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Ostrowski-Type Inequalities for Functions Whose Derivative Modulus is Relatively $(m,h_1,h_2)-$Convex.

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Abstract: We have found some new Ostrowski-type inequalities for functions whose derivative module is relatively $(m,h_1,h_2)-$convex. From the main results some corollaries refereeing to relative convexity, relative $P-$convexity, relative $m-$convexity, relative $s-$convexity in the second sense and relative $(s,m)-$convexity are deduced. Also some inequalities of Hermite-Hadamard type are obtained.

Keywords: Ostrowski type inequalities, Relative convexity, Relative $(m,h_1,h_2)-$convexity

1 Introduction

The Ostrowski inequality is known in the classical literature since 1938 [18], when A. Ostrowski gave an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at the point $x\in[a,b]$ as follows: Let $f: I\subset[0,+\infty)\to\mathbb{R}$ a differentiable function in I°, the interior of the interval I, such that $f'\in\mathscr{L}[a,b]$, where $a,b\in I$ and $a<b$. If $|f'(x)|\leq M$, then the following inequality holds

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{M}{b-a}\left[\frac{(x-a)^2+(b-x)^2}{2}\right].$$

The growing development of the concept of convex function is observed in several studies in which the field of inequalities has a special attention [13,19]. In the area of stochastic processes, these generalizations have been applied with the use of mean square integrals inequalities [11,12]. Also it is studied the classical Hermite-Hadamard inequality and Jensen-type inequalities on fractal sets related with $h-$convex functions as showed in [22]. Recently, many generalizations of the Ostrowski inequality for functions of bounded variation, Lipschitzian, monotone, absolutely continuous, convex functions, s-convex and h-convex functions, $(m,h_1,h_2)-$convex functions, n-times differentiable mappings with error estimates with some special means together with some numerical quadrature are done [1,2,3,4,6,10,21].

Another famous integral inequality is named after those who studied it, J. Hadamard and Ch. Hermite in the years 1893 and 1883, respectively [8,9].

Using a particular convex function generalization established by M. Noor [16], called relative convexity with respect to a function and the so-called $s-$convexity in the second sense, we introduce the definition of $(m,h_1,h_2)-$convexity relative to a function and find some Ostrowski type inequalities, and from these results we deduce some Hermite-Hadamard type inequalities.

2 Preliminaries

As is known in the literature, the classical concept of convex function is as follows.

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Definition 1. *Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be convex, if for every $x, y \in I$ and every $t \in (0,1)$, the inequality*

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y),$$

holds.

If the inequality in (1) holds in the opposite sense, then we say that f is concave.

One of the generalizations of this concept, called $s-$convexity in the first and second sense, is established by W. Orlicz [17], later used by W.W. Breckner [5] and widely studied in applications by M. Alomari et. al. in [3].

Definition 2. *Let $0 < s \leq 1$. A function $f : [0, +\infty) \to \mathbb{R}$ is s-convex in the first sense or s_1-convex if*

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y),$$

for every $x, y \in [0, +\infty)$ and $\alpha, \beta \in (0,1)$ and $\alpha^s + \beta^s = 1$. The function f is s-convex in the second sense or s_2-convex if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y),$$

for every $x, y \in [0, +\infty)$ and $\alpha, \beta \in (0,1)$ and $\alpha + \beta = 1$.

If the inequalities in (2) holds in the opposite sense, then we say that f is s-concave.

Theorem 1. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ a differentiable function in I° such that $f' \in \mathscr{L}[a,b]$ where $a, b \in I$ with $a < b$. If $|f'|$ is s-convex in the second sense in $[a,b]$ for some fixed $s \in (0,1]$ and $|f'(x)| \leq M$, $x \in [a,b]$, then the following inequality holds for each $x \in [a,b]$.*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{s+1} \right].$$

The proof of that theorem can be found in [3].

Theorem 2. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ a differentiable mapping in I° such that $f' \in \mathscr{L}[a,b]$ where $a, b \in I$ with $a < b$. If $|f'|^q$ is s-convex in the second sense in $[a,b]$ for some fixed $s \in (0,1]$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $|f'(x)| \leq M$, $x \in [a,b]$, then the following inequality holds*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right|$$

$$\leq \frac{M}{(1+p)^{1/p}} \left(\frac{2}{s+1} \right)^{1/q} \left[\frac{(x-a)^2 + (b-x)^2}{b-a} \right],$$

for each $x \in [a,b]$.

The proof of that theorem can be found in [4]

Theorem 3. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ a differentiable mapping in I° such that $f' \in \mathscr{L}[a,b]$ where $a, b \in I$ with $a < b$. If $|f'|^q$ is s-convex in the second sense in $[a,b]$ for some fixed $s \in (0,1]$, $q \geq 1$, and $|f'(x)| \leq M$, $x \in [a,b]$, then the following inequality holds*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right|$$

$$\leq M \left(\frac{2}{s+1} \right)^{1/q} \left[\frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right],$$

for each $x \in [a,b]$.

The proof of that theorem can be found in [4].

In [15], M.A. Noor introduced and studied a new class of convex set and convex function with respect to an arbitrary function; which are called relative convex set and relative convex function respectively, as follows. Let K be a non-empty closed set in a real Hilbert spaces H.

Definition 3. *Let K_g a subset of H. K_g is said to be relatively convex with respect to the function $g : H \to H$ if*

$$tg(v) + (1-t)u \in K_g$$

$\forall u, v \in H : u, g(v) \in K_g$, and $t \in [0,1]$.

Definition 4. *Let I be an interval in $\mathbb{R}$. A function $f : K_g \subseteq \mathbb{R} \to \mathbb{R}$ is said to be relatively convex with respect to function $g : \mathbb{R} \to \mathbb{R}$ if the inequality*

$$f(tg(x) + (1-t)y) \leq t f(g(x)) + (1-t) f(y) \quad (1)$$

holds for all $g(x), y \in K_g$, $x, y \in \mathbb{R}$ and $t \in [0,1]$.

If the inequality (1) holds in the opposite sense, then we say that f is relatively concave.

Definition 5. *A function $f : K_g \to [0, +\infty)$ is said to be relatively P-convex with respect to function $g : H \to H$, where $s \in (0,1]$, if the inequality*

$$f(tg(x) + (1-t)y) \leq f(g(x)) + f(y) \quad (2)$$

holds for each $x, y \in [0, +\infty)$, $g(x), y \in K_g$ and $t \in [0,1]$.

If the inequality (2) holds in the opposite sense, then we say that f is relatively P-concave.

Definition 6. *Let $m \in (0,1]$. A function $f : K_g \to [0, +\infty)$ is said to be relatively m-convex with respect to function $g : H \to H$, if the inequality*

$$f(tg(x) + m(1-t)y) \leq t f(g(x)) + m(1-t) f(y) \quad (3)$$

holds for each $x, y \in [0, +\infty)$, $g(x), y \in K_g$ and $t \in [0,1]$.

If the inequality (3) holds in the opposite sense, then we say that f is relatively m-concave.

Definition 7. *A function $f : K_g \to [0,+\infty)$ is said to be relatively s-convex in the second sense with respect to function $g : H \to H$, where $s \in (0,1]$, if the inequality*

$$f(tg(x)+(1-t)y) \leq t^s f(g(x)) + (1-t)^s f(y) \quad (4)$$

holds for each $x, y \in [0,+\infty)$, $g(x), y \in K_g$ and $t \in [0,1]$.

If the inequality (4) holds in the opposite sense, then we say that f is relatively s-concave in the second sense.

Definition 8. *Let $s, m \in (0,1]$. A function $f : K_g \to [0,+\infty)$ is said to be relatively (s,m)-convex in the second sense with respect to function $g : H \to H$, if the inequality*

$$f(tg(x)+m(1-t)y) \leq t^s f(g(x)) + m(1-t)^s f(y) \quad (5)$$

holds for each $x, y \in [0,+\infty)$, $g(x), y \in K_g$ and $t \in [0,1]$.

If the inequality in (5) holds in the opposite sense, then we say that f is relatively $(s,m)-$concave in the second sense.

Definition 9. *Let $h : [0,1] \to \mathbb{R}^+$. A function $f : K_g \to [0,+\infty)$ is said to be relatively $h-$convex with respect to function $g : H \to H$, if the inequality*

$$f(tg(x)+(1-t)y) \leq h(t) f(g(x)) + h(1-t) f(y) \quad (6)$$

holds for each $x, y \in [0,+\infty)$, $g(x), y \in K_g$ and $t \in [0,1]$.

If the inequality in (6) holds in the opposite sense, then we say that f is relatively $h-$concave in the second sense.

In this work we introduce the following definition.

Definition 10. *Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be non-negative and not identically zero functions and $m \in (0,1]$. A function $f : K_g \to [0,+\infty)$ is said to be relatively $(m,h_1,h_2)-$-convex with respect to function $g : H \to H$, if the inequality*

$$f(tg(x)+m(1-t)y) \leq h_1(t) f(g(x)) + m h_2(t) f(y) \quad (7)$$

holds for each $x, y \in [0,+\infty)$, $g(x), y \in K_g$ and $t \in [0,1]$.

To obtain our main results we need the following Lemmas whose proofs are found in [20].

Lemma 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ a differentiable function in I° where $a, b \in I$, $a < b$ and $g : \mathbb{R} \to \mathbb{R}$ is a function. If $f' \in \mathscr{L}[a,b]$, then the next equality holds*

$$\begin{aligned} f(g(x)) - \frac{1}{b-a}\int_a^b f(z)dz &= \frac{(g(x)-a)^2}{b-a}\int_0^1 tf'(tg(x)+(1-t)a)dt \\ &- \frac{(g(x)-b)^2}{b-a}\int_0^1 tf'(tg(x)+(1-t)b)dt, \end{aligned}$$

for every $x \in g^{-1}(I)$.

Lemma 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function in I° where $a, b \in I$ with $a < b$ and $g : \mathbb{R} \to \mathbb{R}$ be a function. if $f' \in \mathscr{L}[a,b]$, then the following equality*

$$f(x) - \frac{1}{b-g(a)}\int_{g(a)}^b f(u)du$$

$$= (g(a)-b)\int_0^1 p(t)f'(tg(a)+(1-t)b)dt$$

holds for every $x \in [a,b]$, where

$$p(t) = \begin{cases} t, & t \in [0, \frac{b-x}{b-g(a)}] \\ t-1, & t \in (\frac{b-x}{b-g(a)}, 1] \end{cases}$$

for every $t \in [0,1]$ and any $x \in [a,b]$.

Using the technique applied in the work of W.D. Jiang et. al. [14] it is easy to prove the following Lemma.

Lemma 3. *If $f^{(n)}(x)$ exists and is integrable on $[a, g(b)]$ for $n \in \mathbb{N}$, then*

$$S(a, g(b); k, n, f)$$

$$= \frac{(g(b)-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+(1-t)g(b))dt$$

where

$$S(a, g(b); k, n, f) =$$

$$= \frac{f(a)+f(g(b))}{2} - \frac{1}{g(b)-a}\int_a^{g(b)} f(u)du$$

$$-\sum_{k=1}^{n-1}\frac{(k-1)(g(b)-a)^k}{2(k+1)!}f^{(k)}(a)$$

3 Main Results

Theorem 4. *Let $f : I \subset [0,+\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in \mathscr{L}[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is relatively $(m,h_1,h_2)-$convex with respect to a function $g : \mathbb{R} \to \mathbb{R}$ in $[a,b]$ and $|f'(x)| \leq M$, the inequality*

$$\left| f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du \right|$$

$$\leq \frac{M}{b-a}\left[(g(x)-a)^2 + (g(x)-b)^2\right](A_1 + mA_2),$$

where

$$A_1 = \int_0^1 th_1(t)dt \ \ and \ \ A_2 = \int_0^1 th_2(t)dt$$

holds for all $x \in g^{-1}(I)$.

Proof. Using Lemma 1 we have

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{(g(x)-a)^2}{b-a}\int_0^1 t\left|f'(tg(x)+(1-t)a)\right|dt + \frac{(g(x)-b)^2}{b-a}\int_0^1 t\left|f'(tg(x)+(1-t)b)\right|dt.$$

Now, since $|f'|$ is relatively $(m,h_1,h_2)-$convex y $|f'(x)|\leq M$ we get

$$\int_0^1 t\left|f'(tg(x)+(1-t)a)\right|dt$$

$$\leq M\int_0^1 th_1(t)dt+mM\int_0^1 th_2(t)dt$$

and similarly

$$\int_0^1 t\left|f'(tg(x)+(1-t)b)\right|dt$$

$$\leq M\int_0^1 th_1(t)dt+mM\int_0^1 th_2(t)dt$$

So we have

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{M}{b-a}\left[(g(x)-a)^2+(g(x)-b)^2\right](A_1+mA_2)$$

where

$$A_1=\int_0^1 th_1(t)dt \text{ and } A_2=\int_0^1 th_2(t)dt$$

and the proof follows. □

*Remark.*Letting $m=1$, $h_1(t)=t$, and $h_1(t)=1-t$ for all $t\in[0,1]$ in Theorem 4 it follows that

$$A_1=\int_0^1 t^2dt=\frac{1}{3} \text{ and } A_2=\int_0^1 t(1-t)dt=\frac{1}{6}$$

so, by replacement we have

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{M}{b-a}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{2}\right],$$

making coincidence with Theorem 5 in [20]

Theorem 5. *Let $f:I\subset\mathbb{R}_+\to\mathbb{R}_+$ a differentiable function in I° such that $f'\in\mathscr{L}[a,b]$ where $a,b\in I$ with $a<b$. If $|f'|^q$ is relatively $(m,h_1,h_2)-$convex with respect to function $g:\mathbb{R}\to\mathbb{R}$ for some $q\geq 1$, $\frac{1}{p}+\frac{1}{q}=1$ and $|f'(x)|\leq M$, $x\in[a,b]$, then the following inequality holds*

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{M}{(p+1)^{1/p}}(A_1+mA_2)^{1/q}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{(b-a)}\right],$$

for each $x\in[a,b]$, where

$$B_1=\int_0^1 h_1(t)dt \text{ and } B_2=\int_0^1 h_2(t)dt.$$

Proof. Suppose that $p>1$ from lemma (2), and using the Hölder inequality, we have:

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\begin{aligned}&\leq \frac{(g(x)-a)^2}{b-a}\int_0^1 t|f'(tg(x)+(1-t)a)|dt + \frac{(g(x)-b)^2}{b-a}\int_0^1 t|f'(tg(x)+(1-t)b)|dt\\ &\leq \frac{(g(x)-a)^2}{b-a}\left(\int_0^1 t^p dt\right)^{1/p}\times \left(\int_0^1 |f'(tg(x)+(1-t)a)|^q dt\right)^{1/q}\\ &+\frac{(g(x)-b)^2}{b-a}\left(\int_0^1 t^p dt\right)^{1/p}\times \left(\int_0^1 |f'(tg(x)+(1-t)b)|^q dt\right)^{1/q}.\end{aligned}$$

Since $|f'|^q$ is relatively $(m,h_1,h_2)-$convex with respect to function g and $|f'(x)|\leq M$, then we have

$$\int_0^1 |f'(tg(x)+(1-t)a)|^q dt$$

$$\leq \int_0^1 h_1(t)|f'(g(x))|^q+mh_2(t)|f'(a)|^q dt$$

$$\leq M^q\left(\int_0^1 h_1(t)+mh_2(t)dt\right)dt$$

and similarly

$$\int_0^1 |f'(tg(x)+(1-t)b)|^q dt$$

$$\leq M^q\left(\int_0^1 h_1(t)+mh_2(t)dt\right)dt.$$

Doing

$$A_1=\int_0^1 th_1(t)dt \text{ and } A_2=\int_0^1 th_2(t)dt,$$

it is attained

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{M}{(p+1)^{1/p}}(A_1+mA_2)^{1/q}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{(b-a)}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$. The proof is complete. □

Remark. Letting $m=1$, $h_1(t)=t$, and $h_1(t)=1-t$ for all $t\in[0,1]$ in Theorem 5 it follows that

$$A_1=\int_0^1 tdt=\frac{1}{2} \text{ and } A_2=\int_0^1(1-t)dt=\frac{1}{2}$$

so, by replacement we have

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{M}{(p+1)^{1/p}}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{(b-a)}\right].$$

Theorem 6. *Let $f:K_g\to\mathbb{R}$ be $n-$times differentiable and integrable on K_g. If $|f^{(n)}|$ is relative $(m,h_1,h_2)-$convex with respect to a function $g:K_g\to\mathbb{R}$, then*

$$|S(a,g(b);k,n,f)|$$

$$\leq \frac{(g(b)-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q}\times$$

$$\left(C_1|f^{(n)}(a)|^q+mC_2|f^{(n)}(g(b))|^q\right)^{1/q},$$

where

$$C_1=\int_0^1 t^{n-1}(n-2t)h_1(t)dt$$

and

$$C_2=\int_0^1 t^{n-1}(n-2t)h_2(t)dt$$

.

Proof. Using Lemma 3, the power mean inequality and the fact that $|f^{(n)}|^q$ is relative $(m,h_1,h_2)-$convex with respect to a function $g:K_g\to\mathbb{R}$ then

$$|S(a,g(b);k,n,f)|$$

$$\leq \left|\frac{(g(b)-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+(1-t)g(b))dt\right|$$

$$\leq \frac{(g(b)-a)^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)dt\right)^{1-1/q}\times$$

$$\left(\int_0^1 t^{n-1}(n-2t)|f^{(n)}(ta+(1-t)g(b))|^q dt\right)^{1/q}$$

$$\leq \frac{(g(b)-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q}\times$$

$$\left(\int_0^1 t^{n-1}(n-2t)\times\right.$$

$$\left.(h_1(t)|f^{(n)}(a)|^q+mh_2(t)|f^{(n)}(g(b))|^q)dt\right)^{1/q}$$

So, doing

$$C_1=\int_0^1 t^{n-1}(n-2t)h_1(t)dt$$

and

$$C_2=\int_0^1 t^{n-1}(n-2t)h_2(t)dt$$

it is attained

$$|S(a,g(b);k,n,f)|$$

$$\leq \frac{(g(b)-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q}\times$$

$$\left(C_1|f^{(n)}(a)|^q+mC_2|f^{(n)}(g(b))|^q\right)^{1/q}.$$

The proof is complete. □

*Remark.*Letting $m=1$, $h_1(t)=t$, and $h_1(t)=1-t$ for all $t\in[0,1]$ in Theorem 5 it follows that

$$C_1=\int_0^1 t^{n-1}(n-2t)tdt=\frac{n}{n+1}-\frac{2}{n+2}=\frac{n^2-2}{(n+1)(n+2)}$$

and

$$C_2=\int_0^1 t^{n-1}(n-2t)(1-t)dt=\frac{n}{(n+1)(n+2)}$$

by replacement we have

$$|S(a,g(b);k,n,f)|$$

$$\leq \frac{(g(b)-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q}\times$$

$$\left(C_1|f^{(n)}(a)|^q+C_2|f^{(n)}(g(b))|^q\right)^{1/q},$$

4 Some Consecuences

Corollary 1. *Let $f : I \subset [0,+\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in \mathscr{L}[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|$ is relatively $P-$convex with respect to a function $g : \mathbb{R} \to \mathbb{R}$ in $[a,b]$ and $|f'(x)| \le M$, the inequality*

$$\left|f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{M}{b-a}\left[(g(x)-a)^2 + (g(x)-b)^2\right].$$

Proof. Letting $m = 1$, $h_1(t) = h_2(t) = 1$ for all $t \in [0,1]$ in Theorem 4 it follows that

$$A_1 = A_2 = \int_0^1 tdt = \frac{1}{2}$$

by replacement we have

$$\left|f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{M}{b-a}\left[(g(x)-a)^2 + (g(x)-b)^2\right],$$

The proof is complete. □

Corollary 2. *Let $f : I \subset [0,+\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in \mathscr{L}[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|$ is relatively $m-$convex with respect to a function $g : \mathbb{R} \to \mathbb{R}$ in $[a,b]$ and $|f'(x)| \le M$, the inequality*

$$\left|f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{M(m+2)}{6(b-a)}\left[(g(x)-a)^2 + (g(x)-b)^2\right].$$

Proof. Letting $h_1(t) = t$, and $h_2(t) = 1-t$ for all $t \in [0,1]$ in Theorem 4 it follows that

$$A_1 = \int_0^1 t^2 dt = \frac{1}{3} \text{ and } A_2 = \int_0^1 t(1-t)dt = \frac{1}{6}$$

So by replacement we get

$$\left|f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{M(m+2)}{6(b-a)}\left[(g(x)-a)^2 + (g(x)-b)^2\right].$$

The proof is complete. □

Corollary 3. *Let $f : I \subset [0,+\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in \mathscr{L}[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|$ is relatively $s-$convex in the second sense with respect to a function $g : \mathbb{R} \to \mathbb{R}$ in $[a,b]$ and $|f'(x)| \le M$, the inequality*

$$\left|f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{M}{(s+1)(b-a)}\left[(g(x)-a)^2 + (g(x)-b)^2\right]$$

Proof. Letting $m = 1$, $h_1(t) = t^s$, and $h_2(t) = (1-t)^s$ for all $t \in [0,1]$ in Theorem 4 it follows that

$$A_1 = \int_0^1 t^{s+1} dt = \frac{1}{s+2}$$

$$A_2 = \int_0^1 t(1-t)^s dt = \frac{1}{s+1} - \frac{1}{s+2}$$

So by replacement

$$\left|f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{M}{(s+1)(b-a)}\left[(g(x)-a)^2 + (g(x)-b)^2\right]$$

The proof is complete. □

Corollary 4. *Let $f : I \subset [0,+\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in \mathscr{L}[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|$ is relatively $(s,m)-$convex in the second sense with respect to a function $g : \mathbb{R} \to \mathbb{R}$ in $[a,b]$ and $|f'(x)| \le M$, the inequality*

$$\left|f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{M}{b-a}\left[(g(x)-a)^2 + (g(x)-b)^2\right]\left(\frac{1-m}{s+2} + \frac{1}{s+1}\right)$$

Proof. Letting $h_1(t) = t^s$, and $h_2(t) = (1-t)^s$ for all $t \in [0,1]$ in Theorem 4 it follows that

$$A_1 = \int_0^1 t^{s+1} dt = \frac{1}{s+2}$$

$$A_2 = \int_0^1 t(1-t)^s dt = \frac{1}{s+1} - \frac{1}{s+2}$$

Replacing

$$\left|f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{M}{b-a}\left[(g(x)-a)^2 + (g(x)-b)^2\right]\left(\frac{1-m}{s+2} + \frac{1}{s+1}\right)$$

The proof is complete. □

Corollary 5. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ a differentiable function in I° such that $f' \in \mathscr{L}[a,b]$ where $a,b \in I$ with $a < b$. If $|f'|^q$ is relatively $P-$convex with respect to function $g : \mathbb{R} \to \mathbb{R}$ for some $q \ge 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $|f'(x)| \le M$, $x \in [a,b]$, then the following inequality holds*

$$\left|f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\le \frac{2^{1/q}M}{(p+1)^{1/p}}\left[\frac{(g(x)-a)^2 + (g(x)-b)^2}{(b-a)}\right],$$

for each $x \in [a,b]$.

Proof. Letting $m = 1$ and $h_1(t) = h_2(t) = 1$ for all $t \in [0,1]$ in Theorem 5 it follows that

$$B_1 = B_2 = \int_0^1 dt = 1.$$

So, by replacement

$$\left|f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{2^{1/q}M}{(p+1)^{1/p}}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{(b-a)}\right],$$

The proof is complete. □

Corollary 6. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ a differentiable function in I° such that $f' \in \mathscr{L}[a,b]$ where $a,b \in I$ with $a < b$. If $|f'|^q$ is relatively $m-$convex with respect to function $g : \mathbb{R} \to \mathbb{R}$ for some $q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $|f'(x)| \leq M$, $x \in [a,b]$, then the following inequality holds*

$$\left|f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{2^{1/q}M}{(p+1)^{1/p}}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{(b-a)}\right],$$

for each $x \in [a,b]$.

Proof. Letting $h_1(t) = t$, $h_2(t) = 1-t$ for all $t \in [0,1]$, and taking $m \in (0,1]$ in Theorem 5 it follows that

$$B_1 = B_2 = \int_0^1 h_1(t)dt = \frac{1}{2}.$$

So, by replacement

$$\left|f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{M}{(p+1)^{1/p}}\left(\frac{m+1}{2}\right)^{1/q}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{(b-a)}\right],$$

The proof is complete. □

Corollary 7. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ a differentiable function in I° such that $f' \in \mathscr{L}[a,b]$ where $a,b \in I$ with $a < b$. If $|f'|^q$ is relatively $s-$convex in the second sense with respect to function $g : \mathbb{R} \to \mathbb{R}$ for some $q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $|f'(x)| \leq M$, $x \in [a,b]$, then the following inequality holds*

$$\left|f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{M}{(p+1)^{1/p}}\left(\frac{2}{s+1}\right)^{1/q}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{(b-a)}\right],$$

for each $x \in [a,b]$.

Proof. Letting $m = 1$, $h_1(t) = t^s$, $h_2(t) = (1-t)^s$ for all $t \in [0,1]$ for some $s \in (0,1]$, in Theorem 5 it follows that

$$B_1 = B_2 = \int_0^1 t^s dt = \frac{1}{s+1}.$$

So, by replacement

$$\left|f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{M}{(p+1)^{1/p}}\left(\frac{2}{s+1}\right)^{1/q}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{(b-a)}\right],$$

The proof is complete. □

Corollary 8. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ a differentiable function in I° such that $f' \in \mathscr{L}[a,b]$ where $a,b \in I$ with $a < b$. If $|f'|^q$ is relatively $(s,m)-$convex in the second sense with respect to function $g : \mathbb{R} \to \mathbb{R}$ for some $q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $|f'(x)| \leq M$, $x \in [a,b]$, then the following inequality holds*

$$\left|f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{M}{(p+1)^{1/p}}\left(\frac{m+1}{s+1}\right)^{1/q}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{(b-a)}\right],$$

for each $x \in [a,b]$.

Proof. Letting $m = 1$, $h_1(t) = t^s$, $h_2(t) = (1-t)^s$ for all $t \in [0,1]$ for some $s \in (0,1]$, in Theorem 5 it follows that

$$B_1 = B_2 = \int_0^1 t^s dt = \frac{1}{s+1}.$$

So, by replacement

$$\left|f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{M}{(p+1)^{1/p}}\left(\frac{m+1}{s+1}\right)^{1/q}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{(b-a)}\right],$$

The proof is complete. □

Corollary 9. *Let $f : K_g \to \mathbb{R}$ be $n-$times differentiable and integrable on K_g. If $|f^{(n)}|$ is relative $P-$convex with respect to a function $g : K_g \to \mathbb{R}$, then*

$$|S(a,g(b);k,n,f)|$$

$$\leq \frac{(g(b)-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)\left(|f^{(n)}(a)|^q + |f^{(n)}(g(b))|^q\right)^{1/q}.$$

Proof. Letting $m=1$, $h_1(t)=h_2(t)=1$, for all $t\in[0,1]$ in Theorem 6 it follows that

$$C_1=C_2=\int_0^1 t^{n-1}(n-2t)dt=\frac{n-1}{n+1}$$

So, by replacement

$|S(a,g(b);k,n,f)|$

$$\leq \frac{(g(b)-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)\left(|f^{(n)}(a)|^q+|f^{(n)}(g(b))|^q)\right)^{1/q}$$

The proof is complete. □

Corollary 10. *Let $f:K_g\to\mathbb{R}$ be $n-$times differentiable and integrable on K_g. If $|f^{(n)}|$ is relative $m-$convex with respect to a function $g:K_g\to\mathbb{R}$, then*

$|S(a,g(b);k,n,f)|$

$$\leq \frac{(g(b)-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)\left(|f^{(n)}(a)|^q+|f^{(n)}(g(b))|^q)\right)^{1/q}.$$

Proof. Letting $h_1(t)=t$ and $h_2(t)=1-t$, for all $t\in[0,1]$ for some $m\in(0,1]$ in Theorem 6 it follows that

$$C_1=\int_0^1 t^{n-1}(n-2t)tdt=\frac{n^2-2}{(n+1)(n+2)}$$

and

$$C_1=\int_0^1 t^{n-1}(n-2t)(1-t)dt=\frac{n}{(n+1)(n+2)}$$

So, by replacement

$|S(a,g(b);k,n,f)|$

$$\leq \frac{(g(b)-a)^n}{2n!((n+1)(n+2))^{1/q}}\left(\frac{n-1}{n+1}\right)^{1-1/q}\times$$

$$\left((n^2-2)|f^{(n)}(a)|^q+mn|f^{(n)}(g(b))|^q)\right)^{1/q},$$

The proof is complete. □

Corollary 11. *Let $f:K_g\to\mathbb{R}$ be $n-$times differentiable and integrable on K_g. If $|f^{(n)}|$ is relative $s-$convex in the second sense with respect to a function $g:K_g\to\mathbb{R}$, then*

$|S(a,g(b);k,n,f)|$

$$\leq \frac{(g(b)-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q}\times$$

$$\left(\frac{n(n-1)+s(n-2)}{(n+s)(n+s+1)}|f^{(n)}(a)|^q\right.$$

$$\left.+\frac{n!(n+s-1)\Gamma(s+1)}{\Gamma(n+s+1)}|f^{(n)}(g(b))|^q)\right)^{1/q}.$$

Proof. Letting $m=1$, $h_1(t)=t^s$ and $h_2(t)=(1-t)^s$, for all $t\in[0,1]$ for some $s\in(0,1]$ in Theorem 6 it follows that

$$C_1=\int_0^1 t^{n-1}(n-2t)t^s dt=\frac{n(n-1)+s(n-2)}{(n+s)(n+s+1)}$$

and

$$C_2=\int_0^1 t^{n-1}(n-2t)(1-t)^s dt=\frac{n!(n+s-1)\Gamma(s+1)}{\Gamma(n+s+1)}$$

So, by replacement

$|S(a,g(b);k,n,f)|$

$$\leq \frac{(g(b)-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q}\times$$

$$\left(\frac{n(n-1)+s(n-2)}{(n+s)(n+s+1)}|f^{(n)}(a)|^q\right.$$

$$\left.+\frac{n!(n+s-1)\Gamma(s+1)}{\Gamma(n+s+1)}|f^{(n)}(g(b))|^q)\right)^{1/q}.$$

The proof is complete. □

Corollary 12. *Let $f:K_g\to\mathbb{R}$ be $n-$times differentiable and integrable on K_g. If $|f^{(n)}|$ is relative $(s,m)-$convex in the second sense with respect to a function $g:K_g\to\mathbb{R}$, then*

$|S(a,g(b);k,n,f)|$

$$\leq \frac{(g(b)-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q}\times$$

$$\left(\frac{n(n-1)+s(n-2)}{(n+s)(n+s+1)}|f^{(n)}(a)|^q\right.$$

$$\left.+\frac{mn!(n+s-1)\Gamma(s+1)}{\Gamma(n+s+1)}|f^{(n)}(g(b))|^q)\right)^{1/q}.$$

Proof. The proof follows after evaluating the coefficients C_1 and C_2 taking $s,m\in(0,1]$, $h_1(t)=t^s$ and $h_2(t)=(1-t)^s$, for all $t\in[0,1]$ in Theorem 6. □

5 Some applications of Hermite-Hadamard type inequalities

Corollary 13. *If in Theorem 4 we choose $m=1$, $h_1(t)=t$, $h_2(t)=1-t$ for all $t\in[0,1]$ and $g(x)=\frac{a+b}{2}$ we get*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{M}{4}(b-a)$$

where $|f'|$ is relatively convex with respect to function $g:\mathbb{R}\to\mathbb{R}$ and $|f'(x)|\leq M$, $x\in[a,b]$.

Corollary 14. *If in Theorem 5 we choose $m=1$, $h_1(t)=t$, $h_2(t)=1-t$ for all $t\in[0,1]$ and $g(x)=\frac{a+b}{2}$ we get*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{M}{4(1+p)^{1/p}}(b-a).$$

where $|f'|^q$ is relatively convex with respect to function $g:\mathbb{R}\to\mathbb{R}$, $q\geq1$, $\frac{1}{p}+\frac{1}{q}=1$ and $|f'(x)|\leq M$, $x\in[a,b]$

6 Conclusions

In this work we have found some Ostrowski-type inequalities for functions whose derivatives in modulus are $(m,h_1,h_2)-$convex. From the main results some Corollaries referring to other generalized convexity types are found. Also some Hermite-Hadamard-type inequalities are deduced. The authors hope that this work serves to stimulate the study in this line of research.

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References

[1] R.P.Agarwal, M.J. Luo and R.K. Raina , On Ostrowski type inequalities. Fasciculli Mathematici, Vol. **56**, No. 1, pp. 5-27 (2016).
[2] M. Alomari and M. Darus, Otrowski type inequalities for quasi-convex functions with applications to special means, RGMIA., Vol. **13**, No. 1, Article No. 3 (2010).
[3] M. Alomari and M. Darus,, S.S. Dragomir, P. Cerone, Otrowski type inequalities for functions whose derivatives are s-convex in the second sense, Appl.Math. Lett., Vol. **23**, No. 9, pp. 1071-1076 (2010).
[4] M. Alomari, Several Inequalities of Hermite-Hadamard, Ostrowski and Simpson type for s-convex, Quasi-convex and r-convex mappings and application, Thesis Submitted in Fulfilment for the degree of Doctor of Philosophy, Faculty of Science and Technology University Kebangsaan Malaysia Bangi.
[5] W. W. Breckner, Stetigkeitsaussagen für eine klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen, Publ. Inst. Math. (Beograd),Vol. **23**, pp. 13-20 (1978).
[6] P. Cerone and S.S. Dragomir, Ostrowski type inequalities for functions whose derivative satisfy certain convexity assumptions, Demonstr. Math., Vol. **37**, No. 2, pp. 299-308 (2004).
[7] S. S. Dragomir and S. Fitzpatrick, The Hadamard's inequality for $s-$convex functions in the first sense, Demonstratio Math., Vol. **31**, No. 3, pp. 633-642 (1998)
[8] J. Hadamard., Etude sur les propriétés des fonctions entiéres et en particulier d'une fonction considérée par Riemann. Journal de Mathématiques Pures et Appliquées,Vol. **58**, pp. 171-216 (1893)
[9] Ch. Hermite, Sur deux limites d'une intégrale d'efinie , Mathesis , Vol. 3, pp. 82, (1883).
[10] J.E. Hernández Hernández, Ostrowski Type Fractional Integral Operator Inequalities for $(m,h_1,h_2)-$Convex Functions, Mayfeb Journal of Mathematics, Vol. **4**, No. 1, 13-28, (2017)
[11] J.E. Hernández Hernández, Some Fractional Integral Inequalities for Stochastic Processes whose First and Second Derivatives are Quasi-Convex, Matua: Revista de Matemática de la Universidad del Atlántico, Vol. **5**, No. 2, 1-13, (2018)
[12] J.E. Hernández Hernández, Hermite Hadamard type inequalities for Stochastic Processes whose Second Derivatives are $(m,h_1,h_2)-$Convex using Riemann-Liouville Fractional Integral, Matua: Revista de Matemática de la Universidad del Atlántico, Vol. **5**, No. 1, 13-28, (2018)
[13] J. E. Hernández Hernández and M. J. Vivas-Cortez, On a Hardy's inequality for a fractional integral operator, Annals of the University of Craiova, Mathematics and Computer Science Series Vol. **45**, No. 2, 232-242, (2018)
[14] W. D. Jiang, D. W. Niu, Y. Hua and F. Qi., Generalizations of Hermite-Hadamard inequality to n-time differentiable functions which are s-convex in the second sense, Analysis, Vol. **32**, pp. 209-220 (2012).
[15] M. A. Noor., Differentiable non-convex functions and general variational inequalities., Appl. Math. Comp., Vol. **199**, No. 82, pp. 623-630 (2008)
[16] M.A. Noor, M. U. Awan and K.I. Noor, On some inequalities for relative semi-convex functions, Journal of Inequalities and Applications, Vol. **2013**:332, pp. 16 (2013).
[17] W. Orlicz, A note on modular spaces, I. Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., Vol. **9** , pp. 157-162 (1962).
[18] A. Ostrowski, Über die Absolutabweichung einer diferentiebaren Funktion von ihrem Integralmittelwert. Comment. Math. Helv., Vol. **10**, pp. 226-227 (1938).
[19] Y.C. Rangel Oliveros and M.J. Vivas-Cortez , On some Hermite-Hadamard type inequalities for functions whose second derivative are convex generalized, Matua: Revista de Matemática de la Universidad del Atlántico, Vol. **4**, No. 2, 21-31, (2018)
[20] M.J. Vivas-Cortez, C. García and J.E. Hernández H., Ostrowski Type Inequalities for Functions Whose Derivative Modulus is Relatively Convex, Appl. Math. Inf. Sci., Vol. **13**, No. 1, 121-127 (2019)
[21] M.J. Vivas-Cortez and J.E. Hernández H., A variant of Jensen-Mercer Inequality for h-convex functions and Operator h-convex functions, Matua: Revista de Matemática de la Universidad del Atlántico, Vol. **4**, No. 2, 62-76, (2017)
[22] M.J. Vivas-Cortez, J.E. Hernández and N. Merentes, New Hermite-Hadamard and Jensen Type Inequalities for h-Convex Functions on Fractal Sets, Revista Colombiana de Matemáticas, Vol. **50**, No. 2, 145-164, (2016)

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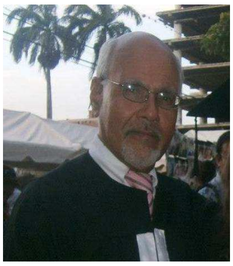

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