Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/130205

Squeeze Operators in Classical Scenarios

Jorge A. Anaya-Contreras[1], *Arturo Zúñiga-Segundo*[1], *Francisco Soto-Eguibar*[2,*], *Víctor Arrizón*[2] *and Héctor M. Moya-Cessa*[2]

[1] Instituto Politécnico Nacional, ESFM, Departamento de Física. Edificio 9, Unidad Profesional Adolfo López Mateos, CP 07738 CDMX, Mexico
[2] Instituto Nacional de Astrofísica, Óptica y Electrónica, Calle Luis Enrique Erro 1, Santa María Tonantzintla, Puebla, 72840, Mexico

Received: 12 Nov. 2018, Revised: 23 Dec. 2018, Accepted: 20 Jan. 2019
Published online: 1 Mar. 2019

Abstract: We analyse the paraxial field propagation in the realm of classical optics, showing that it can be written as the action of the fractional Fourier transform, followed by the squeeze operator applied to the initial field. Secondly, we show that a wavelet transform may be viewed as the application of a displacement and squeeze operator onto the mother wavelet function.

Keywords: Squeezed states, squeeze operator, paraxial propagation, fractional Fourier transform

1 Introduction

In the late seventies, squeezed states were introduced [1, 2]. On the one hand, Yuen [3] defined them squeezing the vacuum and then displacing the resulting state. On the other hand, Caves [4] defined them by displacing the vacuum and then squeezing the produced coherent state. Squeezed states have been shown to produce ringing revivals (a fingerprint that a squeezed state is used) in the interaction between light and matter [5]. Applications of quantum techniques in classical optics have been the subject of many studies during the last years [6,7]. Along the same line, one of the goals of this article is to show, that in a mathematical sense, the squeeze operator could have been introduced in the description of free light propagation, i.e. in the domain of classical optics, at least hundred years earlier. We also show that we can use such squeeze operators to write the continuous wavelet transform as its average with the mother wavelet function, a displacement operator and the function to be transformed.

2 Squeezed states

As we already explained in the introduction, there are two equivalent forms to define the squeezed states. In the first one, introduced by Yuen [3], squeezed states are obtained from the vacuum as

$$|\alpha;r\rangle = \hat{S}(r)\hat{D}(\alpha)|0\rangle = \hat{S}(r)|\alpha\rangle, \tag{1}$$

where

$$\hat{S}(r) = \exp\left[\frac{r}{2}\left(\hat{a}^2 - \hat{a}^{\dagger 2}\right)\right] \tag{2}$$

is the squeeze operator and $\hat{D}(\alpha)$ is the Glauber displacement operator [8]. Here $\hat{a} = \frac{1}{\sqrt{2}}(x + d/dx)$ and $\hat{a}^\dagger = \frac{1}{\sqrt{2}}(x - d/dx)$ are the ladder operators [9]. In this view, squeezed states are created displacing the vacuum, and after, squeezing it. Note that when the squeeze parameter r is set to zero, the squeezed states reduce to the coherent states. In this work, we will consider only real squeeze parameters, as that is enough for our intentions.
In the definition of the squeezed states followed by Caves [4], the vacuum is squeezed and the resulting state is then displaced; which means that in this approach, they are given by the expression

$$|\alpha';r'\rangle = \hat{D}\left(\alpha'\right)\hat{S}\left(r'\right)|0\rangle. \tag{3}$$

Both definitions of the squeezed states agree when the squeeze factor is the same, $r' = r$, and when the modified amplitude α' of the Caves approach is given by

$$\alpha' = \mu\alpha - \nu\alpha^*, \tag{4}$$

* Corresponding author e-mail: feguibar@inaoep.mx

being
$$\mu = \cosh r \tag{5}$$
and
$$\nu = \sinh r. \tag{6}$$
To analyse the uncertainties in the position and in the momentum of the squeezed states, we introduce, following Loudon and Knight [1], the quadrature operators
$$\hat{X} = \frac{\hat{a}+\hat{a}^\dagger}{2} = \frac{\hat{x}}{\sqrt{2}} \tag{7}$$
and
$$\hat{Y} = \frac{\hat{a}-\hat{a}^\dagger}{2i} = \frac{\hat{p}}{\sqrt{2}}. \tag{8}$$

In order to show that really the squeezed states are minimum uncertainty states, we need to calculate the expected values in the squeezed state (1) of the quadrature operators (7) and (8), and its squares. Using (7) and (1), we obtain
$$\langle\alpha;r|\hat{X}|\alpha;r\rangle = \langle\alpha|\hat{S}^\dagger(r)\frac{\hat{a}+\hat{a}^\dagger}{2}\hat{S}(r)|\alpha\rangle. \tag{9}$$
The action of the squeeze operator on the creation and annihilation operators is obtained using the Hadamard's lemma [10,11,12],
$$\hat{S}^\dagger(r)\hat{a}\hat{S}(r) = \mu\hat{a}-\nu\hat{a}^\dagger,\ \hat{S}^\dagger(r)\hat{a}^\dagger\hat{S}(r) = \mu\hat{a}^\dagger-\nu\hat{a}, \tag{10}$$
such that
$$\hat{S}^\dagger(r)\frac{\hat{a}+\hat{a}^\dagger}{2}\hat{S}(r) = e^{-r}\hat{X}. \tag{11}$$
Therefore, as $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and $\langle\alpha|\hat{a}^\dagger = \langle\alpha|\alpha^*$, it is easy to see that
$$\langle\alpha;r|\hat{X}|\alpha;r\rangle = e^{-r}\frac{\alpha+\alpha^*}{2}, \tag{12}$$
and that
$$\langle\alpha;r|\hat{X}^2|\alpha;r\rangle = e^{-2r}\frac{1+2|\alpha|^2+\alpha^2+\alpha^{*2}}{4}. \tag{13}$$
So, we obtain for the uncertainty in the quadrature operator $\hat{X}$,
$$\Delta X \equiv \sqrt{\langle\alpha;r|\hat{X}^2|\alpha;r\rangle - \langle\alpha;r|\hat{X}|\alpha;r\rangle^2} = \frac{e^{-r}}{2}. \tag{14}$$
Proceeding in exactly the same way for the quadrature operator $\hat{Y}$, we obtain
$$\Delta Y \equiv \sqrt{\langle\alpha;r|\hat{Y}^2|\alpha;r\rangle - \langle\alpha;r|\hat{Y}|\alpha;r\rangle^2} = \frac{e^{r}}{2}. \tag{15}$$
As mentioned before, we can then think in the position eigenstates and in the momentum eigenstates as limiting cases of squeezed states.

3 Squeeze and fractional Fourier operators in paraxial optics

The propagation of light in free space can be described by the paraxial equation
$$i\frac{\partial E(x,y,z)}{\partial z} = -\frac{1}{2}\frac{\partial^2 E(x,y,z)}{\partial x^2} - \frac{1}{2}\frac{\partial^2 E(x,y,z)}{\partial y^2}, \tag{16}$$
where we have set the wavevector k equal to one. We define $\hat{p}_\alpha = -i\partial/\partial\alpha$, with $\alpha = x,y$ such that we rewrite the above equation as (we obviate the variables x and y)
$$i\frac{\partial E(z)}{\partial z} = \frac{\hat{p}_x{}^2+\hat{p}_y{}^2}{2}E(z), \tag{17}$$
that allows to give the simple formal solution
$$E(z) = \exp\left[-i\frac{z}{2}(\hat{p}_x{}^2+\hat{p}_y{}^2)\right]E(0). \tag{18}$$
We use the annihilation and creation operators for the harmonic oscillator,
$$\hat{a}_\alpha = \frac{\hat{\alpha}+i\hat{p}_\alpha}{\sqrt{2}}, \qquad \hat{a}_\alpha^\dagger = \frac{\hat{\alpha}-i\hat{p}_\alpha}{\sqrt{2}}, \qquad \alpha = x,y \tag{19}$$
to cast Eq. (18) into
$$\begin{aligned} E(z) = {} & \exp\left[-i\frac{z}{2}\left(\hat{n}_x+\frac{1}{2}-\frac{\hat{a}_x^2}{2}-\frac{\hat{a}_x^{\dagger 2}}{2}\right)\right] \\ & \times \exp\left[-i\frac{z}{2}\left(\hat{n}_y+\frac{1}{2}-\frac{\hat{a}_y^2}{2}-\frac{\hat{a}_y^{\dagger 2}}{2}\right)\right]E(0), \end{aligned} \tag{20}$$
with $\hat{n}_\alpha = \hat{a}_\alpha^\dagger\hat{a}_\alpha$, the number operator for each variable. In the following, we show how to factorize this exponential as the product of a squeeze and a fractional Fourier transform operators[13].

3.1 Evolution operator factorization

Each exponential in (20) may be written as
$$\exp\left[-i\frac{z}{2}\left(2\hat{K}_0-\hat{K}_+-\hat{K}_-\right)\right], \tag{21}$$
with (for simplicity, we drop the α subindexes of the annihilation and creation operators)
$$\hat{K}_0 = \frac{1}{2}\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right), \qquad \hat{K}_+ = \frac{\hat{a}^{\dagger 2}}{2}, \qquad \hat{K}_- = \frac{\hat{a}^2}{2}, \tag{22}$$
which are the elements of a Lie Algebra $su(1,1)$ [12] and satisfy the following commutation relations
$$\left[\hat{K}_-,\hat{K}_+\right] = 2\hat{K}_0, \qquad \left[\hat{K}_0,\hat{K}_+\right] = \hat{K}_+, \qquad \left[\hat{K}_0,\hat{K}_-\right] = -\hat{K}_-. \tag{23}$$

According to Fisher *et al.* [14] the generators of the $su(1,1)$ algebra admit the matrix representation

$$\hat{K}_0 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \quad (24)$$

and

$$\hat{K}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \hat{K}_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad (25)$$

such that Eq. (21) may be rewritten as

$$\exp\left[-i\frac{z}{2}\left(2\hat{K}_0 - \hat{K}_+ - \hat{K}_-\right)\right] = \begin{pmatrix} 1-i\frac{z}{2} & i\frac{z}{2} \\ -i\frac{z}{2} & 1+i\frac{z}{2} \end{pmatrix}, \quad (26)$$

because

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}^n = 0 \qquad \text{for all} \qquad n = 2,3,\ldots. \quad (27)$$

We now assume that two complex numbers, $\xi = re^{i\theta}$ and ω, exist, such that

$$\begin{aligned} \begin{pmatrix} 1-i\frac{z}{2} & i\frac{z}{2} \\ -i\frac{z}{2} & 1+i\frac{z}{2} \end{pmatrix} &= \exp\left[-i\frac{z}{2}\left(2\hat{K}_0 - \hat{K}_+ - \hat{K}_-\right)\right] \\ &= \exp(-i2\omega\hat{K}_0)\exp\left(\xi\hat{K}_+ - \xi^*\hat{K}_-\right) \\ &= \begin{pmatrix} e^{-i\omega}\cosh r & e^{i(\theta-\omega)}\sinh r \\ e^{i(\omega-\theta)}\sinh r & e^{i\omega}\cosh r \end{pmatrix}, \end{aligned} \quad (28)$$

with

$$1 + i\frac{z}{2} = e^{i\omega}\cosh r, \qquad -i\frac{z}{2} = e^{-i\theta}e^{i\omega}\sinh r, \quad (29)$$

or

$$e^{i\omega} = \frac{1+i\frac{z}{2}}{\sqrt{1+\left(\frac{z}{2}\right)^2}}, \qquad e^{i\theta} = ie^{i\omega}, \quad (30)$$

and

$$r = \ln\left(\sqrt{1+\left(\frac{z}{2}\right)^2} - \frac{z}{2}\right). \quad (31)$$

Therefore, we may write

$$\begin{aligned} \exp\left(-i\frac{z}{2}\hat{p}^2\right) &= \hat{S}\left(ire^{-i\omega}\right)\exp\left[-i\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\right] \\ &= \hat{S}(\xi)\hat{F}(\omega), \end{aligned} \quad (32)$$

where $\hat{S}(\xi)$ is the squeeze operator [4,3], Eq. (2), and $\hat{F}(\omega)$ is the fractional Fourier transform (see for instance Namias [13]), with

$$\xi = ire^{-i\omega}. \quad (33)$$

Then the solution to the paraxial wave equation reads

$$E(z) = \hat{S}_x(\xi)\hat{S}_y(\xi)\hat{F}_x(\omega)\hat{F}_y(\omega)E(0), \quad (34)$$

that is nothing but the application of squeeze operators applied to the two-dimensional fractional Fourier transform of the field at $z = 0$. It is not difficult to show that for large z, $\omega \to \pi/2$ such that $\xi \to r$ and the fractional Fourier transform becomes the (complete) Fourier transform. The solution to the paraxial equation for z large therefore reads

$$E(x,y,z) = \hat{S}_x(r)\hat{S}_y(r)\tilde{E}(x,y,0), \quad (35)$$

where $\tilde{E}(x,y,0)$ is the two-dimensional Fourier transform of $E(x,y,0)$. Further application of the squeeze operator yields

$$E(x,y,z) = e^{-r}\tilde{E}(xe^{-r}, ye^{-r}, 0). \quad (36)$$

As can also be shown from Eqs.(30), when z is very large $r \to \ln(z)$, thus

$$E(x,y,z) = \frac{1}{z}\tilde{E}\left(\frac{x}{z}, \frac{y}{z}, 0\right), \quad (37)$$

which, up to a phase, is the expected expression [15,16,17].

4 Wavelet transforms

The integral (continuous) wavelet transform of a function $f(x)$ is given by [18,19,20]

$$\mathscr{F}(a,b) = \frac{1}{\sqrt{|a|}}\int_{-\infty}^{\infty}\psi^*\left(\frac{x-b}{a}\right)f(x)dx, \quad (38)$$

where $\psi(x)$ is the so called mother wavelet function. Because $\exp(ib\hat{p})g(x) = g(x+b)$ the above equation may be written in the form

$$\mathscr{F}(a,b) = \frac{1}{\sqrt{|a|}}\int_{-\infty}^{\infty}\psi^*\left(\frac{x}{a}\right)\exp(ib\hat{p})f(x)dx, \quad (39)$$

and using the squeeze operator introduced above and the equations presented in the Appendix, we may write in Dirac notation the simple form

$$\mathscr{F}(a,b) = \langle\psi|\hat{S}^\dagger(r)\exp(ib\hat{p})|f\rangle. \quad (40)$$

If we choose the very simple mother wavelet function, namely the state $|0\rangle$, i.e., the Hermite-Gaussian $\psi_0(x)$, the wavelet integral transform reduces to [21,22]

$$\mathscr{F}(a,b) = \langle b,r|f\rangle. \quad (41)$$

where $|b,r\rangle$ has the form of a squeezed state, equation (3) with $r = -\ln a$.

5 Conclusions

We have shown that some techniques that are common in quantum mechanics may be applied in classical scenarios used in optics. In particular, we have written the free propagation of a field as the application of the product of squeeze operators corresponding to the variables x and y and the two-dimensional fractional Fourier operator to the field at $z = 0$. Finally we show that it is possible to write the continuous wavelet transform as the application of a ”bra” mother wavelet to a ”ket” that corresponds to the function to be transformed.

Appendix

A function $\psi(x)$ may be expanded in Hermite-Gaussian functions as

$$\psi(x) = \sum_{n=0}^{\infty} c_n \psi_n(x), \tag{42}$$

with

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \exp(-\frac{x^2}{2}) H_n(x), \tag{43}$$

where $H_n(x)$ are the Hermite polynomials of order n. The coefficients c_n may be calculated from the integral $c_n = \int_{-\infty}^{\infty} \psi(x)\psi_n(x)dx$. In Dirac notation the above may be casted as

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \tag{44}$$

where the states $|n\rangle$ are the number or Fock states. The coefficients c_n are calculated by the quantity $c_n = \langle n|\psi\rangle$. If we apply a squeeze operator to the function $\psi(x)$ we obtain

$$S(r)\psi(x)1 = S(r)\psi(x)S^{\dagger}(r)S(r)1, \tag{45}$$

where we have multiplied by 1 and we have introduced an extra 1, namely $S^{\dagger}(r)S(r)$, in the right hand side of the equation above. From (11), we can see that $S(r)\psi(x)S^{\dagger}(r) = \psi(xe^{-r})$ and the action of $S(r)$ on 1 is

$$\begin{aligned} S(r)1 &= \exp\left[-i\frac{r}{2}(x\hat{p} + \hat{p}x)\right] 1 \\ &= \exp\left[i\frac{r}{2}(2x\hat{p} - i)\right] 1 = \exp(-r/2). \end{aligned} \tag{46}$$

Now, the integral of two functions is given by

$$\int_{-\infty}^{\infty} \psi^*(x) f(x) dx = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \int_{-\infty}^{\infty} c_n^{(\psi)*} c_m^{(f)} \psi_n(x)\psi_m(x), \tag{47}$$

where we have made explicitly that the coefficients are related to specific functions. Because of orthogonality it reduces to

$$\int_{-\infty}^{\infty} \psi^*(x) f(x) dx = \sum_{n=0}^{\infty} c_n^{(\psi)*} c_n^{(f)} = \langle \psi | f \rangle. \tag{48}$$

References

[1] R. Loudon and P.L. Knight, Squeezed light, J Mod Opt Vol. 34, No. 6-7, pp. 709-759 (1987).

[2] S.M. Barnett, A. Beige, A. Ekert, B.M. Garraway, *et al.*, Journeys from quantum optics to quantum technology, Progress in Quantum Electronics, Vol. 54, Special Issue, pp. 19-45 (2017).

[3] H.P. Yuen, Two-photon coherent states of the radiation field, Phys Rev A Vol. 13, No. 6, pp. 2226-2243 (1976).

[4] C.M. Caves, Quantum-mechanical noise in an interferometer, Phys Rev D Vol. 23, No. 8, pp. 1693-1708 (1981).

[5] H. Moya-Cessa and A. Vidiella-Barranco, Interaction of squeezed light with two-level atoms, J Mod Optics Vol. 39, No. 1, pp. 2481-2499 (1992).

[6] A. Perez-Leija, R. Keil, A. Szameit, A.F. Abouraddy, H. Moya-Cessa, D.N. Christodoulides, Tailoring the correlation and anticorrelation behavior of path-entangled photons in Glauber-Fock oscillator lattices, Phys Rev A Vol. 85, No. 1, 013848 (2012).

[7] A. Perez-Leija, J.C. Hernandez-Herrejon, H. Moya-Cessa, A. Szameit and D.N. Christodoulides, Generating photon-encoded W states in multiport waveguide-array systems, Phys Rev A Vol. 87, No. 1, 013842 (2013).

[8] R. J. Glauber, Coherent and Incoherent States of the Radiation Field, Phys Rev Vol. 131, No. 6, pp. 2766-2788, (1963).

[9] G. Arfken, *Mathematical methods for physicists,* Academic Press, Inc., 3rd Edition, 1985.

[10] H.M. Moya-Cessa and F. Soto-Eguibar, Introduction to Quantum Optics, Rinton Press Inc., 2011.

[11] W. Miller, Symmetry Groups and their Applications, Academic Press, New York, 1972.

[12] B. C. Hall, Lie Groups, Lie Algebras, and Representations. An Elementary Introduction, Graduate Texts in Mathematics, 222 (2nd ed.), Springer, 2015.

[13] V. Namias, The Fractional order Fourier transform and its application to quantum mechanics, J Inst Maths Applics Vol. 25, No. 3, pp. 241-265 (1980).

[14] R.A. Fisher, M.M. Nieto, and V.D. Snadberg, Impossibility of naively generalizing squeezed coherent states, Phys Rev D Vol. 29, No. 6, pp.1107-1110 (1984).

[15] P. Pellat-Finet, Optique de Fourier. Théorie métaxiale et fractionaire, Springer, 2009

[16] S. A. Akhmanov and S. Y. Nikitin, Physical Optics, Clarendon Press, Oxford 1997.

[17] V. Arrizón, F. Soto-Eguibar, D. Sánchez-De-La-Llave, and H. M. Moya-Cessa, Conversion of any finite bandwidth optical field into a shape invariant beam, OSA Continuum Vol. 1, No. 2, pp. 604-612 (2018).

[18] Y. Meyer, Wavelets and Operators. Cambridge University Press 1992.

[19] Charles K., An Introduction to Wavelets, Academic Press 1992.

[20] P.S. Addison, The illustrated wavelet transform handbook: introductory theory and applications in science, engineering, medicine and finance, 2nd edition, CRC Press 2016.

[21] Go. Torres-Vega and J.H. Frederick, A quantum mechanical representation in phase space, J Chem Phys Vol. 98, No. 4, pp. 3103-3120 (1993).

[22] L.-y. Hu, H.-y. Fan, and H.-l. Lu, Explicit state vector for Torres-Vega-Frederick phase space representation and its statistical behavior, J Chem Phys Vol. 128, No. 5, 054101 (2008).

Jorge. A. Anaya-Contreras is currently PhD student at Escuela Superior de Física y Matemáticas of Instituto Politécnico Nacional in México city, supervised by Arturo Zúñiga-Segundo and Héctor M. Moya-Cessa. He has published 3 papers in international peer reviewed journals.

Arturo Zúñiga-Segundo received his PhD in physics from Centro de Investigación y de Estudios Avanzados (CINVESTAV) in 1996. Since 1990 he is a physics teacher/researcher at Escuela Superior de Física y Matemáticas of Instituto Politécnico Nacional in Mexico City. He has published over 40 papers in international peer reviewed journals.

Francisco Soto-Eguibar is Titular Researcher at the Optics Department at the National Institute of Astrophysics, Optics and Electronics (INAOE) in Puebla, Mexico. He received his PhD degree in Physics at the National University of Mexico, in Mexico City in 1982. His research interest is in quantum optics and in the foundations of quantum mechanics.

Víctor Arrizón's research activities in optics have been mainly concentrated to the areas of diffractive and holographic optics. Specifically, he has been working in the development of efficient and accurate methods for the generation of structured optical fields, either with uniform or spatially variable polarizations. Most of his designs and proposals have been developed for its implementation with liquid-crystal phase SLMs. The developed diffractive and holographic optical elements have been optimized considering both the advantages and limitations of available SLMs. On the other hand he has maintained collaboration with quantum optics researchers, in the task of identifying formal similitudes between classical wave optics and quantum mechanics. The aim has been to re-discover in classical optics (quantum optics) some already known phenomena in the complementary context. Among other achievements, he and coworkers have established the split of fields propagating in a GRIN fiber, based on similar effects in a quantum Kerr medium. They have also translated to the quantum context some classical phenomena related to propagation invariance.

Héctor Moya-Cessa obtained his PhD at Imperial College in 1993 and since then he is a researcher/lecturer at Instituto Nacional de Astrofísica, Óptica y Electrónica in Puebla, Mexico where he works on Quantum Optics. He has published over 170 papers in international peer reviewed journals. He is fellow of the Alexander von Humboldt Foundation.