Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/130301

Effect of Inertia on Viscous Fluid in a wall-driven Corner Flow with Leakage

Asif Mahmood

Department of Mathematics, Pennsylvania State University, York Campus, 1031 Edgecomb Avenue, York, PA 17403, USA

Received: 5 Dec. 2018, Revised: 2 Feb. 2019, Accepted: 12 Feb. 2019
Published online: 1 May 2019

Abstract: A study of the inertial flow of a viscous fluid near the corner of two intersecting walls with leakage is presented. The flow is primarily generated when one of the walls is moved parallel to itself, but a mass source or sink due to leakage at the apex of the corner can also effect it. The non-linear partial differential equations that arise due to inertial forces are solved analytically, and an approximate solution is obtained by applying a recursive approach. Furthermore, the expressions for stream functions, velocity fields, pressure fields, and the tangential and normal stresses upto second approximation have been obtained. Finally, the effect of inertial forces on the stresses, velocity components and streamlines is discussed with the help of graphs.

Keywords: Corner flow, Viscous fluid, Leakage, Inertial flow, Non-linear

1 Introduction

Two-dimensional flow of viscous incompressible fluids in geometries having wedges and corners has been extensively studied by many researchers. The non-inertial flow caused by the steady movement of a wall in a wedge in a tangential direction is described by the well-known solution of Goodier [1] and Taylor [2]. Dean & Montagnon [3], and Moffatt [4] investigated the flow induced in a fixed-sided corner by a general movement away from the vertex. Moffatt showed that the infinite wedge can have eddy structure solutions near the corner. The conditions under which Moffatt eddies in the viscous flow of two fluids in a wedge bounded by two rigid planes of arbitrary angle can be present were found by Anderson and Davis [5]. Sprittles and Shikhmurzaev [6] considered the numerical issues arising in computations of viscous flows in corners formed by a liquid-fluid free surface and a solid boundary. In addition, Deliceoğlu, et al. [7] solved the local singular behavior of Stokes flow near the salient and re-entrant corners by the matching eigenfunction method.

Corner flow of two-dimensional viscous fluid with inertial effects is examined by Hancock and Moffatt [8]. The first-order inertial effect was presented by obtaining an analytical solution for the stream function as a perturbation series, and the significance of inertia at a certain distance from the corner was shown. Recently, Mahmood and Siddiqui [9] extended their work and presented the second-order inertial effect analytically using a recursive approach. In addition, the pressure, and normal and tangential stresses expressions were presented for flow with and without inertial effects.

Another work for the two dimensional corner flow is that of Riedler and Schneider [10]. They studied the effects of leakage of fluid on the flow of a viscous fluid at the intersection of plane rigid walls . Such flows have many practical applications as they occur at the edge of a blade used to scrape up liquid from a surface, in a screw extruder, or in a cylinder with a moving piston.

The aim of current investigation is to extend the work of Riedler and Schneider, and study the effects of inertia on the flow in a corner region of a viscous fluid with leakage. We start by formulating the governing equations for the flow and obtain the analytical solution for the first-order inertial effect using the recursive approach based on Langlois [11]. In addition, we compute the velocity field, the pressure field as well as the tangential and normal stresses for inertial and non-inertial flows, which were not presented by Riedler and Schneider. It is shown that our results subsume Riedler's and Schneider's results as a special case when the inertia terms are ignored. Further, the inertial effects appear in the second-order solution. Finally, a comparison between the

* Corresponding author e-mail: aum54@psu.edu

behavior of non-inertial and inertial flows is presented by showing the graphs of velocities, streamlines and stresses.

2 Governing Equations

The governing equations of motion for an incompressible steady viscous fluid in the absence of body forces and thermal effects are given as follows:

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$\rho\,(\mathbf{V}.\nabla)\,\mathbf{V} = -\nabla p + \mu \nabla \cdot \mathbf{A_1}, \tag{2}$$

where ρ is the constant density of the fluid, μ is viscosity, $\mathbf{V}$ is the velocity vector, p is the pressure, and $\mathbf{A_1}$ is the Rivlin–Ericksen tensor defined through

$$\mathbf{A_1} = \nabla\mathbf{V} + (\nabla\mathbf{V})^T. \tag{3}$$

For steady plane flows in polar coordinates, we choose the velocity and pressure fields as

$$\mathbf{V} = u(r,\theta)\mathbf{e}_r + v(r,\theta)\mathbf{e}_\theta, \quad p = p(r,\theta), \tag{4}$$

where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are unit vectors.

Now we define the stream function $\psi(r,\theta)$ as

$$u = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad v = -\frac{\partial\psi}{\partial r}. \tag{5}$$

The continuity equation (1) is then identically satisfied, and Eq. (2) after making use of Eqs. (3) – (5) can be expressed in components form as

$$\frac{\rho}{2}\frac{\partial}{\partial r}\left[\frac{1}{r^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2 + \left(\frac{\partial\psi}{\partial r}\right)^2\right] - \rho\frac{\partial\psi}{\partial r}\nabla^2\psi = -\frac{\partial p}{\partial r} + \frac{\mu}{r}\frac{\partial}{\partial\theta}\nabla^2\psi, \tag{6}$$

$$\frac{\rho}{2r}\frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2 + \left(\frac{\partial\psi}{\partial r}\right)^2\right] - \frac{\rho}{r}\frac{\partial\psi}{\partial\theta}\nabla^2\psi = -\frac{1}{r}\frac{\partial p}{\partial\theta} - \mu\frac{\partial}{\partial r}\nabla^2\psi, \tag{7}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \tag{8}$$

is the Laplacian operator in polar coordinates.

We follow the Langlois Recursive Approach [11,12], which requires velocity $\mathbf{V}$, pressure p, and hence stream function ψ to be expanded as follows:

$$\mathbf{V} = \varepsilon\mathbf{V}^{(1)} + \varepsilon^2\mathbf{V}^{(2)} + \varepsilon^3\mathbf{V}^{(3)} + ..., \tag{9}$$

$$p = \text{constant} + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + \varepsilon^3 p^{(3)} + ..., \tag{10}$$

$$\psi = \varepsilon\psi^{(1)} + \varepsilon^2\psi^{(2)} + \varepsilon^3\psi^{(3)} + ..., \tag{11}$$

where ε is a small dimensionless constant associated with the parameters of the flow problem.

Here we restrict ourselves upto second degree in ε, as equations become tedious to solve, and neglect all the higher order terms in ε.

Making use of Eqs. (9) – (11), equations of motion (6) and (7) take the form

$$\varepsilon^2\left(\frac{\rho}{2}\frac{\partial}{\partial r}\left[\frac{1}{r^2}\left(\frac{\partial\psi^{(1)}}{\partial\theta}\right)^2 + \left(\frac{\partial\psi^{(1)}}{\partial r}\right)^2\right] - \rho\frac{\partial\psi^{(1)}}{\partial r}\nabla^2\psi^{(1)}\right) = \varepsilon\left(-\frac{\partial p^{(1)}}{\partial r} + \frac{\mu}{r}\frac{\partial}{\partial\theta}\nabla^2\psi^{(1)}\right) + \varepsilon^2\left(-\frac{\partial p^{(2)}}{\partial r} + \frac{\mu}{r}\frac{\partial}{\partial\theta}\nabla^2\psi^{(2)}\right), \tag{12}$$

$$\varepsilon^2\left(\frac{\rho}{2r}\frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\left(\frac{\partial\psi^{(1)}}{\partial\theta}\right)^2 + \left(\frac{\partial\psi^{(1)}}{\partial r}\right)^2\right] - \frac{\rho}{r}\frac{\partial\psi^{(1)}}{\partial\theta}\nabla^2\psi^{(1)}\right) = \varepsilon\left(-\frac{1}{r}\frac{\partial p^{(1)}}{\partial\theta} - \mu\frac{\partial}{\partial r}\nabla^2\psi^{(1)}\right) + \varepsilon^2\left(-\frac{1}{r}\frac{\partial p^{(2)}}{\partial\theta} - \mu\frac{\partial}{\partial r}\nabla^2\psi^{(2)}\right). \tag{13}$$

3 Problem Formulation

Consider the two-dimensional steady flow of an incompressible viscous fluid near the corner between two rigid walls that intersect at a constant angle θ_w as sketched in Fig. (1). Assume that the liquid is moving with wall velocity εU at $\theta = 0$. and the fluid and wall are at rest at $\theta = \theta_w$. We further assume a mass source (or sink) of strength εQ at the apex corner due to leakage.

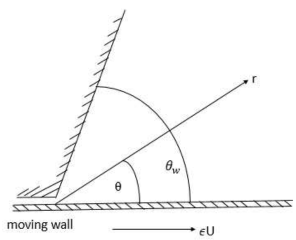

Fig. 1: Walls and coordinate system.

Thus, the boundary conditions for the problem under consideration are

$$\psi = 0, \quad \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \varepsilon U, \quad \text{at} \quad \theta = 0, \tag{14}$$

$$\psi = \varepsilon Q, \quad \frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0, \quad \text{at} \quad \theta = \theta_w. \tag{15}$$

Using Eq. (11) and neglecting the third or higher order ε terms, the boundary conditions (14) and (15) take the form

$$\begin{cases} \varepsilon\psi^{(1)}+\varepsilon^2\psi^{(2)}=0, \\ \varepsilon\left(\frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta}\right)+\varepsilon^2\left(\frac{1}{r}\frac{\partial\psi^{(2)}}{\partial\theta}\right)=\varepsilon U, \end{cases} \quad \text{at} \quad \theta=0 \tag{16}$$

$$\begin{cases} \varepsilon\psi^{(1)}+\varepsilon^2\psi^{(2)}=\varepsilon Q, \\ \varepsilon\left(\frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta}\right)+\varepsilon^2\left(\frac{1}{r}\frac{\partial\psi^{(2)}}{\partial\theta}\right)=0 \end{cases} \quad \text{at} \quad \theta=\theta_w. \tag{17}$$

Equating the coefficients of ε and ε^2 from Eqs. (12), (13), (16), and (17), we get the following system of boundary-value problems:

$O(\varepsilon)$:

$$\frac{\partial p^{(1)}}{\partial r}=\frac{\mu}{r}\frac{\partial}{\partial\theta}\nabla^2\psi^{(1)}, \tag{18}$$

$$\frac{1}{r}\frac{\partial p^{(1)}}{\partial\theta}=-\mu\frac{\partial}{\partial r}\nabla^2\psi^{(1)}, \tag{19}$$

$$\psi^{(1)}=0, \quad \frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta}=U, \quad \text{at} \quad \theta=0, \tag{20}$$

$$\psi^{(1)}=Q, \quad \frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta}=0, \quad \text{at} \quad \theta=\theta_w. \tag{21}$$

$O(\varepsilon^2)$:

$$\frac{\rho}{2}\frac{\partial}{\partial r}\left[\frac{1}{r^2}\left(\frac{\partial\psi^{(1)}}{\partial\theta}\right)^2+\left(\frac{\partial\psi^{(1)}}{\partial r}\right)^2\right]-\rho\frac{\partial\psi^{(1)}}{\partial r}\nabla^2\psi^{(1)} =-\frac{\partial p^{(2)}}{\partial r}+\frac{\mu}{r}\frac{\partial}{\partial\theta}\nabla^2\psi^{(2)}, \tag{22}$$

$$\frac{\rho}{2r}\frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\left(\frac{\partial\psi^{(1)}}{\partial\theta}\right)^2+\left(\frac{\partial\psi^{(1)}}{\partial r}\right)^2\right]-\frac{\rho}{r}\frac{\partial\psi^{(1)}}{\partial\theta}\nabla^2\psi^{(1)} =-\frac{1}{r}\frac{\partial p^{(2)}}{\partial\theta}-\mu\frac{\partial}{\partial r}\nabla^2\psi^{(2)}, \tag{23}$$

$$\psi^{(2)}=0, \quad \frac{1}{r}\frac{\partial\psi^{(2)}}{\partial\theta}=0, \quad \text{at} \quad \theta=0, \tag{24}$$

$$\psi^{(2)}=0, \quad \frac{1}{r}\frac{\partial\psi^{(2)}}{\partial\theta}=0, \quad \text{at} \quad \theta=\theta_w. \tag{25}$$

4 First-Order Problem for non-inertial flow

Elimination of the pressure from Eqs. (18) and (19), gives the following first-order problem for non-inertial flow:

$$\nabla^4\psi^{(1)}=0, \tag{26}$$

$$\psi^{(1)}=0, \quad \frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta}=U, \quad \text{at} \quad \theta=0, \tag{27}$$

$$\psi^{(1)}=Q, \quad \frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta}=0, \quad \text{at} \quad \theta=\theta_w, \tag{28}$$

As discussed in [10], the solution of (26) – (28) is given as

$$\psi^{(1)}=Urf_1(\theta)+Qg_1(\theta), \tag{29}$$

where

$$f_1(\theta)=\frac{1}{\theta_w^2-\sin^2\theta_w}(\theta_w^2\sin\theta-\theta\cos\theta\sin^2\theta_w -(\theta_w-\sin\theta_w\cos\theta_w)\theta\sin\theta), \tag{30}$$

and

$$g_1(\theta)=\frac{4\sin^2\theta_w\sin^2\theta+\sin2\theta_w\,(\sin2\theta-2\theta)}{4\sin^4\theta_w+\sin2\theta_w\,(\sin2\theta_w-2\theta_w)}. \tag{31}$$

We have verified the above mentioned solution presented by Riedler & Schneider [10], and next we find the velocity components, pressure and stresses which were not given in their work.

Using Eq. (29), we obtain the velocity components in terms of $f_1(\theta)$, and $g_1(\theta)$, respectively, as follows:

$$u^{(1)}=\frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta}=Uf_1'+\frac{Q}{r}g_1', \tag{32}$$

$$v^{(1)}=-\frac{\partial\psi^{(1)}}{\partial r}=-Uf_1. \tag{33}$$

On substituting (29) in Eqs. (18) and (19), we obtain pressure field at this order as

$$p^{(1)}(r,\theta)=p_0-\frac{\mu}{r}U\left(f_1'+f_1'''\right)-\frac{\mu}{2r^2}Qg_1''', \tag{34}$$

where p_0 is a positive constant and can be determined by conditions far from the corner.

Finally, we obtain the normal $T_n^{(1)}$, and tangential stresses $T_t^{(1)}$ to the wall as

$$T_n^{(1)}=-p^{(1)}+\frac{2\mu}{r^2}Qg_1', \tag{35}$$

and

$$T_t^{(1)}=\frac{\mu U}{r}\left(f_1+f_1''\right)+\frac{\mu}{r^2}Qg_1''. \tag{36}$$

Here we observe that the normal and tangential stresses become singular as $r\to0$. Further, if $Q=0$, the results coincide with that of Taylor [2].

We now turn to the second-order system to study the inertial effect on the flow.

5 Second-Order Problem (First inertial correction)

We can rewrite the Eqs. (22) and (23) as

$$\frac{\partial p^{*(2)}}{\partial r}=\rho\frac{\partial\psi^{(1)}}{\partial r}\nabla^2\psi^{(1)}+\frac{\mu}{r}\frac{\partial}{\partial\theta}\nabla^2\psi^{(2)}, \tag{37}$$

$$\frac{\partial p^{*(2)}}{\partial\theta}=\rho\frac{\partial\psi^{(1)}}{\partial\theta}\nabla^2\psi^{(1)}-\mu r\frac{\partial}{\partial r}\nabla^2\psi^{(2)}, \tag{38}$$

where

$$p^{*(2)} = p^{(2)} + \frac{\rho}{2}\left[\frac{1}{r^2}\left(\frac{\partial\psi^{(1)}}{\partial\theta}\right)^2 + \left(\frac{\partial\psi^{(1)}}{\partial r}\right)^2\right], \quad (39)$$

is the modified pressure.

On eliminating the pressure from Eqs. (37) – (38), we arrive at the following second-order boundary-value problem

$$\nu\nabla^4\psi^{(2)} = -\frac{1}{r}\frac{\partial(\psi^{(1)},\nabla^2\psi^{(1)})}{\partial(r,\theta)}, \quad (40)$$

$$\psi^{(2)} = 0, \qquad \frac{1}{r}\frac{\partial\psi^{(2)}}{\partial\theta} = 0, \quad \text{at} \quad \theta = 0\,, \quad (41)$$

$$\psi^{(2)} = 0, \qquad \frac{1}{r}\frac{\partial\psi^{(2)}}{\partial\theta} = 0, \quad \text{at} \quad \theta = \theta_w, \quad (42)$$

where

$$\frac{\partial(\psi^{(1)},\nabla^2\psi^{(1)})}{\partial(r,\theta)} = \frac{\partial\psi^{(1)}}{\partial r}\frac{\partial}{\partial\theta}\left(\nabla^2\psi^{(1)}\right) - \frac{\partial\psi^{(1)}}{\partial\theta}\frac{\partial}{\partial r}\left(\nabla^2\psi^{(1)}\right). \quad (43)$$

The second-order system (40) – (42) contains a fourth-order PDE. To change this PDE into a system of ODEs, we consider the stream function of the form

$$\psi^{(2)} = \frac{r^2U^2}{\nu}f_2(\theta) + \frac{rU}{\nu}Qg_2(\theta) + \frac{1}{\nu}Q^2g_3(\theta), \quad (44)$$

where f_2, g_2, and g_3 are unknown functions of argument θ.

Using Eqs. (29) and (44) in Eqs. (40) – (42), we get the following three fourth-order boundary value-problems in terms of f_2, g_2, and g_3:

$$f_2^{iv} + 4f_2'' = \beta_1\sin2\theta + \beta_2\cos2\theta + \beta_3\theta\sin2\theta + \beta_4\theta\cos2\theta,$$
$$f_2 = 0, \qquad f_2' = 0, \quad \text{at} \quad \theta = 0,$$
$$f_2 = 0, \qquad f_2' = 0, \quad \text{at} \quad \theta = \theta_w. \quad (45)$$

$$g_2^{iv} + 2g_2'' + g_2 = T_1\cos3\theta + T_2\sin3\theta + T_3\cos\theta + T_4\sin\theta + T_5\theta\cos3\theta + T_6\theta\sin3\theta,$$
$$g_2 = 0, \qquad g_2' = 0, \quad \text{at} \quad \theta = 0,$$
$$g_2 = 0, \qquad g_2' = 0, \quad \text{at} \quad \theta = \theta_w. \quad (46)$$

and

$$g_3^{iv} + 4g_3'' = S_1\cos2\theta + S_2\sin2\theta + S_3\cos4\theta + S_4\sin4\theta,$$
$$g_3 = 0, \qquad g_3' = 0, \quad \text{at} \quad \theta = 0,$$
$$g_3 = 0, \qquad g_3' = 0, \quad \text{at} \quad \theta = \theta_w. \quad (47)$$

where β's, T's, and S's are known constants whose expressions are given in the Appendix.

After a considerable amount of work, solutions of the boundary-value problems (45) – (47) are respectively reported as follows:

$$f_2(\theta) = R_1 + R_2\theta + R_3\cos2\theta + R_4\sin2\theta + \beta_5\theta\cos2\theta + \beta_6\theta\sin2\theta + \beta_7\theta^2\cos2\theta + \beta_8\theta^2\sin2\theta, \quad (48)$$

$$g_2(\theta) = A_1\cos\theta + A_2\sin\theta + A_3\theta\cos\theta + A_4\theta\sin\theta + B_1\cos3\theta + B_2\sin3\theta + B_3\theta^2\cos\theta + B_4\theta^2\sin\theta + B_5\theta\cos3\theta + B_6\theta\sin3\theta, \quad (49)$$

and

$$g_3(\theta) = D_1 + D_2\theta + D_3\cos2\theta + D_4\sin2\theta + F_1\theta\cos2\theta + F_2\theta\sin2\theta + F_3\cos4\theta + F_4\sin4\theta, \quad (50)$$

where R's, β's, A's, B's, and D's are constants (depending upon angle θ_w) that are given in Appendix.

The velocity components can then be obtained by using Eq. (44) in the following form:

$$u^{(2)} = \frac{1}{r}\frac{\partial\psi^{(2)}}{\partial\theta} = \frac{rU^2}{\nu}f_2' + \frac{U}{\nu}Qg_2' + \frac{Q^2}{\nu r}g_3', \quad (51)$$

and

$$v^{(2)} = -\frac{\partial\psi^{(2)}}{\partial r} = -\frac{2rU^2}{\nu}f_2 - \frac{U}{\nu}Qg_2. \quad (52)$$

Here we note that if leakage is absent i.e. $Q = 0$, then our results match with those of Mahmood and Siddqiui [9], and Hancock et al [8]. Further, if we ignore the leakage term Q^2 then the velocity field has no singularity.

The second-order pressure field can be now obtained from Eqs.(37) – (38), together with Eq. (39), in the following form

$$p^{(2)}(r,\theta) = \frac{\mu U}{\nu r}Q\left(g_2' + \int g_2\,d\theta\right) + \frac{2\mu}{\nu r^2}Q^2g_3' - 4\frac{\rho U^2}{\nu^2}Q^2\int g_3'f_2\,d\theta + \frac{\rho U^3 r}{\nu^2}Q\left(f_2g_2 - 3\int g_2'f_2\,d\theta\right) - \frac{\rho U}{\nu^2 r}Q^3\int g_3'g_2\,d\theta, \quad (53)$$

where f_2, g_2, and g_3, are respectively, given by Eqs. (48) – (50) . The integrals in Eq. (53) are tedious to find by hand, but can be computed easily using a computer software like Matlab, or Mathematica. We observe here that as $r \to 0$, the 2nd-order pressure field becomes singular .

Further, the expressions for normal stress $T_n^{(2)}$, and tangential stress $T_t^{(2)}$ are given by

$$T_n^{(2)} = -p^{(2)} - \frac{2\mu U^2}{\nu}f_2' + \frac{2\mu}{\nu r^2}Q^2g_3', \quad (54)$$

$$T_t^{(2)} = \frac{\mu U^2}{\nu}f_2'' + \frac{\mu U}{\nu r}Qg_2'' + \frac{\mu}{\nu r^2}Q^2g_3''. \quad (55)$$

6 Combined inertial flow solution

Using expressions for first- and second-order stream functions, pressure fields, velocity components, and stresses computed in previous sections, we obtain the combined inertial flow solution as follows:

$$\psi = \varepsilon\psi^{(1)} + \varepsilon^2\psi^{(2)}, \qquad u = \varepsilon u^{(1)} + \varepsilon^2 u^{(2)}, \tag{56}$$

$$v = \varepsilon v^{(1)} + \varepsilon^2 v^{(2)}, \qquad p = \text{constant} + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)}, \tag{57}$$

$$T_t = \varepsilon T_t^{(1)} + \varepsilon^2 T_t^{(2)}, \qquad T_n = \varepsilon T_n^{(1)} + \varepsilon^2 T_n^{(2)}. \tag{58}$$

Following the recursive approach, we take $\varepsilon \to 1$, and make use of Eqs. (29), (32), (34), (35), (36), (44), (51), (52), (53), (54), and (55) to get

$$\psi = Urf_1(\theta) + Qg_1(\theta) + \frac{r^2U^2}{\nu} f_2(\theta) + \frac{rU}{\nu} Qg_2(\theta) + \frac{1}{\nu} Q^2 g_3(\theta), \tag{59}$$

$$u = Uf_1' + \frac{Q}{r} g_1' + \frac{rU^2}{\nu} f_2' + \frac{U}{\nu} Qg_2' + \frac{Q^2}{\nu r} g_3', \tag{60}$$

$$v = -(Uf_1 + \frac{2rU^2}{\nu} f_2 + \frac{U}{\nu} Qg_2), \tag{61}$$

$$p = p_0 - \frac{\mu}{r} U\left(f_1' + f_1'''\right) - \frac{\mu}{2r^2} Qg_1''' + \frac{2\mu}{\nu r^2} Q^2 g_3' + \frac{\mu U}{\nu r} Q\left(g_2' + \int g_2\, d\theta\right) - 4\frac{\rho U^2}{\nu^2} Q^2 \int g_3' f_2\, d\theta + \frac{\rho U^3 r}{\nu^2} Q\left(f_2 g_2 - 3\int g_2' f_2\, d\theta\right) - \frac{\rho U}{\nu^2 r} Q^3 \int g_3' g_2\, d\theta, \tag{62}$$

$$T_t = -p + \frac{2\mu}{r^2} Qg_1 - \frac{2\mu U^2}{\nu} f_2' + \frac{2\mu}{\nu r^2} Q^2 g_3', \tag{63}$$

$$T_n = \frac{\mu U}{r}\left(f_1 + f_1''\right) + \frac{\mu}{r^2} Qg_1'' + \frac{\mu U^2}{\nu} f_2'' + \frac{\mu U}{\nu r} Qg_2'' + \frac{\mu}{\nu r^2} Q^2 g_3''. \tag{64}$$

where f_1, g_1, f_2, g_2, and g_3 are respectively given by Eqs. (30), (31), (48), (49) and (50).

Finally, we note that if $\nu^{-1} = \frac{\rho}{\mu} = 0$, then we recover the results obtained by Riedler and Schneider [10].

7 Graphs and Discussion

We display some graphs in this section to show the variations in the stresses, velocities, and streamlines for inertial as well as non-inertial flows. The figures (2) and (3) compare the velocity components for flows with and without inertial effects. The solid curves represent the velocity components for inertial flow whereas the dashed curves represent the velocity components of the non-inertial flow. It is noticed that, as we move away from the corner the rise of the velocity components for the inertial flow becomes much higher as compared to that for non-inertial flow. This shows that near the corner, the effect of inertial forces is small, but it becomes much more significant for the flow away from the corner.

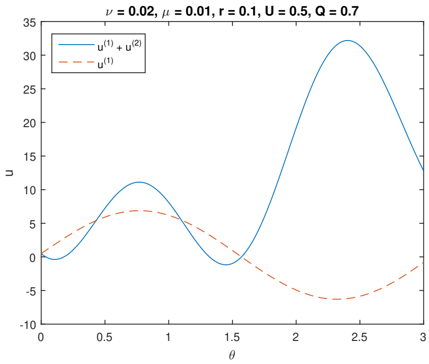

Fig. 2: comparison between u-components of inertial and non-inertial flow velocities. Inertial flow velocity is denoted by solid curve and non-inertial flow velocity by dashed curve.

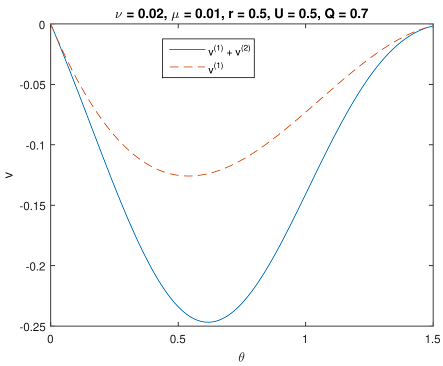

Fig. 3: comparison between v-components of inertial and non-inertial flow velocities. Inertial flow velocity is denoted by solid curve and non-inertial flow velocity by dashed curve.

Fig. (4) compares the normal stresses for inertial and non-inertial flows. It is noticed that with an increase in angle θ, the normal stresses for inertial flow decrease at a higher rate as compared to that of non-inertial flow. We also observe that the stresses shoot up very rapidly near the corner, i.e. as $r \to 0$, and this shows a good correspondence with the mathematical analysis, because there is a singularity in the stress field at this point. In Fig.(5), the tangential stresses at the fixed boundary for inertial and non-inertial flows against the angle θ are sketched. We notice here that the tangential stresses for both inertial and non-inertial flows rise with a rise in θ, but the difference is that for similar data types, the tangential stresses for the inertial flow are greater to those for non-inertial flow.

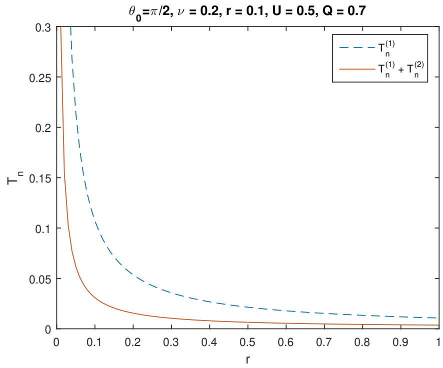

Fig. 4: comparison between inertial and non-inertial flows normal stresses. Normal stress for inertial flow is denoted by solid curve and for non-inertial flow by dashed curve.

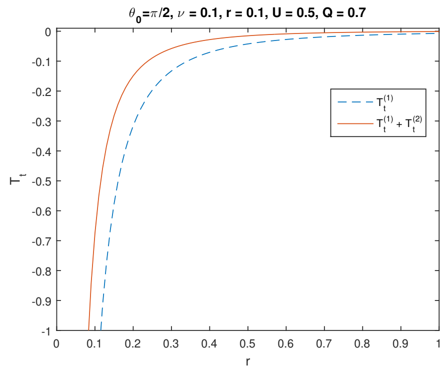

Fig. 5: comparison between inertial and non-inertial flows tangential stresses. Tangential stress for inertial flow is denoted by solid curve and for non-inertial flow by dashed curve.

Finally, Fig. (6) compares the streamline patterns for flows with and without inertia effects. The streamlines ψ_1 = constant, for non-inertial flow are shown by solid curves, and streamlines $\psi_1 + \psi_2$ =constant, including the first inertial correction are represented by dashed curves. It is interesting to see that a remarkably different streamline pattern is observed for different signs of the U/Q ratio. A dividing streamline emanates from the wall at rest in the case of U and Q having same signs. Again, we notice that the effect of inertial forces is weak near the corner, but as we recede the corner the streamlines pattern tends to change as the contributions of the inertia forces come into the play there.

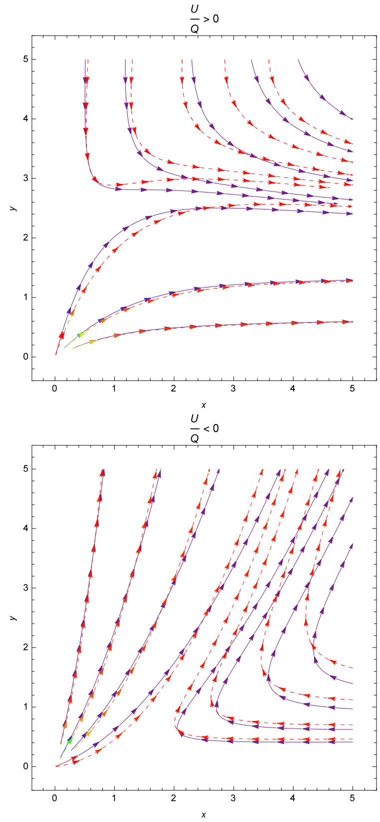

Fig. 6: A plot of the streamlines with a corner angle of $\theta_w = \pi/2$; the streamlines of non-inertial and inertial flow are given by solid and dashed lines respectively. (top) same signs of Q and U, (bottom) opposite signs of Q and U.

8 Conclusion

In this work, we have emphasized the importance of inertial forces on the flow of a two-dimensional viscous fluid with the effects of leakage at the apex of the corner. Using a recursive approach, we've obtained the expressions for velocity fields, stream functions, pressure fields, and stress fields. By comparing inertial and non-inertial flows, we have shown that the inertia effect near the corner is small, but becomes much more significant at distances far from the corner.

The next step is to extend the work on inertial corner flows to non-Newtonian fluids. Future work will investigate the effect of inertia on non-Newtonian flows in corner regions with and without leakage.

Appendix

Here we list the values of the constants discussed in section 5:

$$
\begin{aligned}
R_1 &= \frac{2X(\theta_w)-2X_1(\theta_w)\theta_w-(\beta_5-X_1(\theta_w))\sin 2\theta_w}{4(\cos 2\theta_w+\theta_w\sin 2\theta_w-1)} \\
&- \frac{2(X(\theta_w)-\beta_5\theta_w)\cos 2\theta_w}{4(\cos 2\theta_w+\theta_w\sin 2\theta_w-1)}, \\
R_2 &= \frac{X_1(\theta_w)+\beta_5-(X_1(\theta_w)+\beta_5)\cos 2\theta_w-2X(\theta_w))\sin 2\theta_w)}{2(\cos 2\theta_w+\theta_w\sin 2\theta_w-1)}, \\
R_3 &= -\frac{2X(\theta_w)-2X_1(\theta_w)\theta_w-(\beta_5-X_1(\theta_w))\sin 2\theta_w}{4(\cos 2\theta_w+\theta_w\sin 2\theta_w-1)} \\
&+ \frac{2(X(\theta_w)-\beta_5\theta_w)\cos 2\theta_w}{4(\cos 2\theta_w+\theta_w\sin 2\theta_w-1)}, \\
R_4 &= \frac{\beta_5-X_1(\theta_w)+(X_1(\theta_w)-\beta_5)\cos 2\theta_w}{4(\cos 2\theta_w+\theta_w\sin 2\theta_w-1)} \\
&+ \frac{2(X(\theta_w)-\beta_5\theta_w)\sin 2\theta_w}{4(\cos 2\theta_w+\theta_w\sin 2\theta_w-1)}, \\
\beta_1 &= \frac{\sin^4\theta_w-(\theta_w-\sin\theta_w\cos\theta_w)^2-2\theta_w^2\sin^2\theta_w}{(\theta_w^2-\sin^2\theta_w)^2}, \\
\beta_2 &= \frac{2(\theta_w-\sin\theta_w\cos\theta_w)}{\theta_w^2-\sin^2\theta_w}, \\
\beta_3 &= \frac{4(\theta_w-\sin\theta_w\cos\theta_w)\sin^2\theta_w}{(\theta_w^2-\sin^2\theta_w)^2}, \\
\beta_4 &= 2\frac{\sin^4\theta_w-(\theta_w-\sin\theta_w\cos\theta_w)^2}{(\theta_w^2-\sin^2\theta_w)^2} \\
\beta_5 &= \left(\frac{\beta_1}{16}-\frac{5}{64}\beta_4\right),\ \beta_6=-\left(\frac{\beta_2}{16}+\frac{5}{64}\beta_3\right), \\
\beta_7 &= \frac{1}{32}\beta_3,\quad \beta_8=-\frac{1}{32}\beta_4, \\
X\ (\theta_w) &= \beta_5\theta_w\cos 2\theta_w+\beta_6\theta_w\sin 2\theta_w+\beta_7\theta_w^2\cos 2\theta_w \\
&+ \beta_8\theta_w^2\sin 2\theta_w. \\
X_1\ (\theta_w) &= \beta_5\cos 2\theta_w-2\beta_5\theta_w\sin 2\theta_w+\beta_6\sin 2\theta_w \\
&+ 2\beta_6\theta_w\cos 2\theta_w+2\beta_7\theta_w\cos 2\theta_w \\
&- 2\beta_7\theta_w^2\sin 2\theta_w+2\beta_8\theta_w\sin 2\theta_w+2\beta_8\theta_w^2\cos 2\theta_w. \\
A_1 &= -B_1, \\
A_2 &= -\frac{B_1\theta_w+(3B_2+B_5)\theta_w^2-Y(\theta_w)(\sin\theta_w+\theta_w\cos\theta_w)}{(\theta_w^2-\sin^2\theta_w)} \\
&- \frac{(\theta_w Y_1(\theta_w)+B_1\cos\theta_w)\sin\theta_w}{(\theta_w^2-\sin^2\theta_w)}, \\
A_3 &= \frac{B_1\theta_w+(3B_2+B_5)\sin^2\theta_w-Y(\theta_w)(\sin\theta_w+\theta_w\cos\theta_w)}{(\theta_w^2-\sin^2\theta_w)} \\
&+ \frac{(\theta_w Y_1(\theta_w)+B_1\cos\theta_w)\sin\theta_w}{(\theta_w^2-\sin^2\theta_w)}, \\
A_4 &= \frac{B_1\sin^2\theta_w+(3B_2+B_5)(\theta_w-\sin\theta_w\cos\theta_w)}{(\theta_w^2-\sin^2\theta_w)} \\
&- \frac{Y_1(\theta_w)(\theta_w\cos\theta_w-\sin\theta_w)+Y(\theta_w)\theta_w\sin\theta_w}{(\theta_w^2-\sin^2\theta_w)}, \\
B_1 &= \frac{1}{64}\left(T_1+\frac{3}{2}T_6\right),\quad B_2=\frac{1}{64}\left(T_2-\frac{3}{2}T_5\right), \\
B_3 &= -\frac{1}{8}T_3,\ B_4=-\frac{1}{8}T_4,\ B_5=\frac{1}{64}T_5,\ B_6=\frac{1}{64}T_6, \\
T_1 &= 8BM+6CM-6DN,\ T_2=8BN+6CN+6DM, \\
T_3 &= 2CM-2DL+2DN,\ T_4=2CL+2CN-2DM, \\
T_5 &= 8CN+8DM,\ T_6=8DN-8CM,\ B=\frac{\theta_w^2}{\theta_w^2-\sin^2\theta_w}, \\
C &= -\frac{\sin^2\theta_w}{\theta_w^2-\sin^2\theta_w},\ D=-\frac{(\theta_w-\cos\theta_w\sin\theta_w)}{\theta_w^2-\sin^2\theta_w} \\
K &= \frac{2\sin^2\theta_w}{4\sin^4\theta_w+\sin 2\theta_w(\sin 2\theta_w-2\theta_w)}, \\
L &= -\frac{2\sin 2\theta_w}{4\sin^4\theta_w+\sin 2\theta_w(\sin 2\theta_w-2\theta_w)}, \\
M &= -\frac{2\sin^2\theta_w}{4\sin^4\theta_w+\sin 2\theta_w(\sin 2\theta_w-2\theta_w)}, \\
N &= \frac{\sin 2\theta_w}{4\sin^4\theta_w+\sin 2\theta_w(\sin 2\theta_w-2\theta_w)}, \\
Y(\theta_w) &= B_1\cos 3\theta_w+B_2\sin 3\theta_w+B_3\theta_w^2\cos\theta_w \\
&+ B_4\theta_w^2\sin\theta_w+B_5\theta_w\cos 3\theta_w+B_6\theta_w\sin 3\theta_w, \\
Y_1(\theta_w) &= 3B_2\cos 3\theta_w+B_5\cos 3\theta_w-3B_1\sin 3\theta_w \\
&+ B_6\sin 3\theta_w+3B_6\theta_w\cos 3\theta_w+B_4\theta_w^2\cos\theta_w \\
&- 3B_5\theta_w\sin 3\theta_w-B_3\theta_w^2\sin\theta_w+2B_3\theta_w\cos\theta_w \\
&+ 2B_4\theta_w\sin\theta_w, \\
D_1 &= \frac{4(F_3+Z(\theta_w))\sin^2\theta_w-2\theta_w Z_1(\theta_w)}{4(\cos 2\theta_w+\theta_w\sin 2\theta_w-1)} \\
&- \frac{(F_1+4F_4)(\sin 2\theta_w-2\theta_w\cos 2\theta_w)}{4(\cos 2\theta_w+\theta_w\sin 2\theta_w-1)} \\
&- \frac{(4F_3\theta_w-Z_1(\theta_w))\sin 2\theta_w}{4(\cos 2\theta_w+\theta_w\sin 2\theta_w-1)}, \\
D_2 &= \frac{(F_1+4F_4+Z_1(\theta_w))(1-\cos 2\theta_w)}{2(\cos 2\theta_w+\theta_w\sin 2\theta_w-1)} \\
&+ \frac{(2F_3-2Z(\theta_w))\sin 2\theta_w}{2(\cos 2\theta_w+\theta_w\sin 2\theta_w-1)}, \\
D_3 &= \frac{4(F_3-Z(\theta_w))\sin^2\theta_w+(2\theta_w-\sin 2\theta_w)Z_1(\theta_w)}{4(\cos 2\theta_w+\theta_w\sin 2\theta_w-1)} \\
&+ \frac{(F_1+4F_4)(\sin 2\theta_w-2\theta_w\cos 2\theta_w)}{4(\cos 2\theta_w+\theta_w\sin 2\theta_w-1)},
\end{aligned}
$$

$$
\begin{aligned}
D_4 &= \frac{(F_1+4F_4-Z_1(\theta_w))(1-\cos 2\theta_w)}{4(\cos 2\theta_w+\theta_w \sin 2\theta_w-1)} \\
&\quad - \frac{(2F_3-2Z(\theta_w))\sin 2\theta_w}{4(\cos 2\theta_w+\theta_w \sin 2\theta_w-1)} \\
&\quad - \frac{2(F_1+4F_4)\theta_w \sin 2\theta_w}{4(\cos 2\theta_w+\theta_w \sin 2\theta_w-1)} \\
F_1 &= \frac{1}{16}S_2,\ F_2=-\frac{1}{16}S_1,\ F_3=\frac{1}{192}S_3,\ F_4=\frac{1}{192}S_4, \\
S_1 &= 8LM,\ S_2=8LN,\ S_3=16MN,\ S_4=8(N^2-M^2), \\
Z(\theta_w) &= F_1\theta_w\cos 2\theta_w+F_2\theta_w \sin 2\theta_w+F_3\cos 4\theta_w+F_4\sin 4\theta_w, \\
Z_1(\theta_w) &= F_1\cos 2\theta_w-2F_1\theta_w\sin 2\theta_w+F_2\sin 2\theta_w+2F_2\theta_w\cos 2\theta_w \\
&\quad -4F_3\sin 4\theta_w+4F_4\cos 4\theta_w,
\end{aligned}
$$

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Asif Mahmood is an Assistant Professor of Mathematics at the Pennsylvania State University, USA. He received his M.Sc. and M.Phil. from Quaid-i-Azam University, Islamabad, Pakistan; and his PhD from the University at Buffalo, USA. His research interests are in the areas of applied mathematics, more specifically, fluid mechanics and geophysical mass flows. He has published research articles in reputed international journals of mathematical and engineering sciences.