|_{W_2^3},\ A_i\in\mathbb{R}\end{aligned}$$

and

$$\begin{aligned}\left|\frac{d}{dt}(P_{ir}\varphi_{ir}(\tau))\right| &= \left|\frac{d}{dt}(D^{2\alpha}\varphi_{ir}(\tau))\right|\\ &= \left|\left\langle\varphi_{i,r}(s),\frac{d}{dt}(D^{2\alpha}\mathscr{T}_t^3(s))\right\rangle_{W_2^3}\right|\\ &\le \left\|\frac{d}{dt}(D^{2\alpha}(T)_t^3)\right\|_{W_2^3}\|\varphi_{ir}\|_{W_2^3}\\ &= B_i\|\varphi_{ir}\|_{W_2^3},\ B_i\in\mathbb{R}\end{aligned}$$

Thus,

$$\begin{aligned}\|P_{ir}\varphi_{ir}\|^2_{W_2^1} &= \langle P_{ir}\varphi_{ir},P_{ir}\varphi_{ir}\rangle_{W_2^1}\\ &= \int_a^b\left([P_{ir}\varphi_{ir}(\eta)]^2+\left[\frac{d}{d\xi}P_{ir}\varphi_{ir}(\eta)\right]^2\right)d\eta\\ &\le \int_a^b(A_i^4+B_i^4)\|\varphi_{ir}\|^2_{W_2^3}d\eta\\ &\le (A_i^4+B_i^4)(b-a)\|\varphi_{ir}\|^2_{W_2^3}\end{aligned}$$

Let $C_i^2 = (A_i^4+B_i^4)(b-a)$, then $P_{ir},\ i=1,2$ is bounded. On the other hand, from boundedness of P_{ir}, one can show that P_r is also bounded for each $\varphi_r \in W[a,b]$ so that

$$\begin{aligned}\|P_r\varphi_r\|_{\hat{W}} &= \sqrt{\sum_{i=1}^{2}\|P_{ir}\varphi_{ir}\|^2_{W_2^1}} \le \sqrt{\sum_{i=1}^{2}\|P_{ir}\|^2\|\varphi_{ir}\|^2_{W_2^3}}\\ &\le \sqrt{\left(\sum_{i=1}^{2}\|P_{ir}\|^2\right)\left(\sum_{i=1}^{2}\|\varphi_{ir}\|^2_{W_2^3}\right)}\\ &\le \left(\sqrt{\sum_{i=1}^{2}\|P_{ir}\|^2}\right)\|\varphi_r\|_W \le M\|\varphi_r\|_W\end{aligned}$$

The reproducing kernel method adopted on the set of polynomials $\{\psi_{ir}(\tau)\}_{(i,j)=(0,1)}^{(\infty,2)}$ that forms a complete orthogonal system $W[a,b]$ such that $\psi_{i,j}(\tau) = P_r^*\zeta_{ij} = P_r^*(T_{ti}^1(\tau)e_j),\ i=0,1,2,\cdots,\ j=1,2,$ where η_{ij} is a complete orthogonal system of range space $\hat{W}[a,b],\ P_r^* = diag(\underline{P_r^*},\overline{P_{2r}^*})$ is the adjoint operator of P_r, $e_j,\ j=1,2,$ is the standard base such that

$e_1=(1,0)^T,\ e_2=(0,1)^T$, and $\{t_i\}_{i=0}^{\infty}$ is dense subset on the compact interval $[a,b]$. However, by Gram-Schmidt orthogonalization process of $\{\psi_{ij}(\tau)\}_{(i,j)=(1,1)}^{(\infty,2)}$, a complete orthogonal normal system $\{\overline{\psi}_{ij}(\tau)\}_{(i,j)=(1,1)}^{(\infty,2)}$ can be derived as,

$$\overline{\psi}_{ij}(\tau)=\sum_{l=1}^{i}\sum_{k=1}^{j}\sigma_{lk}^{ij}\psi_{lk}(\tau), \tag{12}$$

where the orthogonalization coefficients $\sigma_{lk}^{ij},\ \sigma_{lk}^{ij}>0$are given as follows:

$$\sigma_{11}^{ij}=\frac{1}{\|\psi_{1,1}\|_W},$$

$$\sigma_{lk}^{ij}=\frac{1}{\sqrt{\|\psi_{lk}\|_W^2-\sum_{p=1}^{l-1}\langle\psi_{lk}(\tau),\overline{\psi_{ij}(\tau)}\rangle_W^2}}\ (l<k),$$

$$\sigma_{lk}^{ij}=-\frac{1}{\sqrt{\|\psi_{lk}\|_W^2-\sum_{p=1}^{l-1}\langle\psi_{lk}(\tau),\overline{\psi_{ij}(\tau)}\rangle_W^2}}\ (\sigma_*^*)\ (k\le l),$$

where $\sigma_*^*=\sum_{p=k}^{l-1}\langle\psi_{lk}(\tau),\overline{\psi_{ij}(\tau)}\rangle_W\sigma_{pk}^{ij}$.

Remark 4.2. Since $\left\{\overline{\psi}_{ij}(\tau)\right\}_{(i,j)=(1,1)}^{(\infty,2)}$ forms a complete orthogonal system in $W[a,b]$, then $\varphi_r(\tau)$ can be expanded in the Fourier series such as $\varphi_r(\tau)=\sum_{i=1}^{\infty}\sum_{j=1}^{2}(x_r(\tau),\bar{\psi}_{ij}(\tau))_W\bar{\psi}_{ij}(\tau)$.

Theorem 4.1 Let $\{\tau_i\}_{i=1}^{\infty}$ be dense subset of $[a,b]$. Then the solution of Eq. (1) has the following form:

$$\varphi_r(\tau)=\sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\sigma_{lk}^{ij}\mathscr{F}_r(\tau_l,\varphi_r(\tau_l),D^{\alpha}\varphi_r(\tau_l))\bar{\psi}_{ij}(\tau). \tag{13}$$

Remark 4.3. Since $\varphi_r(t)\in W[a,b]$ and $W[a,b]$ is a Hilbert space, then $\sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\sigma_{lk}^{ij}\mathscr{F}_r(\tau_l,\varphi_r(\tau_l),D^{\alpha}\varphi_r(\tau_l))\bar{\psi}_{ij}(t)<\infty$.

Remark 4.4. The n-term approximation can be obtained by truncation of a finite term of Eq. (8) as follows

$$\varphi_r^n(\tau)=\sum_{i=1}^{n}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\sigma_{lk}^{ij}\mathscr{F}_r(\tau_l,\varphi_r(\tau_l),D^{\alpha}\varphi_r(\tau_l))\bar{\psi}_{ij}(\tau) \tag{14}$$

Algorithm 4.1. The following iterative process is presented for both linear and nonlinear case depending on the internal structure of the function $\mathscr{F}_r$.

Case 1: If $\mathscr{F}_r(.)$ is linear, then the exact and the approximate solutions can be directly obtained by Eqs. (10) and (11), respectively.

Case 2: If $\mathscr{F}_r(.)$ is nonlinear, then the exact and the approximate solutions can be obtained by the following algorithm:

Step 1: Define the operator π_{ijr} such that $\pi_{ijr}=\sum_{l=1}^{i}\sum_{k=1}^{j}\sigma_{lk}^{ij}\mathscr{F}_r(\tau_l,\varphi_r(\tau_l),D^{\alpha}\varphi_r(\tau_l))$.

Step 2: Set the solution of BVPs (10) and (11) as $\varphi_r(\tau)=\sum_{i=1}^{\infty}\sum_{j=1}^{2}\pi_{ijr}\bar{\psi}_{ij}(\tau)$.

Step 3: Set $\tau_1=a$, and set $\varphi_r^0(\tau_1)=\varphi_r(\tau_1)=\varphi_r(a)$, which is known from the BCs, that is, $\mathscr{F}_r(\tau_1,\varphi_r(\tau_1),\varphi_r'(\tau_1))$ is known.

Step 4: Set $B_{ijr}=\sum_{l=1}^{i}\sum_{k=1}^{j}\sigma_{lk}^{ij}\mathscr{F}_r\left(\tau_l,\varphi_r^{l-1}(\tau_l),D^{\alpha}\varphi_r^{l-1}(\tau_l)\right)$, and set $\varphi_r^{i}(\tau)=\sum_{l=1}^{i}\sum_{j=1}^{2}B_{ijr}\bar{\psi}_{ij}(\tau)$.

Step 5: Obtain the n-term approximations of $\varphi_r(t)$ as $\varphi_r^n(\tau)=\sum_{i=1}^{n}\sum_{j=1}^{2}B_{ijr}\bar{\psi}_{ij}(\tau)$.

Remark 4.5. We can guarantee that the approximate solution $\varphi_r^n(\tau)$ in Algorithm 4.1 satisfying the concerned BCs. As well as the $\varphi_r^n(\tau)=\sum_{i=1}^{n}\sum_{j=1}^{2}B_{ijr}\bar{\psi}_{ij}(\tau)$ converges uniformly to the exact solution $\varphi_r(\tau)$ in sense of the norm of $||.||_{W[a,b]}$ whenever $\{\tau_i\}_{i=1}^{\infty}$ is dense subset of $[a,b]$ and $||\varphi_r^n||_W\le\infty$.

Corollary 4.1. If $||\varphi_r^{n-1}(\tau)-\varphi_r(\tau)||_W\to 0,\ \tau_n\to\tau$, $||\varphi_r^{n-1}||_W\le\infty$ as $n\to\infty$, and $\mathscr{F}_r(\tau,\varphi_r(\tau),D^{\alpha}\varphi_r(\tau))$ is continuous, then $\mathscr{F}_r\left(\tau_n,\varphi_r^{n-1}(\tau_n),D^{\alpha}\varphi_r^{n-1}(\tau_n)\right)\to\mathscr{F}_r(\tau,\varphi_r(\tau),D^{\alpha}\varphi_r(\tau))$ as $n\to\infty$.

Theorem 4.2. Let Δ_n be the error between the exact, $\varphi_r(t)$, and the approximate, $\varphi_r^n(\tau)$, the solutions are represented by Eqs. (10) and (11), respectively, such that $\Delta_n=||\varphi_r-\varphi_r^n||_W$. Then, the sequence of number Δ_n is monotonically decreasing in $||.||_{W[a,b]}$ and vanishes as soon as $n\to\infty$.

5 Numerical Simulation

To show the behavior, properties, efficiency and applicability of the present new method, two examples consisting of linear and nonlinear problems are showed numerically.

Example 5.1 Consider the following linear fuzzy fractional differential equation

$$D^{2\alpha}\varphi(\tau)=D^{\alpha}\varphi(\tau)+[r-1,1-r]+1,\tau\in[0,1],\ 0<\alpha\le 1 \tag{15}$$

with fuzzy boundary conditions

$$[\varphi(0)]^r=[0,0]\,,and,\ [\varphi(1)]^r=[r-1,1-r], \tag{16}$$

Based upon differentiability types, the possible fuzzy approximation of the fuzzy fractional Dirichlet BVPs (5.1) and (5.2) can be written for one of the following cases of fractional system in their corresponding parametric forms:

Table 5.1: Absolute errors of $\underline{\varphi_r}(\tau)$ of Example 5.1, case 1.

τ	$r=0.25$	$r=0.5$	$r=0.75$
0.2	7.6222×10^{-8}	1.5669×10^{-7}	2.3715×10^{-7}
0.4	1.1559×10^{-7}	2.5675×10^{-7}	3.9791×10^{-7}
0.6	1.3031×10^{-7}	2.9951×10^{-7}	4.6870×10^{-7}
0.8	1.0700×10^{-7}	2.4447×10^{-7}	3.8193×10^{-7}

Table 5.2: Absolute errors of $\overline{\varphi_r}(\tau)$ of Example 5.1, case 1.

τ	$r=0.25$	$r=0.5$	$r=0.75$
0.2	6.7732×10^{-8}	1.4820×10^{-7}	2.2867×10^{-7}
0.4	6.4457×10^{-8}	2.0562×10^{-7}	3.4678×10^{-7}
0.6	5.2560×10^{-8}	2.2175×10^{-7}	3.9094×10^{-7}
0.8	4.6088×10^{-8}	1.8355×10^{-7}	3.2101×10^{-7}

Table 5.3: Absolute errors of $\underline{\varphi_r}(\tau)$ of Example 5.1, case 2.

τ	$r=0.25$	$r=0.5$	$r=0.75$
0.2	6.5371×10^{-8}	4.3541×10^{-8}	4.3541×10^{-8}
0.4	9.4895×10^{-8}	6.3026×10^{-8}	6.3026×10^{-8}
0.6	9.2815×10^{-8}	6.1517×10^{-8}	6.1517×10^{-8}
0.8	6.1188×10^{-8}	4.0510×10^{-8}	4.0510×10^{-8}

Case 1. The (1,1)-approximation of the following corresponding (1,1)-fractional system when $\varphi(\tau)$ is (1,1)-differentiability is

$$D^{2\alpha}_{(1,1)}\underline{\varphi_r}(\tau)=D^{\alpha}_{(1,1)}\underline{\varphi_r}(\tau)+r, \tag{17}$$

$$D^{2\alpha}_{(1,1)}\overline{\varphi}_r(\tau)=D^{\alpha}_{(1,1)}\overline{\varphi}_r(\tau)+2-r.$$

The corresponding (1,1)-fractional system has the exact solutions at $\alpha=1$ as follows

$$\begin{aligned}\underline{\varphi_r}(\tau)=&(e-1)^{-1}\left[1-e^{\tau}+r(-2+2e^{\tau}+\tau-e\tau)\right],\\ \overline{\varphi_r}(\tau)=&(1-e)^{-1}\left[3-3e^{\tau}-2\tau+2e\tau-r(2-2e^{\tau}-\tau+e\tau)\right].\end{aligned} \tag{18}$$

Case 2. The (1,2)-approximation of the following corresponding (1,2)-fractional system when $\varphi(\tau)$ is (1,2)-differentiability is

$$\begin{aligned}D^{2\alpha}_{(1,2)}\underline{\varphi_r}(\tau)=&D^{\alpha}_{(1,2)}\overline{\varphi_r}(\tau)+2-r,\\ D^{2\alpha}_{(1,2)}\overline{\varphi_r}(\tau)=&D^{\alpha}_{(1,2)}\underline{\varphi_r}(\tau)+r.\end{aligned} \tag{19}$$

The corresponding (1,2)-fractional system has the exact solutions at $\alpha=1$ as follows

$$\begin{aligned}\underline{\varphi_r}(\tau)=&\frac{e^{-\tau}}{1-e}\left[2(r-1)e-e^{2\tau}+e^{\tau+1}(2-r(2-\tau))\right.\\ &\left.+e^{\tau}(1-r\tau)\right],\\ \overline{\varphi_r}(\tau)=&\frac{e^{-\tau}}{e-1}\left[2(r-1)e+e^{2\tau}+e^{\tau+1}(2+r(-2+\tau)-2\tau)\right.\\ &\left.-e^{\tau}(1-(2-r)\tau)\right].\end{aligned} \tag{20}$$

Case 3. The (2,1)-approximation of the following corresponding (2,1)-fractional system when $\varphi(\tau)$ is (2,1)-differentiability is

$$\begin{aligned}D^{2\alpha}_{(2,1)}\underline{\varphi_r}(\tau)=&D^{\alpha}_{(2,1)}\underline{\varphi_r}(\tau)+2-r,\\ D^{2\alpha}_{(2,1)}\overline{\varphi_r}(\tau)=&D^{\alpha}_{(2,1)}\overline{\varphi_r}(\tau)+r.\end{aligned} \tag{21}$$

The corresponding (2,1)-fractional system has the exact solutions at $\alpha=1$ as follows

$$\begin{aligned}\underline{\varphi_r}(\tau)=&(1-e)^{-1}\left[1-e^{\tau}-(2-r)\tau+(2-r)e\tau\right],\\ \overline{\varphi_r}(\tau)=&(1-e)^{-1}\left[1-e^{\tau}-r\tau+re\tau\right].\end{aligned} \tag{22}$$

Case 4. The (2,2)-approximation of the following corresponding (2,2)-fractional system when $\varphi(\tau)$ is (2,2)-differentiability is

$$\begin{aligned}D^{2\alpha}_{(2,2)}\underline{\varphi_r}(\tau)=&D^{\alpha}_{(2,2)}\overline{\varphi_r}(\tau)+r,\\ D^{2\alpha}_{(2,2)}\overline{\varphi_r}(\tau)=&D^{\alpha}_{(2,2)}\underline{\varphi_r}(\tau)+2-r.\end{aligned} \tag{23}$$

The corresponding (2,2)-fractional system has the exact solutions at $\alpha=1$ as follows

$$\begin{aligned}\underline{\varphi_r}(\tau)=&(1-e)^{-1}\left[1-e^{\tau}-(2-r)\tau+(2-r)e\tau\right],\\ \overline{\varphi_r}(\tau)=&(1-e)^{-1}\left[1-e^{\tau}-r\tau+re\tau\right].\end{aligned} \tag{24}$$

subject to the boundary conditions

$$\begin{cases}\underline{\varphi_r}(0)=0, & \underline{\varphi_r}(1)=r-1,\\ \overline{\varphi_r}(0)=0, & \overline{\varphi_r}(1)=1-r.\end{cases} \tag{25}$$

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Table 5.4: Absolute errors of $\overline{\varphi_r}(\tau)$ of Example 5.1, case 2.

τ	$r=0.25$	$r=0.5$	$r=0.75$
0.2	6.5607×10^{-8}	4.3777×10^{-8}	2.1948×10^{-8}
0.4	9.6315×10^{-8}	6.4447×10^{-8}	3.2579×10^{-8}
0.6	9.4975×10^{-8}	6.3677×10^{-8}	3.2378×10^{-8}
0.8	6.2880×10^{-8}	4.2202×10^{-8}	2.1524×10^{-8}

Table 5.5: Numerical results of $\varphi_r^n(\tau)$ of Example 5.2, case 1.

	τ_i	$r=0.25$	$r=0.50$	$r=0.75$
$\underline{\varphi_r}(\tau)$	0.25	0.1072001858	0.1197001858	0.1322001858
	0.50	0.1072001858	0.1197001858	0.1322001858
	0.75	-0.056908371	-0.031908371	-0.006908371
	τ_i	$r=0.25$	$r=0.50$	$r=0.75$
$\overline{\varphi_r}(\tau)$	0.25	0.2822001858	0.2697001858	0.2572001858
	0.50	0.2828826523	0.2641326523	0.2453826523
	0.75	0.2930916291	0.2680916291	0.2430916291

Table 5.6: Numerical results of $\varphi_r^n(\tau)$ of Example 5.2, case 2.

	τ_i	$r=0.25$	$r=0.50$	$r=0.75$
$\underline{\varphi_r}(\tau)$	0.25	0.1072001874	0.1197001874	0.1322001874
	0.50	0.0203826546	0.0391326546	0.0578826546
	0.75	-0.056908369	-0.031908369	-0.006908369
	τ_i	$r=0.25$	$r=0.50$	$r=0.75$
$\overline{\varphi_r}(\tau)$	0.25	0.2822001874	0.2697001874	0.2572001874
	0.50	0.2828826546	0.2641326546	0.2453826546
	0.75	0.2930916307	0.2680916307	0.2430916307

Applying the RKHS approach with nodes $\tau_i=\frac{i-1}{n-1}, i=1,2,\ldots,n, 0\le\tau\le1$, and using the kernel function $\mathscr{T}_t^2(s)$ in the space $W[0,1]$, whereas the approximate solution, $\varphi_r^n(\tau)\in W[0,1]$, is given as, $\varphi_r^n(\tau)=\sum_{i=1}^n\sum_{j=1}^2 B_{ijr}\bar{\psi}_{ij}(\tau)$. The absolute errors of $\varphi_r^n(\tau)$ are given for some selected grid points, τ_i, at $\alpha=1$, and $n=101$ with different values of r, and are summarized for case 1 in Table 5.1, and Table 5.2, as well as for case 2 in Table 5.3, and Table 5.4.

The existence of (n,m)-RK solution of the corresponding (n,m)-fractional system for fuzzy fractional BVPs (5.1) and (5.2) is discussed as follows,

- For case 1 and case 2, the RK solution, $\varphi_r^{101}(\tau)$ at $\alpha=1$, leads to $\underline{\varphi_r}(\tau)\le\overline{\varphi_r}(\tau)$ for each $\tau\in[0,1]$. Thereby, it can be concluded that $\left[\varphi^{101}(\tau)\right]^r$ has valid level sets for $(1,1)$-fractional system, that is, it defines fuzzy-valued functions for each $\tau\in[0,1]$. In addition, $\left[\varphi^{101}(\tau)\right]^r$ has also valid level sets for for $(1,2)$-fractional system for each $\tau\in[0,1]$.
- For case 3 and case 4, the RK solution, $\varphi_r^{101}(\tau)$ at $\alpha=1$, leads to $\underline{\varphi_r}(\tau)\le\overline{\varphi_r}(\tau)$ and $D^\alpha\underline{\varphi_r}(\tau)\le D^\alpha\overline{\varphi_r}(\tau)$ for each $\tau\in[0,1]$. Thereby, by using characterization theorem, it can be concluded that $[D^{2\alpha}_{(2,1)}\varphi^{101}(\tau)]r$ and $[D^{2\alpha}_{(2,2)}\varphi^{101}(\tau)]r$ have no valid level sets for $(2,1)$-fractional and $(2,2)$-fractional systems.

Example 5.2 Consider the following nonlinear fuzzy fractional differential equation

$$D^{2\alpha}\varphi(\tau)=\varphi(\tau)+\varphi^3(\tau)+f(\tau),\tau\in[0,1],\quad 0<\alpha\le1 \tag{26}$$

with the fuzzy boundary conditions

$$[\varphi(0)]^r=\left[\frac{r}{40}-\frac{1}{20},\frac{1}{20}-\frac{r}{40}\right]\ and\ [\varphi(1)]^r=\left[\frac{r}{8},\frac{1}{2}-\frac{r}{8}\right]. \tag{27}$$

where μ is trapezoidal fuzzy number whose membership function

$$\mu(s)=\begin{cases}10s+2, & -1/5\le s\le-1/10,\\ 2-10s, & 1/10\le s\le1/5, \\ 0 & o.w.,\end{cases}\quad ,s\in\mathbb{R},$$

Here, $[f(\tau)]^r=\Big[\left(\left(\tau+\frac14\right)\overline{\mu_r}-\frac14e^{-\tau}\right)^3+\left(\tau+\frac14\right)\underline{\mu_r}\Big],$
$\left(\left(\tau+\frac14\right)\underline{\mu_r}-\frac14e^{-\tau}\right)^3+\left(\tau+\frac14\right)\overline{\mu_r},$

Based upon differentiability types, the possible fuzzy approximation of the fuzzy fractional BVPs (5.12) and (5.13) can be written for one of the following cases of

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fractional system in their corresponding parametric forms:

Case 1. The corresponding (1,1)-fractional system and (2,2)-fractional system whenever $\varphi(\tau)$ is (1,1)-differentiability and (2,2)-differentiability are given as,

$$\begin{aligned} D^{2\alpha}_{(n,m)}\underline{\varphi_r}(\tau) &= \overline{\varphi_r}(\tau) + \underline{\varphi_r^3}(\tau) + \underline{f_r}(\tau), \\ D^{2\alpha}_{(n,m)}\overline{\varphi_r}(\tau) &= \underline{\varphi_r}(\tau) + \overline{\varphi_r^3}(\tau) + \overline{f_r}(\tau). \end{aligned} \tag{28}$$

Case 2. The corresponding (1,2)-fractional system and (2,1)-fractional system whenever $\varphi(\tau)$ is (1,2)-differentiability and (2,1)-differentiability are given as,

$$\begin{aligned} D^{2\alpha}_{(n,m)}\underline{\varphi_r}(\tau) &= \underline{\varphi_r}(\tau) + \overline{\varphi_r^3}(\tau) + \overline{f_r}(\tau), \\ D^{2\alpha}_{(n,m)}\overline{\varphi_r}(\tau) &= \overline{\varphi_r}(\tau) + \underline{\varphi_r^3}(\tau) + \underline{f_r}(\tau). \end{aligned} \tag{29}$$

subject to the boundary conditions

$$\begin{cases} \underline{\varphi_r}(0) = \frac{r}{40} - \frac{1}{20}, & \underline{\varphi_r}(1) = \frac{r}{8}, \\ \overline{\varphi_r}(0) = \frac{1}{20} - \frac{r}{40}, & \overline{\varphi_r}(1) = \frac{1}{2} - \frac{r}{8}. \end{cases} \tag{30}$$

The corresponding (n,m)-fractional system has the exact solutions at $\alpha = 1$ as follows

$$\varphi(\tau) = \frac{1}{4}e^{-\tau} + \left(\tau + \frac{1}{4}\right)\mu. \tag{31}$$

Applying the RKHS approach with nodes $\tau_i = \frac{i-1}{n-1}$, $i = 1,2,\ldots,n, 0 \le \tau \le 1$, and using the kernel function $\mathscr{T}_t^2(s)$ in the space $W[0,1]$, whereas the approximate solution, $\varphi_r^n(\tau) \in W[0,1]$, is given as, $\varphi_r^n(\tau) = \sum_{i=1}^n \sum_{j=1}^2 B_{ijr}\bar{\Psi}_{ij}(\tau)$. The numerical results of $\varphi_r^n(\tau)$ of Example 5.2, case 1 and case 2, for selected grid points, τ_i, at $n = 101$ and different values of r-cut representation with fixed value of $\alpha = 1$ are given in Table 5.6 and Table 5.7.

6 Perspective

In this paper, we have proposed and applied analytical-numerical technique, called RKM, to handle a class of fuzzy differential equations of fractional in Caputo sense. The solution's methodology is based on generating the orthogonal-basis functions yielding from the reproducing-kernel functions obtained. Such orthonormal-basis is constructed to formulate and utilize the analytic as well as numeric solutions, and to provide efficiently convergent approximate series solutions with easily computable coefficients without employing linearization or perturbation. The behavior of approximate solutions for fixed value of r and different values of fractional-order α or for fixed value of fractional order α and different values of r-cut representation are shown quantitatively as well graphically. The numerical representations indicate the complete validity, reliability and efficiency of the presented method with a great potential in scientific applications.

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Mohammad Abdel all is an assistant professor of Applied Mathematics at the Faculty of Art and Science, Middle East University, Jordan. He received his Ph.D. degree from the University of Jordan. He received his MSc in mathematics from the University of Jordan. His main research interests are in fuzzy differential equations and fractional calculus.

Nisrein Abu-Darwish is a Lecturer of computer science at Ajloun college, Al-Balqa Applied University. She received her BSc degree in computer science from Al-Hussein Bin Talal University, Ma'an, Jordan. She received her MSc degree in computer science from the University of Jordan, Amman, Jordan. Her main research interests are in algorithms, networks, and databases. She is currently working on cloud computing testing research.

Omar Abu Arqub received his Ph.D. degree in the Applied Mathematics from the University of Jordan in 2008. He then began work at Al-Balqa Applied University as assistant professor of applied mathematics and promoted to full Professor in 2017. His research interests focus on the following fields: Numerical Analysis and Computational Optimization.

Mohammed Al-Smadi received his Ph.D. from the University of Jordan, Amman in 2011. He then began work as assistant professor of applied mathematics in Qassim University. Currently, Dr. Al-Smadi is working as assistant professor of applied mathematics (since Sep. 2013) at Al-Balqa Applied University and he promoted to associate professor in May 2017. His research interests include fractional calculus, numerical analysis, applied mathematics, and fuzzy calculus.

Shaher Momani received his Ph.D from the university of Wales (UK) in 1991. He then began work at Mutah university in 1991 as assistant professor of applied mathematics and promoted to full Professor in 2006. He left Mutah university to the university of Jordan in 2009 until now. His research interests focus on the numerical solution of fractional differential equations in fluid mechanics, non-Newtonian fluid mechanics and numerical analysis. Prof. Momani has written over 250 research papers and warded several national and international prizes. Also, he was classified as one of the top ten scientists in the world in fractional differential equations according to ISI web of knowledge.

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