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Optimal Oscillation Conditions for a Delay Differential Equation

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Abstract: Consider the differential equation with a retarded argument of the form

$$x'(t)+p(t)x(\tau(t))=0, \quad t\geq t_0, \tag{1}$$

where the functions $p,\tau\in C([t_0,\infty),\mathbb{R}^+)$, (here $\mathbb{R}^+=[0,\infty)$), $\tau(t)\leq t$ for $t\geq t_0$ and $\lim_{t\to\infty}\tau(t)=\infty$ and the equation with a constant positive delay τ of the form

$$x'(t)+p(t)x(t-\tau)=0, \quad t\geq t_0, \tag{2}$$

Optimal conditions for the oscillation of all solutions to these equations are presented when the well-known oscillation conditions

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds>1 \quad \text{and} \quad \liminf_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds>\frac{1}{e}$$

are not satisfied and also in the critical case where $\liminf_{t\to\infty}p(t)=\frac{1}{e\tau}$ in Eq. (2). In the case that the function $\int_{t-\tau}^{t}p(s)ds$ is slowly varying at infinity, then under mild additional assumptions

$$\limsup_{t\to\infty}\int_{t-\tau}^{t}p(s)ds>\frac{1}{e}$$

is a sharp condition for the oscillation of all solutions to Eq. (2). Examples illustrating the results are given.

Keywords: Oscillation, delay differential equation, slowly varying at infinity function, S-asymprotically periodic function

1 Introduction

Consider the differential equation with a retarded argument of the form

$$x'(t)+p(t)x(\tau(t))=0, \quad t\geq t_0,$$

where the functions $p,\tau\in C([t_0,\infty),\mathbb{R}^+)$, (here $\mathbb{R}^+=[0,\infty)$), $\tau(t)\leq t$ for $t\geq t_0$ and $\lim_{t\to\infty}\tau(t)=\infty$ and the equation with a constant positive delay τ of the form

$$x'(t)+p(t)x(t-\tau)=0, \quad t\geq t_0,$$

By a solution of Eq. (1), we understand a continuously differentiable function defined on $[\tau(T_0),+\infty)$ for some $T_0\geq t$ and such that Eq. (1) is satisfied for $t\geq T_0$, analogously for Eq. (2). Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory*.

It is noteworthy to observe that a first-order linear differential equation of the form Eq. (1) [Eq. (2)] without delay ($\tau(t)\equiv t$) [$\tau=0$] does not possess oscillatory solutions. Therefore the investigation of oscillatory solutions is of interest for equations of the above forms.

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Furthermore, the mathematical modeling of several real-world problems leads to differential equations that depend on the past history (like equations of these forms) rather than only the current state. For the general theory of these equations the reader is referred to [10,13,14,15].

In this paper optimal conditions for the oscillation of all solutions to these equations are presented when the well-known oscillation conditions

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds>1 \quad\text{and}\quad \liminf_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds>\frac{1}{e}$$

are not satisfied, also in the critical case where $\liminf_{t\to\infty}p(t)=\frac{1}{e\tau}$ in Eq. (2) and in the case that the function $\int_{t-\tau}^{t}p(s)ds$ is slowly varying at infinity, where under mild additional assumptions

$$\limsup_{t\to\infty}\int_{t-\tau}^{t}p(s)ds>\frac{1}{e}$$

is a sharp condition for the oscillation of all solutions to Eq. (2).

2 Oscillation Criteria for Equation (1)

In this section we study the delay differential equation

$$x'(t)+p(t)x(\tau(t))=0,\quad t\geq t_0,$$

where the functions $p,\tau\in C([t_0,\infty),\mathbb{R}^+)$, $\tau(t)<t$ for $t\geq t_0$ and $\lim_{t\to\infty}\tau(t)=\infty$. The problem of establishing sufficient conditions for the oscillation of all solutions to the delay differential equation (1) has been the subject of many investigations [2]-[15], [17]-[31]. The first systematic study for the oscillation of all solutions to Eq. (1) was made by Myshkis. In 1950 [26] he proved that every solution of Eq. (1) oscillates if

$$\limsup_{t\to\infty}[t-\tau(t)]<\infty \quad\text{and}\quad \liminf_{t\to\infty}[t-\tau(t)]\liminf_{t\to\infty}p(t)>\frac{1}{e}.$$

In 1972, Ladas, Lakshmikantham and Papadakis [23] proved that the same conclusion holds if τ is a non-decreasing function and

$$A:=\limsup_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds>1. \tag{3}$$

In 1979, Ladas [22] established integral conditions for the oscillation of Eq. (1) with constant delay, while in 1982, Koplatadze and Canturija [20] established the following result. If

$$\mathfrak{a}:=\liminf_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds>\frac{1}{e}, \tag{4}$$

then all solutions of Eq. (1) oscillate; If

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds<\frac{1}{e}, \tag{5}$$

then Eq. (1) has a non-oscillatory solution.

It is obvious that there is a gap between the conditions (3) and (4) when the limit $\lim_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds$ does not exist. How to fill this gap is an interesting problem which has been investigated by several authors.

In 1988, Erbe and Zhang [11] developed new oscillation criteria by employing the upper bound of the ratio $x(\tau(t))/x(t)$ for possible non-oscillatory solutions $x(t)$ of Eq. (1). Their result says that all the solutions of Eq. (1) are oscillatory, if $0<\mathfrak{a}\leq\frac{1}{e}$ and

$$A>1-\frac{\mathfrak{a}^2}{4}. \tag{6}$$

Since then several authors tried to obtain better results by improving the upper bound for $x(\tau(t))/x(t)$.

In 1991, Jian [18] derived the condition

$$A>1-\frac{\mathfrak{a}^2}{2(1-\mathfrak{a})}, \tag{7}$$

while in 1992, Yu, Wang, Zhang and Qian [30] obtained the condition

$$A>1-\frac{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}{2}. \tag{8}$$

In 1990, Elbert and Stavroulakis [8] and in 1991 Kwong [21], using different techniques, improved (6), in the case where $0<\mathfrak{a}\leq\frac{1}{e}$, to the conditions

$$A>1-(1-\frac{1}{\sqrt{\lambda_1}})^2 \tag{9}$$

and

$$A>\frac{\ln\lambda_1+1}{\lambda_1}, \tag{10}$$

respectively, where λ_1 is the smaller real root of the equation $\lambda=e^{\mathfrak{a}\lambda}$.

In 1998, Philos and Sficas [27] and in 1999, Zhou and Yu [31] and Jaroš and Stavroulakis [17] improved further the above conditions in the case where $0<\mathfrak{a}\leq\frac{1}{e}$ as follows

$$A>1-\frac{\mathfrak{a}^2}{2(1-\mathfrak{a})}-\frac{\mathfrak{a}^2}{2}\lambda_1, \tag{11}$$

$$A>1-\frac{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}{2}-(1-\frac{1}{\sqrt{\lambda_1}})^2, \tag{12}$$

and

$$A>\frac{\ln\lambda_1+1}{\lambda_1}-\frac{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}{2}, \tag{13}$$

respectively.

Consider Eq. (1) and assume that $\tau(t)$ is continuously differentiable and that there exists $\theta>0$ such that $p(\tau(t))\tau'(t)\geq\theta p(t)$ eventually for all t. Under this additional assumption, in 2000, Kon, Sficas and Stavroulakis [19] and in 2003, Sficas and Stavroulakis [29] established the conditions

$$A > 2\mathfrak{a} + \frac{2}{\lambda_1} - 1, \tag{14}$$

and

$$A > \frac{\ln \lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2\mathfrak{a}\lambda_1}}{\lambda_1}, \tag{15}$$

respectively. In the case where $\mathfrak{a} = \frac{1}{e}$, then $\lambda_1 = e$, and (15) leads to

$$A > \frac{\sqrt{7-2e}}{e} \approx 0.459987065.$$

It is to be noted that for small values of $\mathfrak{a}$ ($\mathfrak{a} \to 0$), all the previous conditions (6)-(14) reduce to the condition (3), i.e. $A > 1$. However, the condition (15) leads to

$$A > \sqrt{3} - 1 \approx 0.732,$$

which is an essential improvement. Moreover (15) improves all the above conditions for all values of $\mathfrak{a} \in (0, \frac{1}{e}]$. Note that the value of the lower bound on A can not be less than $\frac{1}{e} \approx 0.367879441$. Thus, the aim is to establish a condition which leads to a value *as close as possible to* $\frac{1}{e}$.

For illustrative purpose, we give the values of the lower bound on A under these conditions when (i) $\mathfrak{a} = 1/1000$ and (ii) $\mathfrak{a} = 1/e$.

	(i)	(ii)
(6):	0.999999750	0.966166179
(7):	0.999999499	0.892951367
(8):	0.999999499	0.863457014
(9):	0.999999749	0.845181878
(10):	0.999999499	0.735758882
(11):	0.999998998	0.709011646
(12):	0.999999249	0.708638892
(13):	0.999998998	0.599215896
(14):	0.999999004	0.471517764
(15):	0.733050517	0.459987065

We see that the condition (15) essentially improves all the known results in the literature.

3 Oscillation Criteria for Equation (2) in a critical case

In 1995 Elbert and Stavroulakis [9] established sufficient conditions under which all solutions to Eq. (1) oscillate in the critical case where

$$\int_{\tau(t)}^{t} p(s)ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \frac{1}{e}.$$

In 1996 Domshlak [2,3] investigated Eq. (2) in the *critical case* where

$$\liminf_{t\to\infty} p(t) = \frac{1}{\tau e}$$

and sufficient conditions for the oscillation of all solutions were established in spite of the fact that the corresponding limiting equation

$$x'(t) + \frac{1}{\tau e} x(t-\tau) = 0, \quad t \geq t_0,$$

admits a non-oscillatory solution $x(t) = e^{-t/\tau}$. Indeed, in [2,3] it was proved that if

$$\liminf_{t\to\infty} p(t) = \frac{1}{\tau e} \quad \text{and} \quad \liminf_{t\to\infty} \left[(p(t) - \frac{1}{\tau e}) t^2 \right] > \frac{\tau}{8e} \tag{16}$$

then all solutions of Eq. (2) oscillate.

Also in 1996 this result was improved by Domshlak and Stavroulakis [4] as follows.

Theorem 3.1 ([4]) Assume that

$$\liminf_{t\to\infty} p(t) = \frac{1}{\tau e}, \quad \liminf_{t\to\infty} \left[(p(t) - \frac{1}{\tau e}) t^2 \right] = \frac{\tau}{8e}$$

and

$$C := \liminf_{t\to\infty} \left\{ \left[(p(t) - \frac{1}{\tau e}) t^2 - \frac{\tau}{8e} \right] \ln^2 t \right\} > \frac{\tau}{8e}. \tag{17}$$

Then all solutions of Eq. (2) oscillate.

Example 3.1 ([4]) Consider the equation (cf. Theorem 3 in [9])

$$x'(t) + p(t)x(t-1) = 0, \quad t \geq 1,$$

where

$$p(t) = \frac{(2t-1)\ln t - 1}{2e\sqrt{t(t-1)\ln t \ln(t-1)}}.$$

It is easy to see that

$$x(t) = e^{-t}\sqrt{t \ln t}$$

is a non-oscillatory solution. In this case one can check that

$$\liminf_{t\to\infty} \left\{ \left[(p(t) - \frac{1}{\tau e}) t^2 - \frac{\tau}{8e} \right] \ln^2 t \right\} = \frac{1}{8e},$$

that is, condition (17) is not satisfied (as expected). Thus the inequality $C > \frac{\tau}{8e}$ can not be replaced by the corresponding equality.

Later in 1998 and 2000 the above results were extended by Diblik [5,6,7] using the *iterated* logarithm as follows. Call the expression $\ln_k t$, $k \geq 1$, defined by the formula

$$\ln_k t = \underbrace{\ln \ln \ldots \ln}_{k} t, \; k \geq 1$$

the k-th *iterated* logarithm if $t > \exp_{k-2} 1$ where

$$\exp_k t \equiv (\underbrace{\exp(\exp(\ldots \exp}_{k} t))), \; k \geq 1,$$

$\exp_0 t \equiv t$ and $\exp_{-1} t \equiv 0$. Moreover, let us define $\ln_0 t \equiv t$ and also instead of expressions $\ln_0 t, \ln_1 t$, we write only t and $\ln t$. Then the folowing results are established.

Theorem 3.2 ([5,6,7]) If for some integer $k \geq 0$

$$p(t) \leq \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t\ln t)^2} + \frac{\tau}{8e(t\ln t\ln_2 t)^2} + \cdots + \frac{\tau}{8e(t\ln t\ln_2 t\ldots\ln_k t)^2} \quad \text{as } t \to \infty,$$

then there exists a positive solution $x = x(t)$ of Eq. (2) and moreover,

$$x(t) < e^{-t/\tau}\sqrt{t\ln t\ln_2 t\ldots\ln_k t} \quad \text{as } t \to \infty,$$

while if for a constant $\theta > 1$,

$$p(t) \geq \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t\ln t)^2} + \ldots + \frac{\tau}{8e(t\ln t\ln_2 t\ldots\ln_{k-1} t)^2} + \frac{\theta\tau}{8e(t\ln t\ln_2 t\ldots\ln_k t)^2} \tag{18}$$

as $t \to \infty$, then all solutions of Eq. (2) oscillate.

4 Sharp Oscillation Criteria for Eq (1)

Observe that in the case of the equation

$$x'(t) + p(t)x(t-\tau) = 0, \quad t \geq t_0,$$

$$A = \limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)ds$$

and

$$\mathfrak{a} = \liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds.$$

Set

$$A(t) := \int_{t-\tau}^{t} p(s)ds,$$

$$\mathscr{P} := \limsup_{t\to\infty} p(t)$$

and

$$\mathfrak{p} := \liminf_{t\to\infty} p(t).$$

Then the results by Myshkis [26] reduce to the following conditions: If

$$\mathfrak{p}\tau > \frac{1}{e}, \tag{19}$$

then all solutions of Eq. Eq. (2) oscillate, while

$$\mathscr{P}\tau < \frac{1}{e} \tag{20}$$

implies the existence of a non-oscillatory solution of Eq. (2). Thus, for the oscillation of all solutions to Eq. (2) a necessary condition is the following

$$\mathscr{P}\tau \geq \frac{1}{e}. \tag{21}$$

At this point it should be pointed out that in the special case of the delay equation with a constant positive coefficient p and a constant positive delay τ, that is in the case of the equation

$$x'(t) + px(t-\tau) = 0, \quad t \geq t_0, \tag{22}$$

$$p\tau > \frac{1}{e} \tag{23}$$

is a *necessary and sufficient condition* ([24]) for all solutions to Eq. (22) to oscillate.

In 2017, Pituk [28] studied the delay Eq. (2) in the case where the function $p \in C([t_0,\infty), \mathbb{R}^+)$ is slowly varying at infinity. Recall (see, e.g. [1]) that a function $f : [t_{0,},\infty) \to \mathbb{R}$ is slowly varying at infinity if for every $s \in \mathbb{R}$,

$$f(t+s) - f(t) \to 0 \quad as\ t \to \infty. \tag{24}$$

Generally speaking, a slowly varying function is a function of a real variable whose behavior at infinity is in some sense similar to the behavior of a function converging at infinity. As noted in [28], a continuous function $f : [t_{0,},\infty) \to \mathbb{R}$ is slowly varying at infinity if and only if there exists $t_1 \geq t_0$ such that on the interval $[t_1,\infty)$ the function f can be decomposed into the sum

$$f(t) = g(t) + h(t), \quad t \geq t_1, \tag{25}$$

where $g : [t_{1,},\infty) \to \mathbb{R}$ is a continous function which tends to a finite limit as $t \to \infty$ and $h : [t_{1,},\infty) \to \mathbb{R}$ is a continuously differentiable function such that $h'(t) \to 0$ as $t \to \infty$. Since $A'(t) = p(t) - p(t-\tau)$ for $t \geq t_0 + \tau$, this implies that the condition

$$p(t+\tau) - p(t) \to 0 \quad as\ t \to \infty \tag{26}$$

is sufficient for the function $A(t)$ to be slowly varying at infinity. Functions p satisfying (26) are sometimes called *S−asymptotically* $\tau-$periodic [16]. In [28] Pituk proved the following theorem.

Theorem 4.1 ([28]) Suppose that the function p is *slowly varying at infinity* and $\mathfrak{p} > 0$. Then

$$\mathscr{P}\tau > \frac{1}{e}, \tag{27}$$

implies that all solutions of Eq. (2) oscillate.

Remark 4.1 ([28]) It is easy to see that in the case of Eq. (2)

$$\mathfrak{p}\tau \leq \mathfrak{a} \leq A \leq \mathscr{P}\tau.$$

Thus the above oscillation results by Ladas [22] and Koplatadze and Chanturija [20] imply the results by Myshkis [26]. As it is shown in [28], when the function p is slowly varying at infinity, then

$$\mathfrak{p}\tau = \mathfrak{a} \quad \text{and } \mathscr{P}\tau = A. \tag{28}$$

Therefore in that case both results are equivalent. Moreover, condition (3) together with (28) implies that if p is slowly varying at infinity, then the condition

$$\mathscr{P}\tau > 1, \tag{29}$$

guarantees the oscillation of all solutions to Eq. (2) Consequently, if instead of (27) the stronger condition (29) is assumed, then the uniform positivity condition $\mathfrak{p} > 0$ can be omitted.

Note the analogy of the conditions (29), (3) also (27), (23), (19), (4), (20) and (5).

Remark 4.2 The conclusion of Theorem 4.1 does not hold if (27) is replaced by (21). Indeed, if

$$p(t) = \frac{1}{\tau e},$$

is identical for $t \geq t_0$, then the function p is slowly varying at infinity with $\mathfrak{p} = \mathscr{P} = \frac{1}{\tau e}$ so that $\mathscr{P}\tau = \frac{1}{e}$. Observe that in this case Eq. (2) admits a non-oscillatory solution given by $x(t) = e^{-t/\tau}$ for $t \geq t_0$. Furtheremore in the case that $\mathfrak{p} = \mathscr{P} = \frac{1}{\tau e}$ so that $\mathscr{P}\tau = \frac{1}{e}$ and

$$p(t) \to \frac{1}{\tau e} \quad \text{as } t \to \infty,$$

although p is slowly varying at infinity, Theorem 4.1 does not apply because in this case the oscillation of all solutions depends on the rate of convergence of $p(t)$ to the limit $\frac{1}{\tau e}$ as $t \to \infty$ (see Theorems 3.1, 3.2).

Let's also recall that, according to the condition (5), for the oscillation of Eq. (2) it is necessary that

$$A = \limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)ds \geqq \frac{1}{e}. \tag{30}$$

Thus, if all solutions of Eq. (2) are oscillatory, then A can not be less than

$$\frac{1}{e} \approx 0.367879441$$

See, for example, the values of A in the tables at the end of Section 2 and in [29]. It is therefore of great interest to find a sufficient condition for all solutions to Eq. (2) such that the value of A is as close as possible to $\frac{1}{e}$. Indeed, A. Garab et al. [12] showed the following.

Theorem 4.2 ([12]) Let $p : [t_0, \infty) \to \mathbb{R}^+$ be a nonnegative, bounded and uniformly continuous function such that

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > 0. \tag{31}$$

Assume that the function $A(t) = \int_{t-\tau}^{t} p(s)ds$ is slowly varying at infinity and

$$A = \limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > \frac{1}{e}. \tag{32}$$

Then all solutions of Eq. (2) oscillate.

Remark 4.3 It is noteworthy to observe that under mild additional assumptions the "almost *necessary*" condition $A > \frac{1}{e}$ is *sufficient* for the oscillation of all solutions to Eq. (2). Moreover it is clear the essential improvement of the condition (32) over (3) and (4) and of course over all the known conditions (6-15).

As noted above, if p is an S-asymptotically $\tau-$periodic continuous function, then the function $A(t)$ is slowly varying at infinity and Theorem 4.2 yields the following corollaries.

Corollary 4.1 ([12]) Let $p : [t_0, \infty) \to \mathbb{R}^+$ be a nonnegative, bounded and uniformly continuous function such that Eq. (31) holds. If p is S-asymptotically $\tau-$periodic, then Eq. (32) implies that all solutions of Eq. (2) oscillate.

Corollary 4.2 ([12]) Assume that $p : [t_0, \infty) \to \mathbb{R}^+$ is a nonnegative bounded function which can be decomposed into a sum

$$p(t) = q(t) + r(t), \quad t \geq t_0, \tag{33}$$

where $q : [t_0, \infty) \to \mathbb{R}$ is a continuous $\tau-$periodic function and $r : [t_0, \infty) \to \mathbb{R}$ is continuous and slowly varying at infinity. If

$$\int_{t_0}^{t_0+\tau} q(s)ds + \tau \liminf_{t\to\infty} r(t) > 0,$$
$$\int_{t_0}^{t_0+\tau} q(s)ds + \tau \limsup_{t\to\infty} r(t) > \frac{1}{e}, \tag{34}$$

then all solutions of Eq. (2) oscillate.

5 Examples

In this section we present examples which illustrate the significance of the results.

Example 5.1 ([12]) Consider the delay equation

$$x'(t) + \left(\frac{1}{2\pi e} + \delta(\cos t + \cos\sqrt{t})\right) x(t - 2\pi) = 0, \quad t \geq 0, \tag{35}$$

where $\delta \in (0, \frac{1}{4\pi e})$. Observe that this equation is a special case of Eq. (1) with $\tau = 2\pi, t_0 = 0$, and

$$p(t) = \frac{1}{2\pi e} + \delta(\cos t + \cos\sqrt{t}).$$

Note that the function $\cos\sqrt{t}$ is slowly varying at infinity, because its derivative vanishes there. Therefore the decomposition Eq. (33) holds where $q(t) = \frac{1}{2\pi e} + \delta\cos t$ and $r(t) = \delta\cos\sqrt{t}$. The conditions in (34) reduce to $1/e - 2\pi\delta > 0$ and $1/e + 2\pi\delta > 1/e$ respectively. By the choice of δ both inequalities are satisfied and therefore by Corollary 2 all solutions of Eq. (35) oscillate.

We are going to see, however, that neither the conditions Eq. (4-15) nor the conditions of Theorem 4.1 are satisfied. Indeed, taking into consideration that p is S-asymptotically τ-periodic and combining the periodicity of q with [[28], Lemma 2] we find,

$$\liminf_{t\to\infty}\int_{t-2\pi}^{t} p(s)ds = \int_0^{2\pi}\left(\frac{1}{2\pi e}+\delta\cos s\right)ds + 2\pi\liminf_{t\to\infty}(\delta\cos\sqrt{t}) = \frac{1}{e}-2\pi \tag{36}$$

and

$$\limsup_{t\to\infty}\int_{t-2\pi}^{t} p(s)ds = \int_0^{2\pi}\left(\frac{1}{2\pi e}+\delta\cos s\right)ds + 2\pi\limsup_{t\to\infty}(\delta\cos\sqrt{t}) = \frac{1}{e}+2\pi\delta. \tag{37}$$

From (36) it follows that condition (4) is not satisfied. Moreover, from (37) we see that choosing $\delta>0$ small enough none of the conditions (6-15) is satisfied.

As far as the conditions of Theorem 4.1 is concerned, note that the function $\cos\sqrt{t}$ is slowly varying at infinity and so $\cos\sqrt{t+\pi}-\cos\sqrt{t}\to 0$ as $t\to\infty$ and therefore

$$p(t+\pi)-p(t) = -2\delta\cos t+\delta[\cos\sqrt{t+\pi}-\cos\sqrt{t}\nrightarrow 0$$

as $t\to\infty$. This implies that p is not slowly varying at infinity and therefore Theorem 4.1 cannot be applied.

Example 5.2 ([12]) Consider the delay equation

$$x'(t)+\left(\frac{1}{\pi}+\delta\cos t\right)e^{-1-2\delta\sin t}x(t-\pi)=0,\quad t\geq 0, \tag{38}$$

where $\delta\in(0,\frac{1}{\pi})$. Observe that this equation is a special case of Eq.(1) with $\tau=\pi, t_0=0$ and

$$p(t)=(\frac{1}{\pi}+\delta\cos t)e^{-1-2\delta\sin t}.$$

Here it is easy to see that Eq. (38) has a positive solution given by

$$x(t)=e^{-\frac{t}{\pi}-\delta\sin t}.$$

In this case we are going to see that all the assumptions of Theorem 4.2 are satisfied except that the function $A(t)$ is not slowly varying at infinity.

Observe that p is nonegative and bounded. Further it is $2\pi-$periodic and continuous and therefore it is uniformly continuous on $[t_0,\infty)$.

Since $p(t)\geq(\frac{1}{\pi}-\delta)e^{-1-2\delta}$ for $t\in\mathbb{R}$, we have that

$$\liminf_{t\to\infty}\int_{t-\pi}^{t} p(s)ds \geq (1-\pi\delta)e^{-1-2\delta} > 0,$$

that is, the condition (31) of Theorem 4.2 is satisfied. Also, since p is $2\pi-$periodic, we have

$$A((2n+2)\pi)=\int_{(2n+1)\pi}^{(2n+2)\pi}p(s)ds=\int_{\pi}^{2\pi}p(s)ds=A(2\pi)$$

for all $n\in\mathbb{N}$. Taking into account that $sint<0$ for $t\in(\pi,2\pi)$, we find that

$$A(2\pi)=\int_{\pi}^{2\pi}p(s)ds=\frac{1}{\pi e}\int_{\pi}^{2\pi}(1+\pi\delta\cos s)e^{-2\delta\sin t}\ ds > \frac{1}{\pi e}\int_{\pi}^{2\pi}(1+\pi\delta\cos s)ds=\frac{1}{e}.$$

Thus, $A((2n+n)\pi)=A(2\pi)>1/e$ for all $n\in\mathbb{N}$ and therefore

$$\limsup_{t\to\infty}\int_{t-\pi}^{t}p(s)ds=\limsup_{t\to\infty}A(t)\geq A(2\pi)>\frac{1}{e},$$

that is, the condition (32) of Theorem 4.2 is also satisfied. Finally it is shown that the function $A(t)$ is not slowly varying at infinity, that is there exists $s_0\in\mathbb{R}$ and a sequence $t_n\to\infty$ such that $A(t_n+s_0)-A(t_n)\nrightarrow 0$ as $t\to\infty$. Thereto, note that the function p is $2\pi-$periodic and so is $A(t)$. Thus, it is sufficient to show that $A(t)$ is nonconstant. Indeed, if there exist $t_0\leq s_1\leq s_2<\infty$ such that $A(s_1)\neq A(s_2)$, then letting $s_0=s_2-s_1$ and $t_n=s_1+2n\pi$, we get $A(t_n+s_0)-A(t_n)=A(s_2)-A(s_1)\neq 0$ for all $n\in\mathbb{N}$ Since $A(t)$ is continuously differentiable and periodic, in order to see that it is nonconstant, it suffices to find a $t\in\mathbb{R}$ such that $A'(t)\neq 0$. Choosing $t=\pi/2$ we have

$$A'(\frac{\pi}{2})=p(\frac{\pi}{2})-p(-\frac{\pi}{2})=\frac{e^{1-2\delta}-e^{-1+2\delta}}{\pi}\neq 0.$$

This completes the proof that $A(t)$ is not slowly varying at infinity.

It is to be pointed out that this example shows that in Theorem 4.2 the assumption that $A(t)$ is slowly varying at infinity cannot be ommited.

6 Applications

6.1 Nicholson's blowflies

The delay differential equation

$$\dot{N}(t)=-\delta N(t)+PN(t-\tau)e^{-aN(T-\tau)},\ \ t\geq 0$$

was used by Gurney et al. [[14], p.51] to describe the dynamics of Nicholson's blowflies. Here P is the maximum per capita daily egg production rate, $1/a$ is the size at which the population reproduces at its maximum rate, δ is the per capita daily adult death rate, τ is the generation time and $N(t)$ is the size of the population at time t.

6.2 Delay-logistic equation

The delay differential equation

$$\dot{N}(t)=\tau N(t)[1-N(t-\tau)/K],$$

where $r, \tau, K \in (0, \infty)$, is known as *delay-logistic equation* and has been investigated by numerous authors [[14], p.85]. This equation is a prototype in modelling the dynamics of single-specie population systems whose biomass or density is denoted by a differentiable function N of the time variable t. The constant r is called the *growth rate* and the constant K is called the *carrying capacity* of the habitat.

6.3 The Lasota-Wazewska model for the survival of red blood cells

The delay differential equation

$$\dot{N}(t) = -\mu N(t) + pe^{-\gamma N(T-\tau)}, \quad t \geq 0$$

has been used by Wazewska-Czyzewska and Lasota [[14], p.89] as a model for the survival of red blood cells in an animal. Here $N(t)$ denotes the number of red blood cells at time t, μ is the probability of death of a red cell, p and γ are positive constants related to the production of red blood cells per unit of time, and τ is the time required to produce a red blood cell.

6.4 Slowly varying functions

These classes of functions (introduced by Karamata) have found several important applications, for example in probabiliy theory, in particular in game theory and in the theory of stochastic branching processes (see [32] and the references cited therein).

7 Perspective

We conclude that under mild additional assumptions the almost necessary condition $A > \frac{1}{e}$ is sufficient for the oscillation of all solutions to Eq. (2). Moreover it is clear the essential improvement of this condition over all the known sufficient oscillation conditions (3)-(15).

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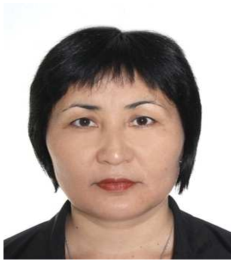

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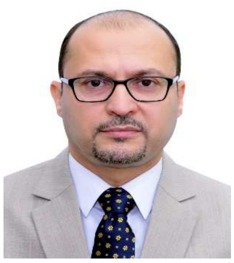

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