Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/130323

Quintic Non-polynomial Spline Method for Solving the Time Fractional Biharmonic Equation

Adel R. Hadhoud

Department of Mathematics and Computer Science, Faculty of Science, Menoufia University, Egypt

Received: 2 Mar. 2019, Revised: 12 Apr. 2019, Accepted: 25 Apr. 2019
Published online: 1 May 2019

Abstract: In this paper, we introduce a general framework of the quintic non-polynomial spline method for solving the time fractional biharmonic equation. Applying the concept of Von Neumann, the stability of the method is investigated and it is shown that the proposed technique is unconditionally stable. The effectiveness and the accuracy of the method are illustrated using a test problem.

Keywords: Quintic non-polynomial spline method, time dependent fractional biharmonic equation, Von Neumann method

1 Introduction

Biharmonic partial differential equation is considered to be one of the important classes of partial differential equations (PDEs) which arise in both physics and in the modeling of many engineering applications. For example, it appears in the bending behavior of a thin elastic rectangular plate, as might be encountered in ship design and manufacture, or the equilibrium of an elastic rectangle can be formulated in terms of the biharmonic equation. Also Stokes flow of a viscous fluid in a rectangular cavity under the influence of the motion of the walls can be described in terms of the solution of this equation. A more recent application of the biharmonic equation has been in the area of geometric and functional design, where it has been used as a mapping to produce efficient mathematical descriptions of surfaces in a physical space. Let Ω be a bounded simply connected region in the plane. We then consider the following boundary value problem for the biharmonic equation

$$\Delta^2 u = f, \qquad \text{on} \qquad \Omega,$$

$$u = g, \qquad \text{on} \qquad \Gamma,$$

$$\frac{\partial u}{\partial n} = h, \qquad \text{on} \qquad \Gamma,$$

where, Δ^2, Ω is the biharmonic operator in the plane, $\frac{\partial u}{\partial n}$ denotes the outward normal derivative of u on Γ and f, g and h are the known functions, for more details of this biharmonic partial differential equation, see Heydari et al. [1]. Many of the applications of the biharmonic equation stem from the consideration of the more complex mechanical and physical processes involving solids and fluids. One of the earliest applications of the biharmoic equation deals with the classical theory of flexure of elastic plates [2]. Fractional differential equations are generalized from integer order ones, which are obtained by replacing integer order derivatives by fractional order ones [3]. The fractional differential equations show many advantages over integer order differential equations in the simulation of natural physical processes and dynamical systems. Many researchers introduced different effective methods for solving fractional partial differential equations such as differential transform method, the homotopy perturbation method, the fractional complex and the variational iteration method, and others, see for example [4] and references therein. Recently, a computational method for approximating solution of the above fractional biharmonic equation with fractional derivatives in the domain Ω, concerning the solution satisfying the equation by combining the shifted Chebyshev polynomials together with their operational matrix of fractional derivatives was proposed by Heydari et al. [1]. In this paper, we shade lights on the attempt of using non-polynomial spline to get an accurate numerical solution to the time fractional biharmonic equation given

* Corresponding author e-mail: adelhadhoud_2005@yahoo.com

by [5,6]

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+\frac{\partial^{4}u(x,t)}{\partial x^{4}}=g(x,t),\ a\leq x\leq b,\ 0\leq\alpha\leq 1, \tag{1}$$

where $g(x,t)$ is a source term. The boundary conditions associated with (1) are assumed to be of the form

$$\begin{aligned} &u(a,t)=\beta_1(t),\ u(b,t)=\beta_2(t),\ u_{xx}(a,t)=\beta_3(t),\\ &u_{xx}(b,t)=\beta_4(t)\ t>0, \end{aligned} \tag{2}$$

and the initial condition is

$$u(x,0)=f(x),\ a\leq x\leq b. \tag{3}$$

The non-polynomial spline function, T_5, as defined in [7]-[12] has the form

$$T_5=span\left\{1,x,x^2,x^3,\sin\omega x,\cos\omega x\right\},$$

where ω is the frequency of the trigonometric part of the spline functions. A novel method depending on the use of the non-polynomial splines T_5 is derived in the first section. In the second section, the truncation error of the method is theoretically analyzed. In addition, in the third section the stability analysis is discussed. Using Von Neumann method, the proposed method is shown to be conditionally stable. In the end we give a numerical example for the biharmonic equation.

2 Analysis of the method

To derive the method, we select the mesh constants h and k, so that we have the mesh points (x_i,t_j) which are defined by $x_i=a+ih$, $i=0,1,\ldots,N+1$ and $t_j=jk$, $j=0,1,\ldots$. Let ${Z_i}^j$ be an approximation to $u(x_i,t_j)$, obtained by the segment $P_i(x,t_j)$ of the mixed spline function passing through the points (x_i,Z_i^j) and (x_{i+1},Z_{i+1}^j). Each segment has the form [4]

$$\begin{aligned} P_i(x,t_j)=&a_i(t_j)\sin\omega(x-x_i)+b_i(t_j)\cos\omega(x-x_i)\\ &+c_i(t_j)(x-x_i)^3+d_i(t_j)(x-x_i)^2\\ &+e_i(t_j)(x-x_i)+f_i(t_j),\quad i=0,1,\cdots,N. \end{aligned} \tag{4}$$

To obtain expressions for the coefficients of (4) in terms of ${Z_i}^j$, Z_{i+1}^j, ${D_i}^j$, D_{i+1}^j, ${S_i}^j$, and S_i^{j+1}, we first define

$$P_i(x_i,t_j)=Z_i^j, P_i(x_{i+1},t_j)=Z_{i+1}^j, {P_i}^{(1)}(x_i,t_j)=D_i^j,$$

$${P_i}^{(1)}(x_{i+1},t_j)=D_{i+1}^j, {P_i}^{(4)}(x_i,t_j)=S_i^j, {P_i}^{(4)}(x_{i+1},t_j)=S_{i+1}^j. \tag{5}$$

Using Eqs. in (5) and Eq. (4), we get

$$b_i+f_i=Z_i^j,$$

$$a_i\sin\theta+b_i\cos\theta+c_ih^3+d_ih^2+e_ih+f_i=Z_{i+1}^j,$$

$$a_i\omega+e_i=D_i^j,$$

$$a_i\theta\cos\theta-b_i\theta\sin\theta+3c_ih^2+2d_ih+e_i=D_{i+1}^j,$$

$$\omega^4b_i=S_i^j,$$

$$\omega^4a_i\sin\omega+\omega^4b_i\cos=S_{i+1}^j,$$

where $\theta=\omega h$. After solving the previous six equations, we get expressions as follow:

$$a_i=\frac{h^4\left(S_{i+1}^j-S_i^j\cos\theta\right)}{\theta^4\sin\theta\}},\ b_i=\frac{h^4S_i^j}{\theta^4},$$

$$\begin{aligned} c_i=&-\frac{2\left(Z_{i+1}^j-Z_i^j\right)}{h^3}+\frac{D_{i+1}^j+D_i^j}{h^2}+\frac{2h\left(S_{i+1}^j-S_i^j\right)}{\theta^4}\\ &-\frac{h\left(S_{i+1}^j-S_i^j\right)(1+\cos\theta)}{\theta^3\sin\theta}, \end{aligned}$$

$$\begin{aligned} d_i=&\frac{3\left(Z_{i+1}^j-Z_i^j\right)}{h^2}-\frac{D_{i+1}^j+2D_i^j}{h}-\frac{3h^2\left(S_{i+1}^j-S_i^j\right)}{\theta^4}\\ &+\frac{h^2\left(S_{i+1}^j(2+\cos\theta)-S_i^j(1+2\cos\theta)\right)}{\theta^3\sin\theta}, \end{aligned} \tag{6}$$

$$e_i=D_i^j-\frac{h^3\left(S_{i+1}^j-S_i^jcos\theta\right)}{\theta^3\quad sin\theta},\ f_i=Z_i^j-\frac{h^4S_i^j}{\theta^4}.$$

From the continuity condition of the second and third derivatives at $x=x_i$, that is $P_{i-1}^{(n)}(x_i,t_j)=P_i^{(n)}(x_i,t_j)$ where n=2 and 3, we obtain the relations:

$$\begin{aligned} &-(\omega^2\sin\omega)a_{i-1}-(\omega^2\cos\omega)b_{i-1}+6hc_{i-1}+2d_{i-1}\\ &=-\omega^2b_i+2d_i, \end{aligned} \tag{7}$$

$$-(\omega^3\cos\theta)a_{i-1}+(\omega^3\sin\theta)b_{i-1}+6c_{i-1}=-\omega^3a_i+6c_i. \tag{8}$$

Using expressions (6), Eqs. (7) and (8) give us

$$\begin{aligned} D_{i+1}^j+4D_i^j+D_{i-1}^j=&\frac{3\left(Z_{i+1}^j-Z_{i-1}^j\right)}{h}-\frac{3h^3\left(S_{i+1}^j-S_{i-1}^j\right)}{\omega^4}\\ &+\frac{h^3\left(S_{i+1}^j-S_{i-1}^j\right)(2+\cos\theta)}{\theta^3\sin\theta}, \end{aligned} \tag{9}$$

$$
\begin{aligned}
D_{i+1}^j - D_{i-1}^j =& \frac{2\left(Z_{i+1}^j - 2Z_i^j + Z_{i-1}^j\right)}{h} \\
&+ \frac{h^3\left(S_{i+1}^j - 2S_i^j\cos\theta + S_{i-1}^j\right)}{6\theta\sin\theta} \\
&- \frac{2h^3\left(S_{i+1}^j - 2S_i^j + S_{i-1}^j\right)}{\theta^4} \\
&+ \frac{h^3\left(S_{i+1}^j - 2S_i^j + S_{i-1}^j\right)(1+\cos\theta)}{\theta^3\sin\theta}.
\end{aligned} \tag{10}
$$

To eliminate D's, we use the continuity of the second and third derivatives at $x = x_{i-1}$ and $x = x_{i+1}$, that are

$$P_{i-2}^{(n)}(x_{i-1}, t_j) = P_{i-1}^{(n)}(x_{i-1}, t_j),$$

and

$$P_i^{(n)}(x_{i+1}, t_j) = P_{i+1}^{(n)}(x_{i+1}, t_j),$$

where $n = 2$ and 3**.** In other words ***i*** is replaced by ***i-1*** and ***i+1*** in Eqs. (9) and (10) respectively:

$$
\begin{aligned}
D_i^j + 4D_{i-1}^j + D_{i-2}^j =& \frac{3\left(Z_i^j - Z_{i-2}^j\right)}{h} - \frac{3h^3\left(S_i^j - S_{i-2}^j\right)}{\theta^4} \\
&+ \frac{h^3\left(S_i^j - S_{i-2}^j\right)(2+\cos\theta)}{\theta^3\sin\theta},
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
D_i^j - D_{i-2}^j =& \frac{2\left(Z_i^j - 2Z_{i-1}^j + Z_{i-2}^j\right)}{h} \\
&+ \frac{h^3\left(S_i^j - 2S_{i-1}^j\cos\theta + S_{i-2}^j\right)}{6\theta\sin\theta} \\
&- \frac{2h^3\left(S_i^j - 2S_{i-1}^j + S_{i-2}^j\right)}{\theta^4} \\
&+ \frac{h^3\left(S_i^j - 2S_{i-1}^j + S_{i-2}^j\right)(1+\cos\theta)}{\theta^3\sin\theta},
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
D_{i+2}^j + 4D_{i+1}^j + D_i^j =& \frac{3\left(Z_{i+2}^j - Z_i^j\right)}{h} \\
&- \frac{3h^3\left(S_{i+2}^j - S_i^j\right)}{\theta^4} \\
&+ \frac{h^3\left(S_{i+2}^j - S_i^j\right)(2+\cos\theta)}{\theta^3\sin\theta},
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
D_{i+2}^j - D_i^j =& \frac{2(Z_{i+2}^j - 2Z_{i+1}^j + Z_i^j)}{h} \\
&+ \frac{h^3(S_{i+2}^j - 2S_{i+1}^j\cos\theta + S_i^j)}{6\theta\sin\theta} \\
&- \frac{2h^3(S_{i+2}^j - 2S_{i+1}^j + S_i^j)}{\theta^4} \\
&+ \frac{h^3(S_{i+2}^j - 2S_{i+1}^j + S_i^j)(1+\cos\theta)}{\theta^3\sin\theta}.
\end{aligned} \tag{14}
$$

If we multiply Eqs. (10) and (13) and add Eqs. (10)-(14), we obtain the following relation

$$
\begin{aligned}
&(1-1)D_{i+2}^j + (4-4)D_{i+1}^j + (1+1-1-1)D_i^j + (-4+4)\times \\
&D_{i-1}^j + (-1+1)D_{i-2}^j = \frac{1}{h}[(-3+2)Z_{i-2}^j + (8-4)Z_{i-1}^j \\
&+ (-16+3+2+3+2)Z_i^j + (8-4)Z_{i+1}^j + (-3+2)Z_{i+2}^j] \\
&+ h^3[(\frac{3}{\theta^4} - \frac{2+\cos\theta}{\theta^3\sin\theta} + \frac{1}{6\theta\sin\theta} - \frac{2}{\theta^4} + \frac{1+\cos\theta}{\theta^3\sin\theta})\times \\
&(S_{i-2}^j + S_{i+2}^j)] + h^3[(\frac{4}{6\theta\sin\theta} - \frac{8}{\theta^4} + \frac{4+4\cos\theta}{\theta^3\sin\theta} \\
&- \frac{2\cos\theta}{6\theta\sin\theta} + \frac{4}{\theta^4} - \frac{2+2\cos\theta}{\theta^3\sin\theta})(S_{i-1}^j + S_{i+1}^j)] \\
&+ h^3\left[\left(\begin{array}{l} \frac{-8\cos\theta}{6\theta\sin\theta} + \frac{16}{\theta^4} - \frac{8+8\cos\theta}{\theta^3\sin\theta} - \frac{3}{\theta^4} + \frac{2+\cos\theta}{\theta^3\sin\theta} + \frac{1}{6\theta\sin\theta} - \frac{2}{\theta^4} \\ + \frac{1+\cos\theta}{\theta^3\sin\theta} - \frac{3}{\theta^4} + \frac{2+\cos\theta}{\theta^3\sin\theta} + \frac{1}{6\theta\sin\theta} - \frac{2}{\theta^4} + \frac{1+\cos\theta}{\theta^3\sin\theta} \end{array}\right) S_i^j\right].
\end{aligned}
$$

After simple calculations, the last equation becomes

$$
\begin{aligned}
&Z_{i-2}^j - 4Z_{i-1}^j + 6Z_i^j - 4Z_{i+1}^j + Z_{i+2}^j = \gamma S_{i-2}^j + \beta S_{i-1}^j \\
&+ \delta S_i^j + \beta S_{i+1}^j + \gamma S_{i+2}^j, \quad i = 0, 1, \cdots, N+1,
\end{aligned} \tag{15}
$$

where

$$
\begin{aligned}
\gamma &= h^4\left(\frac{-1}{\theta^3\sin\theta} + \frac{1}{6\theta\sin\theta} + \frac{1}{\theta^4}\right), \\
\beta &= h^4\left(\frac{2(1+\cos\theta)}{\theta^3\sin\theta} - \frac{\cos\theta - 2}{3\theta\sin\theta} - \frac{4}{\theta^4}\right),
\end{aligned}
$$

and

$$\delta = h^4\left(-\frac{2(1+2\cos\theta)}{\theta^3\sin\theta} - \frac{4\cos\theta - 1}{3\theta\sin\theta} + \frac{6}{\theta^4}\right).$$

3 Remarks

1-The above relations that enable us to choose γ, β and δ can be computed using simple calculations by expanding Eq. (15) in terms of u_i^j and its derivatives. Hence, the local truncation error of Eq. (15) can be obtained as follows:

$$
\begin{aligned}
T^{*j}_i =& (u_{i-2}^j + u_{i+2}^j) - 4(u_{i-1}^j + u_{i+1}^j) \\
&+ 6u_i^j - \gamma(D_x^4 u_{i-2}^j + D_x^4 u_{i+2}^j) \\
&- \beta(D_x^4 u_{i-1}^j + D_x^4 u_{i+1}^j) - \delta D_x^4 u_i^j,
\end{aligned}
$$

$$T^{*j}_i=(h^4-(\delta+2\gamma+2\beta))D_x^4u_i^j+h^2(\frac{h^4}{6}-(\beta+4\gamma))D_x^6u_i^j$$
$$+h^4(\frac{504h^4}{8!}-\frac{1}{12}(\beta+16\gamma))D_x^8u_i^j+ \quad +\cdots.$$

2-As $\omega \to 0$, that is $\theta \to 0$, then $(\gamma,\beta,\delta) \to (\frac{h^4}{120},\frac{26h^4}{120},\frac{66h^4}{120})$, and Eq.(15) reduce to the following ordinary quintic spline:

$$Z_{i-2}^j-4Z_{i-1}^j+6Z_i^j-4Z_{i+1}^j+Z_{i+2}^j=$$
$$\frac{h^4}{120}(S_{i-2}^j+26S_{i-1}^j+66S_i^j+26S_{i+1}^j+S_{i+2}^j),$$
$$i=0,1,\cdots,N+1,$$

Using Eq. (1), we can write S_i^j in the form

$$S_i^j=\frac{\partial^4 Z_i^j}{\partial x^4}=g_i^j-\frac{\partial^\alpha Z_i^j}{\partial t^\alpha}. \tag{16}$$

Using the Caputo partial fractional derivative [11-13], we have

$$\frac{\partial^\alpha Z(x_i,t_j)}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t_{j+1}}\frac{\partial Z(x_i,s)}{\partial t}(t_{j+1}-s)^{-\alpha}ds,$$
$$t_{j+1}=(j+1)k,\ 0<\alpha<1. \tag{17}$$

Using the concept of to the short memory principle [15], the fractional derivative with the lower limit t_0 is approximated by the fractional derivative with moving lower limit $t_{j+1}-\Delta t$, which means taking into account the behavior of $Z(x,t)$ in the interval $[t_{j+1}-\Delta t,t_{j+1}]$, where Δt is the memory length.

$${}_{t_0}D^\alpha_{t_{j+1}}Z(x,t)\approx(t_{j+1}-\Delta t)D^\alpha_{t_{j+1}}Z(x,t),t_{j+1}>t_0+\Delta t$$

where $t_j=t_{j+1}-\Delta t$, then

$$\int_{t_0}^{t_{j+1}}\frac{\partial Z(x_i,s)}{\partial t}(t_{j+1}-s)^{-\alpha}ds\approx\int_{t_j}^{t_{j+1}}\frac{\partial Z(x_i,s)}{\partial t}(t_{j+1}-s)^{-\alpha}ds,$$
$$j=0,1,2,\cdots$$

and

$$\frac{\partial^\alpha Z(x_i,t_j)}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_{t_j}^{t_{j+1}}\frac{\partial Z(x_i,s)}{\partial t}(t_{j+1}-s)^{-\alpha}ds,$$
$$0<\alpha<1, \tag{18}$$

since $(t_{j+1}-s)^{-\alpha}$does not change sign on $[t_j,t_{j+1}]$, the weighted mean value theorem for integrals [13], can be applied to each integration in the last summation as follows [10,16],

$$\int_{t_j}^{t_{j+1}}\frac{\partial Z(x_i,s)}{\partial t}(t_{j+1}-s)^{-\alpha}ds=$$
$$\frac{\partial Z(x_i,s^*)}{\partial t}\int_{t_j}^{t_{j+1}}(t_{j+1}-s)^{-\alpha}ds,\ t_j<s^*<t_{j+1}.$$

This implies that

$$\int_{t_j}^{t_{j+1}}\frac{\partial Z(x_i,s)}{\partial t}(t_{j+1}-s)^{-\alpha}ds\approx\frac{Z_i^j-Z_i^{j-1}}{k}\int_{t_j}^{t_{j+1}}(t_{j+1}-s)^{-\alpha}ds$$
$$=[\frac{Z_i^j-Z_i^{j-1}}{k}][\frac{(t_{j+1}-t_j)^{1-\alpha}}{1-\alpha}]$$
$$=\frac{1}{k^\alpha(1-\alpha)}[Z_i^j-Z_i^{j-1}].$$

After that, the partial fractional derivative (18) of the discrete approximation can be written as follows:

$$\frac{\partial^\alpha Z(x_i,t_j)}{\partial t^\alpha}\approx\frac{1}{\lambda}[Z_i^j-Z_i^{j-1}],\ 0<\alpha<1, \tag{19}$$

where $\lambda=(1-\alpha)\Gamma(1-\alpha)k^\alpha$. From formula (19) S_i^j can be expressed as in the form

$$S_i^j=g_i^j-\frac{1}{\lambda}[Z_i^j-Z_i^{j-1}] \tag{20}$$

Using Eq. (1), we can write S_{i-2}^j,S_{i-1}^j, S_i^j,S_{i+1}^j and S_{i+2}^j, in the form:

$$S_{i-2}^j=\frac{\partial^4 Z_{i-2}^j}{\partial x^4}=(g_{i-2}^j-\frac{\partial^\alpha Z_{i-2}^j}{\partial t^\alpha}),$$
$$S_{i-1}^j=\frac{\partial^4 Z_{i-1}^j}{\partial x^4}=(g_{i-1}^j-\frac{\partial^\alpha Z_{i-1}^j}{\partial t^\alpha})$$
$$S_i^j=\frac{\partial^4 Z_i^j}{\partial x^4}=(g_i^j-\frac{\partial^\alpha Z_i^j}{\partial t^\alpha}),$$
$$S_{i+1}^j=\frac{\partial^4 Z_{i+1}^j}{\partial x^4}=(g_{i+1}^j-\frac{\partial^\alpha Z_{i+1}^j}{\partial t^\alpha})$$
$$S_{i+2}^j=\frac{\partial^4 Z_{i+2}^j}{\partial x^4}=(g_{i+2}^j-\frac{\partial^\alpha Z_{i+2}^j}{\partial t^\alpha}).$$

These equations can be discretized in the form

$$S_{i-2}^j\approx(g_{i-2}^j-(\frac{Z_{i-2}^j-Z_{i-2}^{j-1}}{\lambda})),$$
$$S_{i-1}^j\approx(g_{i-1}^j-(\frac{Z_{i-1}^j-Z_{i-1}^{j-1}}{\lambda})),$$
$$S_i^j\approx(g_i^j-(\frac{Z_i^j-Z_i^{j-1}}{\lambda})), \tag{21}$$
$$S_{i+1}^j\approx(g_{i+1}^j-(\frac{Z_{i+1}^j-Z_{i+1}^{j-1}}{\lambda})),$$
$$S_{i+2}^j\approx(g_{i+2}^j-(\frac{Z_{i+2}^j-Z_{i+2}^{j-1}}{\lambda})).$$

The use of Eq. (21) in Eq. (15) gives us the following system

$$Z_{i-2}^{j}-4Z_{i-1}^{j}+6Z_{i}^{j}-4Z_{i+1}^{j}+Z_{i+2}^{j}=$$
$$\gamma(g_{i-2}^{j}-\frac{Z_{i-2}^{j}-Z_{i-2}^{j-1}}{\lambda})+\beta(g_{i-1}^{j}-\frac{Z_{i-1}^{j}-Z_{i-1}^{j-1}}{\lambda})$$
$$+\delta(g_{i}^{j}-\frac{Z_{i}^{j}-Z_{i}^{j-1}}{\lambda})+\beta(g_{i+1}^{j}-\frac{Z_{i+1}^{j}-Z_{i+1}^{j-1}}{\lambda})$$
$$+\gamma(g_{i+2}^{j}-\frac{Z_{i+2}^{j}-Z_{i+2}^{j-1}}{\lambda}),$$

or

$$A_iZ_{i-2}^{j}+B_iZ_{i-1}^{j}+C_iZ_{i}^{j}+D_iZ_{i+1}^{j}+E_iZ_{i+2}^{j}=$$
$$\gamma Z_{i-2}^{j-1}+\beta Z_{i-1}^{j-1}+\delta Z_{i}^{j-1}+\beta Z_{i+1}^{j-1}+\gamma Z_{i+2}^{j-1}+\tau_i^j,\ i=2,\dots,N, \tag{22}$$

where

$$A_i=\lambda+\gamma,$$
$$B_i=-4\lambda+\beta,$$
$$C_i=6\lambda+\delta,$$
$$D_i=-4\lambda+\beta,$$
$$E_i=\lambda+\gamma,$$

and

$$\tau_i^j=\lambda(\gamma g_{i-2}^{j}+\beta g_{i-1}^{j}+\delta g_{i}^{j}+\beta g_{i+1}^{j}+\gamma g_{i+2}^{j}).$$

System (22) consists of *N-2* equations in *N+2* unknowns. Now, 4-additional equations are needed to get a solution to this system. From conditions (2) these equations are obtained. The first two boundary conditions in (2) are replaced by

$$Z_0^j=\beta_0, Z_{N+1}^j=\beta_1,\ j\geq 0, \tag{23}$$

but the last two boundary conditions in (2) are discretized by the following equations

$$45Z_0^j-154Z_1^j+214Z_2^j-156Z_3^j+61Z_4^j-10Z_5^j=12h^2\frac{\partial^2 Z_0^j}{\partial x^2}$$
$$=12h^2L_0-10Z_{N-4}^j+61Z_{N-3}^j-156Z_{N-2}^j+214Z_{N-1}^j$$
$$-154Z_N^j+45Z_{N+1}^j=12h^2\frac{\partial^2 Z_{N+1}^j}{\partial x^2}=12h^2L_1,\ j\geq 0. \tag{24}$$

4 Stability analysis

Applying the Von Neumann concept, the stability analysis of the technique can be investigated. According to this method the solution of the difference equation (22) has a solution of the form [13, 16, 17]

$$Z_i^j=\zeta^j exp(q\varphi ih)\,, \tag{25}$$

where φ is the wave number, $q=\sqrt{-1}$, h is the element size, and ζ^j is the amplification factor at time level $\boldsymbol{j}$. Inserting Z_i^j given by (25) in scheme (22), we obtain

$$\zeta^j\{A_iexp((i-2)q\varphi h)\ +B_iexp((i-1)q\varphi h)\ +C_iexp(iq\varphi h)$$
$$+D_iexp((i+1)q\varphi h)\ +E_iexp((i+2)q\varphi h)\ \}=$$
$$\zeta^{j-1}\{\gamma exp((i-2)q\omega h)\ +\beta exp((i-1)q\varphi h)\ +\delta exp(iq\varphi h)+$$
$$\beta exp((i+1)q\varphi h)\ +\gamma exp((i+2)q\varphi h)\ \}.$$

After simple calculations, we obtain

$$\zeta=$$
$$\frac{\gamma exp(-2q\varphi h)\ +\beta exp(-q\varphi h)\ +\delta+\beta\exp(q\varphi h)\ +\gamma\exp(2q\varphi h)}{A_i\exp(-2q\varphi h)\ +B_i\exp(-q\varphi h)\ +C_i+D_i\exp(q\varphi h)\ +E_i\exp(2q\varphi h)}. \tag{26}$$

Using Euler's formula, that is

$$\exp[\,q\phi]=\cos\phi\ +q\sin\phi\ ,\phi=\varphi h.$$

Eq.(26) becomes

$$\zeta=\frac{2\gamma\cos 2\ \phi+2\beta\cos\phi\ +\delta}{2\lambda\cos 2\ \phi+2\gamma\cos 2\ \phi+2\beta\cos\phi\ +\delta-8\lambda\cos\phi\ +6\lambda}. \tag{27}$$

Since the **condition of stability** is $|\varepsilon|\leq 1$. We can rewrite this condition as

$$\zeta^2\leq 1. \tag{28}$$

From Eq. (27), **inequality** (28) becomes

$$[2\gamma\cos 2\phi+2\beta\cos 2\phi+\delta]^2\leq$$
$$[2\lambda\cos 2\phi+2\gamma\cos 2\phi+2\beta\cos\phi+\delta \tag{29}$$
$$-8\lambda\cos\phi+6\lambda]^2,$$

Squaring and arranging Eq. (29) with some simplification of the resulting equation, we get

$$-24\beta\delta\cos\phi-8\beta\lambda\cos 2\phi\cos\phi+32\beta\lambda\cos^2\phi$$
$$-24\gamma\lambda\cos 2\phi-8\gamma\lambda\cos^2 2\phi+32\gamma\lambda\cos 2\phi\cos\phi$$
$$-12\delta\lambda-4\delta\lambda\cos 2\phi+16\delta\lambda\cos\phi-36\lambda^2 \tag{30}$$
$$-24\lambda^2\cos 2\phi-4\lambda^2\cos^2 2\phi+96\lambda^2\cos\phi$$
$$+32\lambda^2\cos 2\phi\cos\phi-64\lambda^2\cos^2\phi\leq 0.$$

Dividing by ω the inequality (30) takes the form:

$$32\beta\cos^2\phi+32\gamma\cos 2\phi\cos\phi+16\delta\cos\phi$$
$$+96\lambda\cos\phi+32\lambda\cos 2\phi\cos\phi\leq 24\beta\cos\phi$$
$$+8\beta\cos 2\phi\cos\phi+24\gamma\cos 2\phi+8\gamma\cos^2 2\phi \tag{31}$$
$$+12\delta+4\delta\cos 2\phi+36\lambda+24\lambda\cos 2\phi$$
$$+4\lambda\cos^2 2\phi+64\lambda\cos^2\phi,$$

since

$$\phi = \varphi h, \qquad \omega = (1-a)G(1-a)k^{a},$$

the inequality (31) is satisfied**.** For h being small enough and $k < h, \delta > 0, \beta \ > 0 \ and \ \gamma < 0,\ 0 = a = 1$, the condition of stability (31) is satisfied.

5 Numerical example

We obtain numerical solutions of Eq. (1) for a test example. Given the time fractional biharmonic problem [15];

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + \frac{\partial^{4} u}{\partial x^{4}} = \frac{4!}{\Gamma(5-\alpha)} t^{4-\alpha}(\sin 4\ \pi x) + 256\pi^{4} t^{4}(\sin 4\ \pi x)$$
$$\text{with} 0 \le x \le 1,\ t \ge 0,\ 0 \le \alpha \le 1, \tag{32}$$

where the initial conditions

$$u(x,0) = 0 \quad 0 \le x \le 1, \tag{33}$$

and the boundary conditions

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0,\ t \ge 0. \tag{34}$$

This problem has an exact solution given as

$$u(x,t) = t^{4}\sin(4px) \tag{35}$$

Using a variety of time steps and for fixed values for h and k ($k = 0.0005$) and for certain values ofδ,β and $\omega$the numerical results obtained by implementing our technique are summarized in the tables below where all calculations are carried out using Mathematica. The accuracy is measured by computing the maximum absolute error.

Table 1$h = 0.025, \alpha = 0.1, \delta + 2\omega + 2\beta = h^{4}, \beta + 4\gamma = \frac{h^{4}}{6}, and \beta + 16\gamma = \frac{12\times504h^{4}}{8!}$

Time	0.05	0 .1	0 .15	0.2
$L_8 - error$	6.98625×10^{-8}	1.11768×10^{-6}	5.65779×10^{-6}	$1.78802\text{x}10^{-5}$

Table2$h = 0.025, \alpha = 0.5, \delta + 2\gamma + 2\beta = h^{4}, \beta + 4\gamma = \frac{h^{4}}{6}, and \beta + 16\gamma = \frac{12\times504h^{4}}{8!}$

Time	0.05	0.1	0.15	0.2
$L_8 - error$	7.06453×10^{-8}	1.12688×10^{-6}	5.77173×10^{-6}	$1.81617\ \text{x}10^{-5}$

Table 3$h = 0.025, \alpha = 0.99, \delta + 2\gamma + 2\beta = h^{4}, \beta + 4\gamma = \frac{h^{4}}{6}, and \beta + 16\gamma = \frac{12\times504h^{4}}{8!}$

Time	0.05	0 .1	0 .15	0.2
$L_8 - error$	6.70731×10^{-8}	1.0931×10^{-6}	5.56942×10^{-6}	$1.78369\text{x}10^{-5}$

Table 4$h = 0.025, \alpha = 1.0, \delta + 2\omega + 2\beta = h^{4}, \beta + 4\gamma = \frac{h^{4}}{6}, and \beta + 16\gamma = \frac{12\times504h^{4}}{8!}$

Time	0.05	0 .1	0 .15	0.2
$L_8 - error$	6.65568×10^{-8}	1.0882×10^{-6}	5.55146×10^{-6}	$1.76143\ \text{x}10^{-5}$

In the following, we introduce some figures to illustrate the behavior of the numerical and exact solutions for some values of the parameters $h, \alpha, \delta, \gamma$ and β with varying time level step.

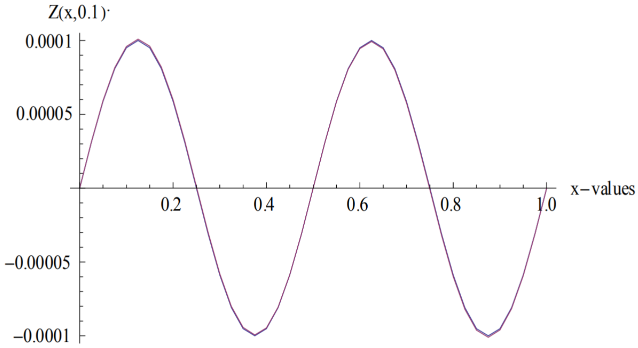

Fig. 1: The behavior of exact solution and numerical solution at α= 0.1

Fig 1$h = 0.025, \alpha = 0.1, \delta + 2\omega + 2\beta = h^{4}, \beta + 4\gamma = \frac{h^{4}}{6}, and \beta + 16\gamma = \frac{12\times504h^{4}}{8!}$.

Fig 2 $h = 0.025, \alpha = 0.5, \delta + 2\gamma + 2\beta = h^{4}, \beta + 4\gamma = \frac{h^{4}}{6}, and \beta + 16\gamma = \frac{12\times504h^{4}}{8!}$

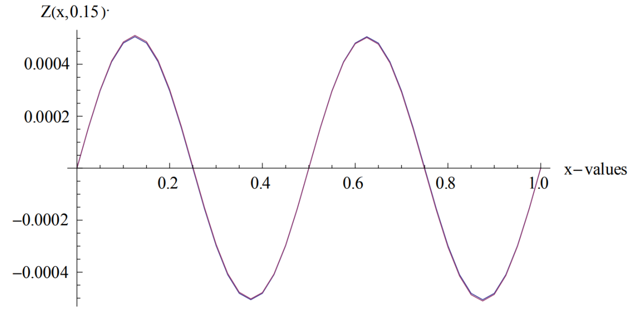

Fig. 2: The behavior of exact solution and numerical solution at α= 0.5

Fig 3$h = 0.025, \alpha = 0.99, \delta + 2\gamma + 2\beta = h^{4}, \beta + 4\gamma = \frac{h^{4}}{6}, and \beta + 16\gamma = \frac{12\times504h^{4}}{8!}$

From Fig. 1, Fig 2 and Fig 3, respectively, it can be seen that the proposed method is very accurate for solving the time-dependent fractional biharmonic equation because the numerical solution tends to the exact solutions as the fractional order of the time-derivative is close to $\alpha = 1$.

6 Conclusion

A numerical technique based on quintic non-polynomial spline method for solving the time dependent fractional biharmonicequation is suggested. The stability analysis has revealed that the meth od is unconditionally stable. The numerical results clarify that the proposed method

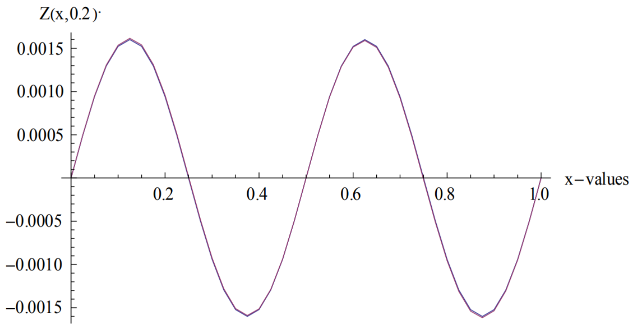

Fig. 3: The behavior of exact solution and numerical solution at $\alpha = 0.99$

has a high accuracy when compared with the exact solutions.

Acknowledgements

The author is grateful to Prof. Mohamed A. Ramadan, Mathematics department, Faculty of Science, Menoufia University, for his valuable suggestions and comments which enhanced the paper.

Competing interests

The author declares that he has no competing interests.

Authors' contributions

The author contributed, prepared and wrote the material of this paper alone. He read and approved the final manuscript.

References

[1] M. H. Heydari, M. R. Hooshmandas and G. F. M. Ghaini, An efficient computational method for solving fractional biharmonic equation, Comput. Math. Appl., Vol. 68, pp. 269-287 (2014).
[2] P. A. S. Selvadurai, Partial Differential Equations in Mechanics 2: The Biharmonic Equation, Poisson's Equation, Springer Science & Business Media, June 29, 2013.
[3] Y. Chen, M. Yi and C. Yu, Error analysis for numerical solution of fractional differential equation by Haar wavelets method, J. Comput. Sci., Vol. 3, No. 5, pp. 367-373 (2012).
[4] V. D. Beibalaev and R. P. Meilanov, The Dirichlet problem for the fractional Poisson?s equation with Caputo derivatives: a finite difference approximation and a numerical solution, Therm. Sci., Vol. 16, No. 2, pp. 385?394 (2012).
[5] O. P. Agrawal, A general solution for a fourth-order fractional diffusion-wave equation defined in a bounded domain, Comput. Struct., Vol. 79, pp. 1497- 1501 (2001).
[6] T. S. Aboelenen, Numerical Treatment For Fractional Differential Equations by Fractional Spectral Methods and Nodal Discontinuous Galerkin method, PhD Thesis, Assuit University, Egypt (2016).
[7] M. Van Daele, V. Berghe and H. D. Meyer, A smooth approximation for the solution of a fourth-order boundary value problem based on non-polynomial splines, J. Comput. Appl. Math., Vol. 51, pp. 383-394 (1994).
[8] M. A. Ramadan, T. S. El Danaf and F. E. I. Abd Alaal, Application of non-polynomial spline approach to the solution of Burgers' equation, The Open Appl. Math. J., Vol. 1, pp. 15-20 (2007).
[9] J. Rashidinia and R. Mohammadi, Non-polynomial cubic spline methods for the solution of parabolic equations, Int. J. Comput. Math., Vol. 85, No. 5, pp. 843-850 (2008).
[10] T. S. El-Danaf and A. R. Hadhoud, Computational method for solving space fractional Fisher's nonlinear equation, Math. Methods Appl. Sci., Vol. 36, pp. 4557-4564 (2013).
[11] T. S. El-Danaf and A. R. Hadhoud, Parametric spline functions for the solution of one time fractional Burgers' equation, Appl. Math. Model., Vol. 36, pp. 4557-4564 (2012).
[12] Mohamed A. Ramadan, E. F. Lashien and W. K. Zahra, High order accuracy non-polynomial spline solutions for 2 th order two point boundary value problems, Appl. Math. Comput., Vol. 204 , pp. 920-927 (2008).
[13] T. S. El-Danaf, Efficient and Accurate Numerical Methods for the Burgers' and Related Partial Differential Equations, Ph. D. Thesis, Menoufia University (1998).
[14] A. M. Diego, Implicit finite difference approximation for time fractional diffusion equations, Comput. Math. Appl., Vol. 56, pp. 1138-1145 (2008).
[15] I. Podlubny, Fractional differential equations, Academic Press (1999).
[16] K. B. Oldham and J. Spanier, The fractional calculus, Academic Press, New York (1999).
[17] R. Kress, Numerical Analysis, Springer-Verlag , New York (1998).

Adel R. Hadhoud: He is an Assistance Professor of numerical analysis at Mathematics and Computer Science Department, Faculty of Science, Menoufia University, Egypt. His research interest Numerical Analysis, solving ordinary and fractional differential equations.