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Star Edge Coloring of Subcubic Graphs

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Abstract: A proper edge coloring of a graph G is called star edge coloring if there is no bi-colored path or cycle of length four in G. The minimum number of colors needed to star color the edges of G is called the star chromatic index of G, denoted by $\chi_s'(G)$. In 2013 [1], Dvořák et. al. proved that for a subcubic graph G, $\chi_s'(G) \leq 7$ and conjectured that it is less than or equal to 6. In this paper, we show that if a subcubic graph G has maximum average degree less than $\frac{8}{3}$ then $\chi_s'(G) \leq 6$.

Keywords: Star edge coloring, star chromatic index, subcubic graphs, maximum average degree

1 Introduction

All the graphs considered in this paper are finite and simple. For a graph G with vertex set $V(G)$ and edge set $E(G)$ a proper edge coloring of G is an assignment of colors to the edges of G so that no two adjacent edges receive the same color. A *star edge coloring* is a proper edge coloring with an additional condition that any path or cycle of length four is not bi-colored. The *star chromatic index* of G is the least number of colors needed for a star edge coloring of G, denoted by $\chi_s'(G)$. This coloring was introduced by Liu and Deng [2] in 2008. In 2013 [1], Dvořák, Mohar and Šámal found the following bound for the star chromatic index of complete graphs.

Theorem 1.*The star chromatic index of the complete graph K_n satisfies*

$$(2+o(1))n \leq \chi_s'(K_n) \leq n\frac{2^{2\sqrt{2}(1+o(1))\sqrt{logn}}}{(logn)^{\frac{1}{4}}}.$$

In particular, for every $\varepsilon > 0$ there exists a constant c such that $\chi_s'(K_n) \leq cn^{1+\varepsilon}$ for every $n \geq 1$.

They also obtained a near-linear upper bound in terms of the maximum degree Δ for general graphs. In addition, they considered subcubic graph - a graph in which each vertex has degree at most three and showed that the star chromatic index of every subcubic graph is at most 7. L. Bezegová et. al.[3] proved that the star chromatic index of every subcubic tree and subcubic outerplanar graph is at most 4 and 5 respectively.

Motivated by the strong list edge coloring of subcubic graphs [4], [5], list version of star edge coloring is also studied and bounds are given in terms of maximum average degree of the graph. The maximum average degree of a graph G, denoted by $mad(G)$ is defined as $mad(G) = max_{H\subseteq G, |V(H)|\geq 1}\{\frac{2|E(H)|}{|V(H)|}\}$. S. Kerdjoudj et.al. [6] proved that the list star chromatic index of a subcubic graph G with $mad(G) < \frac{7}{3}$ and $mad(G) < \frac{5}{2}$ is at most 5 and 6 respectively. They also showed that the list star chromatic index for every subcubic graph is at most 8. S. Kerdjoudj and A. Raspaud [7] proved that for subcubic graphs with $mad(G) < \frac{30}{11}$ it is at most 7. Recently, B. Lužar et.al.[8] proved that 7 colors suffice for the list star edge coloring of every subcubic graph G.

There are examples for subcubic graphs with the star chromatic index equal to 6, (e.g. $K_{3,3}$ and K_4 with one subdivided edge) but no example of a subcubic graph requiring 7 colors is known. Based on this fact, Dvořák et al.[1] proposed the following conjecture.

Conjecture 1 If G is a subcubic graph, then $\chi_s'(G) \leq 6$.

In this paper we prove the following theorem in support of the Conjecture 1.

Theorem 2.*Let G be a subcubic graph with $mad(G) < \frac{8}{3}$. Then, $\chi_s'(G) \leq 6$.*

The *girth* of a graph G is the length of a shortest cycle in G. For every planar graph with girth g, $mad(G) < \frac{2g}{g-2}$. This, together with the theorem above gives the following corollary.

Corollary 1.*Let G be a planar subcubic graph with girth $g \geq 8$. Then $\chi_s'(G) \leq 6$.*

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Notations: Let $d(v)$ denote the degree of the vertex v in G. A vertex of degree k is called a k-vertex. A k-vertex adjacent to a vertex v is a k-neighbor of v. A 3_k-vertex is a 3-vertex adjacent to exactly k $(0 \le k \le 3)$ vertices of degree 2. A 3-vertex adjacent to a 1-vertex is a *light* 3-vertex. For an edge coloring φ of a graph G, let $\varphi(v)$ denote the set of colors used on the edges incident with the vertex $v \in V(G)$ in the coloring φ. Similarly, for an edge $uv \in E(G)$, $\varphi(uv)$ denotes the color used on the edge uv. We say that a color c is an *available color* for an edge uv, if c is not assigned to any of its neighbors and there is no bi-colored path of length four or cycle of length four involving uv when colored with c. Otherwise, it is a *forbidden color*. The set of available colors for the edge uv is denoted by $A(uv)$.

Proof of Theorem 2

Let G' be a minimum counterexample minimising $|E(G)+V(G)|$. Then the edges of G' are not star colorable with six colors and $mad(G') < \frac{8}{3}$. By minimality of G', we can assume that it is connected. Otherwise, we can star color independently the edges of each connected component of G' with six colors. We list some structures or set of some subgraphs called reducible configurations in G'. We prove all the claims by contradiction. For each of the claim, we suppose that the described structure exists in G', then we remove a certain number of edges from G' to form a graph H, which by minimality of G' is star edge colorable with six colors. Let φ be such a star edge coloring of H. We show a contradiction by extending the star edge coloring φ of H with six colors to a star edge coloring of G' with same number of colors. Before going into the proof we need the following observation.

Observation:
Let u and v be two adjacent 3-vertices in G'. Let $N(v) = \{v_1, v_2, u\}$ and $N(u) = \{u_1, u_2, v\}$. Let φ be a star edge coloring of $H = G' \setminus \{uu_i, i = 1,2\}$ with six colors. Let v_1 not adjacent to v_2 (in worst case) be two 3-vertices. If $|\varphi(v_1) \cap \varphi(v_2)| \ge 1$, then there is at least one color that is not present on the edges incident to the vertices v_1 and v_2. If the edge uv is not colored with this color, then we can recolor it with this color. If $|\varphi(v_1) \cap \varphi(v_2)| = 0$, then we can assume that any available color for the edge uv is such that, either $\varphi(uv) \in \varphi(v_1)$ or $\varphi(uv) \in \varphi(v_2)$. In the worst case, in order to exclude greater number of colors for the edges incident with u, we always assume the latter part $(|\varphi(v_1) \cap \varphi(v_2)| = 0)$.
When v_1 is adjacent to v_2 or either of v_1 or v_2 are 2-vertices then we can always recolor the edge uv with a color that is not present on the edges incident to the vertices v_1 and v_2.

Claim 1. G' does not contain a 1-vertex adjacent to a 2-vertex.

Suppose there is a 1-vertex u adjacent to a 2-vertex v in G'. As G' is a minimum counterexample, $H = G' \setminus \{uv\}$ is star edge colorable with six colors. By counting the number of maximum possible forbidden colors for uv, it is easy to see that we have at least three colors available for it. So, the coloring of H can be extended to G', a contradiction.

Claim 2. G' does not contain a 3-vertex adjacent to two 1-vertices.

Let a 3-vertex u be adjacent to the 1-vertices v and w in G'. Consider $H = G' \setminus \{uv\}$, which by minimality of G' is star edge colorable with six colors. By counting the number of maximum possible forbidden colors for uv, again it is easy to see that we have at least two colors available for it. So, the coloring of H can be extended to G', a contradiction.

Claim 3. G' does not contain a 3-vertex adjacent to a 1-vertex and a 2-vertex.

Suppose u be such 3-vertex adjacent to the 1-vertex v and a 2-vertex w. The graph $H = G' \setminus \{uv\}$ has a star edge coloring with six colors. Again, by counting the number of maximum possible forbidden colors for the edge uv it is easy to see that there is at least one color available for uv, a contradiction.

Claim 4. G' does not contain two adjacent *light* 3-vertices.

Let u and v be two adjacent *light* 3-vertices adjacent to the 1-vertices u_1 and v_1 respectively in G'. Let x and y be the other neighbors of u and v respectively. By claim 3, x and y are 3-vertices. Consider $H = G' \setminus \{uu_1\}$, which by minimality of G' is star edge colorable with six colors. By counting the number of available colors it is easy to see that we have at least one color available for uu_1. So, the coloring of H can be extended to G', a contradiction.

Claim 5. G' does not contain two adjacent 2-vertices.

Suppose u_1 and u_2 be the two adjacent 2-vertices in G'. For $i = 1,2$, let v_i be the other neighbor of u_i. By minimality of G', $H = G' \setminus \{u_1u_2\}$ has a star edge coloring φ with six colors. When v_1 is adjacent to v_2 or when $v_1 = v_2$, by counting the number of available colors for u_1u_2, we can easily extend φ to G'. So let v_1 be distinct and not adjacent to v_2 with $N(v_i) = \{w_i, w_i', u_i\}$, $i = 1,2$. If there is an available color for the edge u_1u_2 we are done. Otherwise, all the six colors are forbidden for this edge, this means, $|\{\varphi(v_1) \cap \varphi(v_2)\}| = 0$ and $\varphi(u_iv_i) \in \{\varphi(w_i) \cap \varphi(w_i')\}$. So, we recolor the edge u_1v_1 using the Observation . This gives $|A(u_1u_2)| \ge 1$. Hence, φ can be extended to G', a contradiction.

Claim 6. G' does not contain a 3-vertex adjacent to three 2-vertices.

Let u be such a 3-vertex in G'. Let u_1, u_2 and u_3 be the 2-neighbors of u. Let each u_i be adjacent to x_i. By claim 1 and claim 5, x_i is a 3-vertex for each i. By minimality of G', $H = G' \setminus \{uu_i, i = 1,2,3\}$ has a star edge coloring φ with six colors. We extend this coloring to the edges uu_1, uu_2 and uu_3 in order. First, we color the edge uu_1 with a color such that it does not appear on the colored edges incident to the vertices x_1, u_2 and u_3. Then, we color the edge uu_2 with a color which does not appear on the colored edges incident to the vertices x_2, u and u_3. Finally, we have at least one color available for the edge uu_3. Hence, φ is extended to $G^{'}$, a contradiction.

Claim 7. $G^{'}$ does not contain a cycle $uvwu$, where u is either a *light* 3-vertex or a 2-vertex.

Suppose $uvwu$ be such a cycle in $G^{'}$. When u is adjacent to a 1-vertex u_1, consider $H = G^{'} \setminus \{uu_1\}$, which by minimality of $G^{'}$ is star edge colorable with six colors. By counting the number of maximum possible forbidden colors it is easy to see that we have at least one color available for uu_1. When u is a 2-vertex, consider $H = G^{'} \setminus \{uv\}$ having a star edge coloring φ with six colors. Let v_1 and w_1 be the neighbors of v and w respectively. If v_1 is adjacent to w_1 or $v_1 = w_1$, then it is easy to see that uv can be easily colored. Otherwise, (when v_1 and w_1 are non adjacent) if uv cannot be colored, then $|\varphi(v_1) \cap \varphi(w)| = 0$. So, we can recolor uw with a color other than $\varphi(uw)$ such that there is an available color for the edge uv, which is a contradiction.

Claim 8. $G^{'}$ does not contain a cycle $xuvwx$, where v and x are 2-vertices.

Suppose $G^{'}$ contains such a cycle $xuvwx$. Let v and x be the 2-vertices. Consider $H = G^{'} \setminus \{uv, ux, wv, wx\}$, which by minimality of $G^{'}$ has a star edge coloring φ with six colors. Clearly, u is not adjacent to w. Let y_1 and y_2 be the neighbors of u and w respectively (other than v and x). If $y_1 = y_2$, it can be observed that φ can be extended to the edges of $G^{'}$ easily. So, we can assume that y_1 is distinct from y_2. For the worst case, let $d(y_i) = 3, i = 1,2$. Let z_i and $z_i^{'}$ be the other neighbors of y_i. Recolor the edges uy_1 and wy_2 using the Observation. In worst case, we can assume that $\varphi(uy_1) \in \varphi(z_1)$ and $\varphi(wy_2) \in \varphi(z_2)$. So, we color the edges ux and uv with c_1 and c_2 respectively, such that $c_1 \notin \{\varphi(y_1), \varphi(wy_2)\}$ and $c_2 \notin \{\varphi(y_1), \varphi(wy_2), c_1\}$. We have, $|A(ux)| \geq 2$ and $|A(uv)| \geq 1$. This gives $|A(wx)| \geq 3$. There is an appropriate color for wx so that there is at least one color available for the edge wv. Hence, φ can be extended to the edges of $G^{'}$, a contradiction.

Claim 9. $G^{'}$ does not contain a 3-vertex adjacent to two *light* 3-vertices.

Suppose $G^{'}$ contains such a vertex u. Let $N(u) = \{v_1, v_2, x_3\}$. Let v_1 and v_2 be the *light* 3-vertices adjacent to the 1-vertices v and w respectively. Let x_1 and x_2 be the other 3-neighbors of v_1 and v_2 respectively. (By claim 3, both are 3-vertices). For $i = 1,2$, let $N(x_i) = \{z_i, z_i^{'}, v_i\}$ and (let $d(x_3) = 3$) $N(x_3) = \{z_3, z_3^{'}, u\}$. Consider $H = G^{'} \setminus \{v_1 v\}$, which by minimality has a star edge coloring φ with six colors. If $x_1 = x_2$, then by counting the number of maximum possible forbidden colors it is easy to see that we have at least one color available for vv_1. So, we assume that x_1 is distinct from x_2. Suppose there is no color available for $v_1 v$. Then, $|\{\varphi(u) \cap \varphi(x_1)\}| = 0$ and $\varphi(v_1 x_1) \in \{\varphi(z_1) \cap \varphi(z_1^{'})\}$ and $\varphi(v_1 u) \in \{\varphi(x_3) \cap \varphi(v_2)\}$ making bi-colored paths of length three.

Without loss of generality, we can assume a coloring that depicts this situation. Let $\varphi(v_1 x_1) = 1$, $\varphi(x_1 z_1) = 2$, $\varphi(x_1 z_1^{'}) = 3$, $\varphi(v_1 u) = 4$, $\varphi(ux_3) = 5$ and $\varphi(uv_2) = 6$.

When $\varphi(v_1 u) = \varphi(v_2 w) = 4$, if there is an available color for $v_2 w$ other than 4, we are done. Otherwise, we can assume that $\varphi(x_2) \subseteq \{1,2,3\}$, $\varphi(v_2 x_2) \in \{\varphi(z_2) \cap \varphi(z_2^{'})\}$ and $6 \in \varphi(x_3)$ forming bi-colored paths of length three. This situation is shown in Figure 1. Remove the colors of the edges uv_2 and $v_2 w$, then recolor the edge $v_2 x_2$ using the Observation. After recoloring, let the color of $v_2 x_2 = c$. Clearly, $c \in \{4,5,6\}$. If $c \in \{5,6\}$, we first color uv_2 with an available color from $\{1,2,3\}$ then, we color the edges $v_1 v$ and $v_2 w$ in order. Otherwise, we uncolor the edge $v_1 u$, then color uv_2 with an available color from $\{1,2,3\}$. Finally, we color $v_1 u$, $v_1 v$ and $v_2 w$ in order. At each step there is a color available for each of the edge.

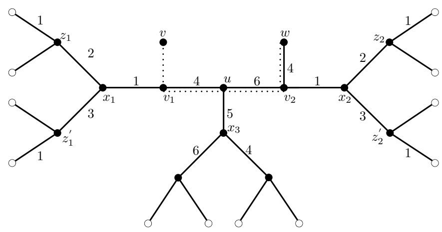

Fig. 1: Configuration of claim 9

When $\varphi(v_1 u) = \varphi(v_2 x_2) = 4$, we remove the colors of the edges uv_2 and $v_2 w$ and recolor the edge $v_2 x_2$ using the Observation. After recoloring, let the color of $v_2 x_2 = c$. In the worst case, let $c \in \varphi(z_2)$. If $c = 4$, we uncolor the edge $v_1 u$, this gives at least two colors available for uv_2. We choose a color other than 6 for uv_2. Then color uv_1, $v_1 v$ and $v_2 w$ in order. Otherwise, (when $c \neq 4$) we color uv_2, $v_1 v$ and $v_2 w$ in order. At each step there is at least one color available for each of the edge.

From the above, we can also see that a 3-vertex is not adjacent to a *light* 3-vertex and a 2-vertex.

Claim 10. $G^{'}$ does not contain a 3_2-vertex adjacent to either (i) a 3_1-vertex or (ii) a 3-vertex which is adjacent to a *light* 3-vertex.

Suppose to the contrary that $G^{'}$ contains a 3_2-vertex u adjacent to such a 3-vertex v_1. Let $N(v_1) = \{u, v_2, x_1\}$. Let v_2 be a 2-vertex. Let u_1 and u_2 be 2-neighbors of u. By claim 7, $v_2 \neq u_1$ or u_2. By claim 5, v_2 is not adjacent to u_1 or u_2. Let w_i be the other neighbor of u_i that is distinct from u and x_2 be a 3-neighbor of v_2 which is distinct from v_1. (By claim 5, x_2 and w_i, $i = 1,2$ are three vertices). Let $N(w_i) = \{z_i, z_i^{'}, u_i\}$ and $N(x_i) = \{y_i, y_i^{'}, v_i\}$. (If v_2 is a *light* 3-vertex, let the adjacent 1-vertex be v.) Since, $G^{'}$ is a minimum counterexample, $H = \{G^{'} \setminus \{uu_i\}, i = 1,2\}$ has a star edge coloring φ with six colors. To extend φ to $G^{'}$, we remove the colors of the edges uv_1 and $v_1 v_2$. (When v_2

is a *light* 3-vertex then, we remove the color of the edge v_2v also.) By claim 7, $x_1 \neq x_2$. By using the Observation for the edges v_ix_i, $i = 1,2$ we can assume that $\varphi(v_ix_i) \in \varphi(y_i)$.

When $\varphi(v_1x_1) = \varphi(v_2x_2)$, there are at most five forbidden colors for the edge v_1v_2 so, we color it with the available color. (Then we color the edge v_2v, when v_2 is a *light* 3-vertex, with the color $\varphi(x_1y_1)$, if it is available. Otherwise, $\varphi(x_1y_1) = \varphi(x_2y_2)$. So, we set the color of v_2v $= \varphi(x_2y_2'))$. This makes at most three colors forbidden for the edge uv_1. Therefore, $|A(uv_1)| \geq 3$.

When $\varphi(v_1x_1) \neq \varphi(v_2x_2)$, there are at least two colors available for the edge v_1v_2. We choose a color that appears on $\varphi(x_1)$ for the edge v_1v_2, if it is available. Otherwise, $|\varphi(x_1) \cap \varphi(x_2)| \geq 1$, so we get a color which does not appear on $\varphi(x_i)$, for both $i = 1,2$. We set this color for v_1v_2. (Then we color the edge v_2v as above when v_2 is a *light* 3-vertex.) Therefore, again there are at most three forbidden colors for the edge uv_1, $|A(uv_1)| \geq 3$.
In both the cases above we can observe that there is at least one color in $A(uv_1)$ which is not present in $\varphi(x_1)$ and $\varphi(v_2)$. Let $A(uv_1) = \{\alpha_1, \alpha_2, \alpha_3\}$.

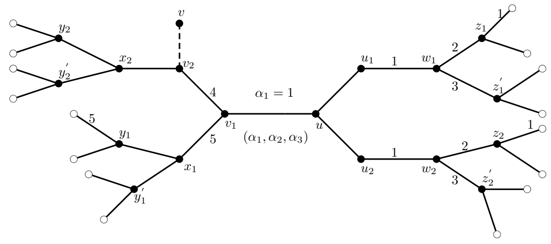

Fig. 2: Configuration of claim 10

By claim 8, $w_1 \neq w_2$, so, by using the Observation for the edges u_iw_i, we can assume that $\varphi(u_iw_i) \in \varphi(z_i)$, for $i = 1,2$. We choose a color, say $\alpha_1 \in A(uv_1)$, which is not present on the edges incident to the vertices x_1 and v_2. Set the color of $uv_1 = \alpha_1$. If $\alpha_1 \neq \varphi(u_iw_i)$ for both i, we can further extend the coloring easily to the edges of G'. If $\alpha_1 = \varphi(u_iw_i)$, for one i, say, $i = 1$, then color uu_1 and uu_2 in this order. At each step there is an available color for each edge. If $\alpha_1 = \varphi(u_iw_i)$, for both i, and φ cannot be extended then, $\{\varphi(w_i) \cap \varphi(v_1)\} = \{\alpha_1\}$. This coloring is shown in Figure 2. So, we replace α_1 by α_2, such that $\alpha_2 \notin \{\varphi(x_1), \varphi(v_2)\}$, if such a color exists, else we choose a color which is either in $\varphi(x_1)$ or in $\varphi(v_2)$. This makes the availability of at least two colors for one of the edges and at least one for the other. Hence, φ can be extended to G', which is a contradiction.

Claim 11. G' does not contain a 3-vertex adjacent to two 3_2-vertices.

Suppose u is a 3-vertex adjacent to two 3_2-vertices u_1 and u_2. Let u_3 be the third neighbor of u. Let $N(u_i) = \{v_i, v_i', u\}, i = 1,2,3$. For $i = 1,2$ let v_i and v_i' be the 2-neighbors of u_i. Let $N(v_i) = \{u_i, w_i\}$, $N(v_i') = \{u_i, w_i'\}$, $N(w_i) = \{x_i, x_i', v_i\}$ and $N(w_i') = \{y_i, y_i', v_i'\}$.
Consider $H = G' \setminus \{u_1v_1, u_1v_1'\}$. By minimality of G', H has a star edge coloring φ using six colors. If $v_1 = v_2$ and we cannot extend φ to the edges of G'. We recolor the edges $v_1'w_1'$ and uu_1 using the Observation. In the worst case, we assume that $\varphi(uu_1) \in \varphi(u_2)$ or $\varphi(uu_1) \in \varphi(u_3)$ and $\varphi(v_1'w_1') \in \varphi(y_1)$. If $\varphi(uu_1) = \varphi(v_1'w_1')$, then we color the edges u_1v_1' and u_1v_1 in this order. There exists at least one color for each of them. Otherwise, $|A(u_1v_1')| \geq 2$, so, we set the color of $u_1v_1' = \alpha \in A(u_1v_1')$, such that $\alpha \in \varphi(u)$ and $\alpha \notin \varphi(w_1')$, if such α exists. Else, we get a color $\alpha' \notin \{\varphi(u) \cup \varphi(w_1')\}$. So, we set the color of $u_1v_1' = \alpha'$. This gives $|A(u_1v_1)| \geq 1$ so, φ can be extended to G'. Similarly, we can extend φ to G' when $v_1' = v_2'$. Therefore, in G', v_1 is distinct from v_2 and v_1' is distinct from v_2'.
Also, it can be observed that for $i = 1,2$, v_i is not adjacent to v_i' (by claim 5) and w_i is distinct from w_i' (by claim 8).

Now, suppose φ cannot be extended to the edges of G'. We recolor the edges v_1w_1 and $v_1'w_1'$ using the Observation. In the worst case we can assume that the color of $v_1w_1 = (say)t_1 \in \varphi(x_1)$ and the color of $v_1'w_1' = (say)t_2 \in \varphi(y_1)$. If we get colors available for the edges u_1v_1 and u_1v_1', we are done. Otherwise, we recolor the edge uu_1. After recoloring, let the color of $uu_1 = \alpha$. We consider two cases.

Case 1: Suppose $\alpha \notin \{\varphi(u_2) \cup \varphi(u_3)\}$.

When $\alpha \neq t_1 \neq t_2$ or $\alpha \neq t_1 = t_2$ we can easily get available colors for the edges of G'. When $\alpha = t_1 \neq t_2$ ($\alpha = t_2 \neq t_1$ respectively), first we color u_1v_1 (u_1v_1' respectively), then we have $A(u_1v_1') \geq 2$ ($A(u_1v_1) \geq 2$ respectively).

When $\alpha = t_1 = t_2$ and there is no color available for one of the edges in G'. We extend this coloring to G' in the following way. Before recoloring the edge uu_1 with α -

Case 1.1: If $\varphi(uu_1)$ is different from α then, $\varphi(uu_1) \neq t_1 = t_2$. As φ could not be extended, $|\{\varphi(w_1) \cap \varphi(w_1')\}| = 3$ and all the three colors incident with the vertex u must be forbidden. So, $\varphi(uu_1)$ must appear on one of the edges incident to u_2 and also on u_3. So, the coloring can be assumed as follows. $\varphi(w_1v_1) = 1$, $\varphi(w_1x_1) = 2$, $\varphi(w_1x_1') = 3$, $\varphi(w_1'v_1') = 1$, $\varphi(w_1'y_1) = 2$ $\varphi(w_1'y_1') = 3$ and $\varphi(uu_1) = 4$, $\varphi(uu_2) = 5$, $\varphi(uu_3) = 6$. Let $\varphi(u_3v_3') = 4$ and $\varphi(u_2v_2') = 4$. After recoloring, now we get the case that the color of $uu_1 = \alpha \notin \{\varphi(u_2) \cup \varphi(u_3)\}$ and $\alpha = t_1 = t_2 = 1$. If there is any color available for the edge uu_1 other than 1 and 4, we are done. Otherwise, we can assume that $\varphi(v_2w_2) = 5$ and $6 \in \varphi(v_3)$. If $\varphi(v_2'w_2') \neq 1$ (or 1 is an available color for uu_2) we can swap the colors of the edges uu_1 and uu_2. Then it is easy to see that, there is at least one color available for each of the two edges u_1v_1 and u_1v_1' of G' and we are done. Therefore, we assume that $\varphi(v_2'w_2') = 1$. Now, we remove the colors of the edges u_2v_2, u_2v_2', uu_2

and uu_1. We set $\varphi(u_2v_2^{'}) = \alpha_1$ and $\varphi(u_2v_2) = \alpha_2$, such that $\alpha_1 \notin \{\varphi(w_2^{'}) \cup (\varphi(v_2w_2), \varphi(uu_3))\}$, (Clearly $\alpha_1 \in \{2,3,4\}$) and $\alpha_2 \notin \{\varphi(w_2) \cup (\varphi(uu_3), \alpha_1)\}$. For $\alpha_2 \neq 1$, we set $\varphi(uu_2) = 1$ and $\varphi(uu_1) = 5$. For $\alpha_2 = 1$ and $\alpha_1 = 2$ or 3, we set $\varphi(uu_2) = 4$ and $\varphi(uu_1) = 5$. For $\alpha_2 = 1$ and $\alpha_1 = 4$, we set $\varphi(uu_2) = 5$ and $\varphi(uu_1) \in \{2,3\}$. Therefore, in each case, the color of $uu_1 \neq t_1 = t_2$. So, φ can be easily extended to $G^{'}$.

Case 1.2: If $\varphi(uu_1)$ is the same as α then $|\{\varphi(u_2) \cap \varphi(u_3)\}| = 1$. If we get a color other than α for uu_1 we can easily extend the coloring to $G^{'}$. Otherwise, we can assume the coloring as in (i) except that the colors of the three edges out of u_2v_2, $u_2v_2^{'}$, u_3v_3 and $u_3v_3^{'}$ are from $\{2,3,4\}$ and $\alpha = t_1 = t_2 = 1$. If $\{\varphi(u_2v_2), \varphi(u_2v_2^{'})\} \in \{2,3,4\}$, then we can swap the colors of the edges uu_2 and uu_1 and hence, we are done. If $\{\varphi(u_3v_3), \varphi(u_3v_3^{'})\} \in \{2,3,4\}$, then we can again swap the colors of uu_2 and uu_1, whenever $\varphi(v_2^{'}w_2^{'}) \neq 1$ or 1 is available for uu_2. Otherwise, (when $\varphi(v_2^{'}w_2^{'}) = 1$), we remove the colors of the edges uu_1 and uu_2 and then recolor the edge uu_3 using the Observation. After recoloring $\varphi(uu_3) \in \{1,4,5\}$ (Observe that before recoloring $\varphi(uu_3) \in \{\varphi(v_3) \cap \varphi(v_3^{'})\}$). If $\varphi(uu_3) = 1$, retain $\varphi(uu_2) = 5$ and set $\varphi(uu_1) = 6$. If $\varphi(uu_3) = \{4,5\}$ and $\varphi(u_2v_2^{'}) \neq 6$, then set $\varphi(uu_2) = 6$ and $\varphi(uu_1) \in \{4,5\}$. When $\varphi(u_2v_2^{'}) = 6$, then $6 \in \varphi(w_2^{'})$ (as 1 is not available for uu_2). So, we remove the color of the edge $u_2v_2^{'}$ and recolor it with a color which is not in $\{\varphi(u) \cup \varphi(v_2) \cup \varphi(w_2^{'})\}$. There is at least one such color. Observe that after recoloring $\varphi(u_2v_2^{'}) \in \{2,3\}$, so we set $\varphi(uu_2) = 1$ and $\varphi(uu_1) = 6$. Therefore, in each case, the color of $uu_1 \neq t_1 = t_2$. So, φ can be easily extended to $G^{'}$.

Case 2: Suppose $\alpha \in \varphi(u_2)$ or $\alpha \in \varphi(u_3)$.

Case 2.1: When $\alpha \in \varphi(u_3)$. (and there is no color available for one of the edges).

From the Observation $|\{\varphi(u_2) \cap \varphi(u_3)\}| = 0$ so, without loss of generality we can assume that $\alpha = \varphi(u_3v_3) = 1$, $\varphi(uu_3) = 2$, $\varphi(u_3v_3^{'}) = 3$, $\varphi(uu_2) = 4$, $\varphi(u_2v_2) = 5$, $\varphi(u_2v_2^{'}) = 6$. In order to extend φ to the edges of $G^{'}$, we try to get a color for the edge uu_1 other than 1. If such color is available, we are done. Otherwise, we can assume that $\varphi(v_2w_2) = \varphi(v_2^{'}w_2{'}) = 4$. So, we swap the colors of the edges uu_1 and uu_2 and set the color of $uu_2 = 1$ and the color of $uu_1 = 4$. This situation is shown in Figure 3.

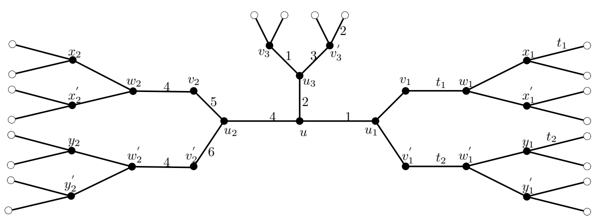

Fig. 3: Configuration of claim 11, case 2.1

Case 2.2: When $\alpha \in \varphi(u_2)$ and there is no color available for one of the edges.

Again from the Observation $|\{\varphi(u_2) \cap \varphi(u_3)\}| = 0$. Let $\alpha = \varphi(u_2v_2) = 1$, $\varphi(uu_3) = 2$, $\varphi(u_3v_3^{'}) = 3$, $\varphi(uu_2) = 4$, $\varphi(u_3v_3) = 5$, $\varphi(u_2v_2^{'}) = 6$. If there is an available color for the edge uu_1 other than 1 then we are done. Otherwise, we can assume that $2 \in \{\varphi(v_3) \cap \varphi(v_3^{'})\}$ and $\varphi(v_2^{'}w_2^{'}) = 4$. This situation is depicted in Figure 4. So, we remove the colors of the edges u_2v_2, uu_1 and uu_2. Then, we recolor the edge uu_3 using the observation. Clearly, after recoloring $\varphi(uu_3) \in \{1,4,6\}$.

When $\varphi(uu_3) = 4$, we retain the color of u_2v_2 and recolor the edge uu_2 with an available color from $\{3,5\}$ and set $\varphi(uu_1) = 2$.

When $\varphi(uu_3) = 6$, we set $\varphi(uu_2) = 1$ and color u_2v_2 with its available color. There is at least one available color for it, as $6 \notin \varphi(w_2^{'})$. If $\varphi(u_2v_2) \neq 2$, we set $\varphi(uu_1) = 2$. Otherwise, we set $\varphi(uu_1) = 4$.

When $\varphi(uu_3) = 1$, as $\varphi(v_2w_2) \neq 4$, we retain the colors of the edges u_2v_2 and uu_2 and set $\varphi(uu_1) = 2$.

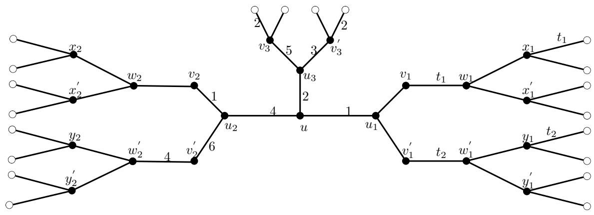

Fig. 4: Configuration of claim 11, case 2.2

Now, in cases 2.1 and 2.2 above, when $\alpha = 1 = t_1 = t_2$, after recoloring, $\varphi(uu_1) \neq t_1 = t_2$, so, there are at least two colors available for u_1v_1 and at least one for $u_1v_1^{'}$. When $\alpha = 1 \neq t_1 \neq t_2$, after recoloring, if $\varphi(uu_1) \neq t_1 \neq t_2$, there are at least three colors available for u_2v_2 and at least one for $u_1v_1^{'}$. Otherwise, if $\varphi(uu_1) = t_1 \neq t_2$ (resp. $= t_2 \neq t_1$), we color the edges u_1v_1 (resp. u_2v_2) and $u_1v_1^{'}$ (resp. $u_2v_2^{'}$) in order. At each step there is a color available for each of the edges. Hence, φ can be extended to $G^{'}$, a contradiction.

Now, we delete all the vertices of degree one from $G^{'}$ and obtain a graph $H^{'} = G^{'} \setminus \{v \in V(G^{'}) \mid d(v) = 1\}$. Clearly, it is connected and has maximum average degree $< \frac{8}{3}$. Using all the above claims we get the following reducible configurations in $H^{'}$.

C1) The minimum degree of $H^{'} \geq 2$. (from claims 1 & 2)

C2) $H^{'}$ does not contain -

C2.1) two adjacent 2-vertices. (from claims 3, 4 & 5)

C2.2) a 3-vertex adjacent to three 2-vertices. (from claims 6 & 9)

C2.3) a 3_1-vertex adjacent to a 3_2-vertex. (from claims 7, 8, 9 & 10)

C2.4) a 3-vertex adjacent to two 3_2-vertices. (from claims 8, 9 & 11)

Next, we set a weight function $w : V(H^{'}) \to \mathbb{R}$ with $w(v) = d(v) - \frac{8}{3}$, $\forall\ v \in V(H^{'})$. It can be easily observed that as $mad(H^{'}) < \frac{8}{3}$, the total sum of weights of all the vertices in the graph $H^{'}$ is strictly negative. That is, $\sum_{v\in V(H')} w(v) < 0$.

Then, we redistribute the weights among the vertices according to the discharging rules described below, to obtain the weight function $w^{'}$. During the discharging process, the sums of the values of $w^{'}$ and w, counting over all the vertices remain same.

Let $v \in V(H^{'})$ be a k-vertex, $k = 2,3$. The initial weight assigned to the 2-vertices is -$\frac{2}{3}$ units and to the 3-vertices is $\frac{1}{3}$ units. We move the positive weights from 3-vertices to 2-vertices using the following discharging rules:
A 3-vertex sends-
R1 $\frac{1}{3}$ units to the adjacent 3_2-vertex.
R2 $\frac{1}{3}$ units to each of the adjacent 2-vertices.
Case 1: Let $v \in V(H^{'})$ be a 2-vertex.
From C2.1 and C2.2, a 2-vertex v is adjacent to 3-vertices only. Therefore, using R2, the 3-vertices send $\frac{1}{3}$ units to v. So, $w^{'}(v) = -\frac{2}{3} + 2\times\frac{1}{3} = 0$.
Case 2: Let $v \in V(H^{'})$ be a 3-vertex.
From C2.2, a 3-vertex v can be adjacent to at most two 2-vertices.
When v is a 3_2-vertex, by C2.3 and C2.4, v is adjacent to at least one 3_0-vertex. So, by R1 it receives $\frac{1}{3}$ units from the 3_0-vertex, then by R2 it sends $\frac{1}{3}$ units to each of the adjacent 2-vertices. Therefore, $w^{'}(v) = \frac{1}{3} + \frac{1}{3} - 2\times\frac{1}{3} = 0$.
When v is a 3_1-vertex, it sends $\frac{1}{3}$ units to the adjacent 2-vertex. Therefore, by using R2, $w^{'}(v) = \frac{1}{3} - \frac{1}{3} = 0$.
When v is a 3_0-vertex, by C2.4 it can be adjacent to at most one 3_2-vertex. So, it sends $\frac{1}{3}$ units to it. Therefore, by using R1, $w^{'}(v) = \frac{1}{3} - \frac{1}{3} = 0$. If v is not adjacent to any 3_2-vertex, then its weight remains unchanged. So, $w^{'}(v) = w(v) = \frac{1}{3}$.

Therefore, $w^{'}(v) \geq 0$ for every $v \in V(H^{'})$. This leads us to the contradiction $0 \leq \sum_{v\in V(H')} w^{'}(v) = \sum_{v\in V(H')} w(v) < 0$. Therefore, $H^{'}$ cannot exist. Hence, the minimal counter example $G^{'}$ also does not exist. This completes the proof.

2 Conclusion

We have determined the star chromatic index of graphs with maximum degree $\Delta \leq 3$ and maximum average degree less than $\frac{8}{3}$. We have shown that $\chi_s^{'}(G) \leq 6$ for such graphs.

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