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Common Fixed Point Theorems for m-weak** Commuting Mappings in 2-metric Spaces

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In this paper, we introduce the notion of m-weak** commuting for a pair of self-maps on a 2-metric space. Using this general class of mappings we obtain two common fixed point theorems satisfying contractive and rational contractive type conditions in complete 2-metric spaces. Our results in this paper are the generalizations of some known theorems due to Rathore et al. [24] and others.

Keywords: Fixed points, m-weak** commuting mappings, 2-metric space.

1 Introduction

Let (M, d) be a metric space. Two single-valued mappings $S, T : M \to M$ are said to commute iff $ST = TS$. For single-valued mappings the notion of weak commutativity is introduced by Sessa [26].

Definition 1.1. *Let $S, T : M \to M$ be two mappings on a metric space (M, d). Then the pair $\{S, T\}$ is said to be weakly commuting pair if*

$$d(STx, TSx) \leq d(Sx, Tx), \quad \textit{for all } \ x \in M.$$

It can be shown that two commuting mappings are weakly commuting but the converse is false as shown in [26].

In 1989, Pathak [19] gave the following definition:

Definition 1.2. *Two self-mappings A and B of a complete metric space (M, d) are called weak** commuting if $A \subset B$ and for any $x \in M$, we have*

$$d(A^2B^2x, B^2A^2x) \leq d(A^2Bx, BA^2x) \leq d(AB^2x, B^2Ax) \leq d(ABx, BAx) \leq d(A^2x, B^2x).$$

Let A, B be mappings of M into itself. Then the map A is called rotative w. r. t. B (where, "w. r. t." means with respect to) if

$$d(Ax, B^2x) \leq d(Bx, A^2x) \quad \text{for all x in M} \ \ \text{(see [24])}.$$

Pathak et al. [20, 21] modified and extended Definition 1.2 as defined below and gave the following definition.

Definition 1.3. *Two self-mappings A and B of a complete metric space (M, d) are m-weak** commute, if $A(M) \subset B(M)$ and for any $x \in M$, we have*

$$d(A^mB^mx, B^mA^mx) \leq d(A^mBx, BA^mx) \leq d(AB^mx, B^mAx) \leq d(A^mx, B^mx),$$

where $m \in I^+$, set of positive integers.

The concept of a 2-metric space is one of the possible generalizations of the metric space and it has been investigated initially by Gähler [9] and has been developed extensively by Gähler and many others.

Definition 1.4. *A 2-metric space is a set X with a real-valued function d on $X \times X \times X$ satisfying the following conditions:*

(m_1) *For distinct points $x, y \in X$, there exists a point $a \in X$ such that $d(x, y, a) \neq 0$,*

(m_2) $d(x, y, a) = 0$ *if at least two of x, y, a are equal,*

(m_3) $d(x, y, a) = d(x, a, y) = d(y, a, x)$,

(m_4) $d(x, y, a) \leq d(x, y, u) + d(x, u, a) + d(u, y, a)$ *for all $a, u \in X$.*

The function d is called a 2-metric for the space X and (X, d) denotes a 2-metric space. It has shown by Gähler [9] that a 2-metric d is non-negative and although d is a continuous function of any one of its three arguments, it need not be continuous in two arguments. A 2-metric d which is continuous in all of its arguments is said to be continuous. Geometrically a 2-metric $d(x, y, a)$ represents the area of a triangle with vertices are x, y and a. Iseki [12], for the first time, developed a fixed point theorem in 2-metric spaces. Since then a quite number of authors (see e.g. [2, 3, 5, 15–17, 27], [29-33] and others) have introduced some different results in these spaces.

Definition 1.5. *[9] A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be convergent to a point $x \in X$, denoted by $\lim_{n\to\infty} x_n = x$, if*

$$\lim_{n\to\infty} d(x_n, x, a) = 0,$$

for all $a \in X$.

The point x is called the limit of the sequence $\{x_n\}$ in X.

Definition 1.6. *[9] A sequence $\{x_n\}$ in a 2-metric space (X, d) is called a Cauchy sequence, if*

$$\lim_{m,n\to\infty} d(x_m, x_n, a) = 0,$$

for all $a \in X$.

Definition 1.7. *[9] A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.*

Remark 1. *We note that, in a metric space a convergent sequence is a Cauchy sequence and in a 2-metric space (X, d) a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the 2-metric d is continuous on X (see [18]).*

Definition 1.8. *[9] A mapping S from a 2-metric space (X, d) into itself is said to be sequentially continuous at x if for every sequence $\{x_n\}$ in X such that,*

$$\lim_{n\to\infty} d(x_n, x, a) = 0,$$

for all $a \in X$,

$$\lim_{n\to\infty} d(Sx_n, Sx, a) = 0.$$

Now we give the following definitions.

Definition 1.9. *Two self-mappings S and T of a complete 2-metric space (X, d) are m-weak** commute, if $S(X) \subset T(X)$ and for any $x \in X$, we have*

$$d(S^mT^mx, T^mS^mx, a) \le d(S^mTx, TS^mx, a) \le d(ST^mx, T^mSx, a) \le d(S^mx, T^mx, a),$$

where $m \in I^+$, set of positive integers.

Definition 1.10. *Let (X, d) be a 2-metric space and let T, T_1 be mappings of X into itself. Then the map T is called m-rotative w. r. t. T_1 if*

$$d(Tx, T_1^m x, a) \le d(T_1x, T^mx, a) \text{ for all x,a in X.}$$

In 1979 Rhoades [25] proved the following theorem :

Theorem 1.11. *Let S, T be two self-mappings of a complete metric space (M, d) with T continuous and satisfying*

$$d(STx, TSy) \le c \max\{d(Tx, TSy) + d(Sy, STx), d(Tx, STx) + d(Sy, STy)\}$$

for all $x, y \in M$ and $0 \le c < \frac{1}{2}$. Then S and T have a unique common fixed point.

The intent of the present paper is to improve and extend the results of Diviccaro et al [8], Rathore et al [24] and Rhoades [25] by using the concept of m-weak** commuting pair of mappings in complete 2-metric spaces.

2 Main results

In this section, we extend two fixed-point theorems by transforming the three points x, y, a in complete 2-metric spaces and using the concept of m-weak** commuting pair of mappings as defined in this paper. The extensions are subjected by the help of particular inequalities for two and three maps. The inequalities constructed are generalized the ideas of Rathore et al. [24].

Theorem 2.1. *Let S, T be two self-mappings of a complete 2-metric space (X, d) with T is sequentially continuous and satisfying the following conditions:*

$$\{S, T\} \qquad \text{is m-weak**commuting pair}, \tag{2.1}$$

$$\begin{aligned} &d(S^mT^mx, T^mS^my, a) \le c\max\{d(T^mx, T^mS^my, a) + d(S^my, S^mT^mx, a),\\ &d(T^mx, S^mT^mx, a) + d(S^my, T^mS^my, a)\}, \end{aligned} \tag{2.2}$$

for all x, y, a in X, where $0 \le c < 1/2$. Then S and T have a unique common fixed point.

Proof. Let x be an arbitrary point in X. Define

$$(S^mT^m)^nx = x_{2n}, \quad T^m(S^mT^m)^nx = x_{2n+1}, \;\; n = 0, 1, 2, 3, \dots\, .$$

Then, we have

$$\begin{aligned} &d(x_{2n}, x_{2n+1}, a) = d((S^mT^m)^nx, T^m(S^mT^m)^nx, a)\\ &= d(S^mT^m(S^mT^m)^{n-1}x, T^mS^m(T^m(S^mT^m)^{n-1}x), a), \end{aligned}$$

which implies that,

$$\begin{aligned} &d(x_{2n}, x_{2n+1}, a) \le c\ \max\{d(T^m(S^mT^m)^{n-1}x, T^mS^m(T^m(S^mT^m)^{n-1}x), a)\\ &+d(S^m(T^m(S^mT^m)^{n-1}x), S^mT^m(S^mT^m)^{n-1}x, a),\\ &d(T^m(S^mT^m)^{n-1}x, S^mT^m(S^mT^m)^{n-1}x, a)\\ &+d(S^m(T^m(S^mT^m)^{n-1}x), T^mS^m(T^m(S^mT^m)^{n-1}x), a)\}. \end{aligned}$$

i.e.,

$$\begin{aligned} &d(x_{2n}, x_{2n+1}, a) \le c\ \max\{d(x_{2n-1}, x_{2n+1}, a) + d(x_{2n}, x_{2n}, a),\\ &d(x_{2n-1}, x_{2n}, a) + d(x_{2n}, x_{2n+1}, a)\}, \end{aligned}$$

which implies that,

$$\begin{aligned} d(x_{2n}, x_{2n+1}, a) &\leq c \max\{d(x_{2n-1}, x_{2n+1}, a), d(x_{2n-1}, x_{2n}, a) + d(x_{2n}, x_{2n+1}, a)\} \\ &\leq c\,(d(x_{2n-1}, x_{2n}, a) + d(x_{2n}, x_{2n+1}, a)), \end{aligned}$$

and so

$$d(x_{2n}, x_{2n+1}, a) \leq \frac{c}{1-c} d(x_{2n-1}, x_{2n}, a),$$

it follows that

$$d(x_{2n}, x_{2n+1}, a) \leq (\frac{c}{1-c})^{2n-1} d(x_1, x_2, a),$$

since $\frac{c}{1-c} \leq 1$, we deduce that the sequence $\{x_{2n}\}$ is a Cauchy sequence in the complete 2-metric space X and so has a limit z in X, that is,

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} (S^m T^m)^n x = z.$$

Since T is sequentially continuous, we have

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} T^m x_{2n} = T^m z = z.$$

Further,

$$\begin{aligned} & d(z, S^m z, a) = d(z, S^m T^m z, a) \\ \leq\; & d(z, x_{2n+1}, a) + d(x_{2n+1}, S^m T^m z, a) + d(z, S^m T^m z, x_{2n+1}) \\ \leq\; & d(z, x_{2n+1}, a) + d(S^m T^m z, T^m S^m (T^m (S^m T^m)^{n-1} x, a) + d(z, S^m T^m z, x_{2n+1}) \\ \leq\; & d(z, x_{2n+1}, a) + c \max\{d(T^m z, x_{2n+1}, a) + d(x_{2n}, S^m T^m z, a), \\ & d(S^m z, S^m T^m z, a) + d(x_{2n}, x_{2n+1}, a)\} + d(z, S^m T^m z, ux_{2n+1}). \end{aligned}$$

Letting $n \to \infty$, we have

$$d(z, S^m z, a) \leq c\, d(z, S^m z, a),$$

which is a contradiction as $c < \frac{1}{2}$. It follows that $d(z, S^m z, a) = 0$, so $z = S^m z = T^m z$. Now by using the definition of $m-$weak** commutativity of pair $\{S, T\}$, we have that

$$S^m T^m z = T^m S^m z,\ \ S^m T z = T S^m z \ \text{ and } \ S T^m z = T^m S z.$$

So, $S^m T z = T z$ and $T^m S z = S z$. Now,

$$\begin{aligned} d(z, Sz, a) &= d(S^m T^m z, T^m S z, a) = d(S^m T^m z, T^m S^m (Sz), a) \\ &\leq c \max\{d(T^m z, T^m S^m (Sz), a) + d(S^m S z, S^m T^m z, a), \\ &\quad d(T^m z, S^m T^m z, a) + d(S^m (Sz), T^m S^m (Sz), a)\}. \end{aligned}$$

i.e.,

$$d(z,Sz,a) \le c\,\max\{d(z,Sz,a)+d(z,Sz,a),0\} = 2c\,d(z,Sz,a),$$

which is a contradiction, as $c < \frac{1}{2}$, so $Sz = z$.

Similarly we can show that $z = Tz$. Hence z is a common fixed point of S and T.

To prove the uniqueness let u($u \ne z$) be another fixed point of S and T. Then

$$\begin{aligned} d(z,u,a) &= d(S^mT^mz, T^mS^mu, a) \\ &\le c\,\max\{d(T^mz, T^mS^mu, a) + d(S^mu, S^mT^mz, a), \\ &\quad d(T^mz, S^mT^mz, a) + d(S^mu, T^mS^mu, a)\}. \end{aligned}$$

i.e.,

$$d(z,u,a) \le c\,\max\{2d(z,u,a),0\} = 2c\,d(z,u,a).$$

Since $c \le \frac{1}{2}$, it follows that $z = u$. □

As a consequence of Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let S, T be two self-mappings of a complete 2-metric space (X,d) with T is sequentially continuous and satisfying the following conditions*

$$\{S,T\} \quad \textit{is a weak**commuting pair}, \tag{2.3}$$

$$\begin{aligned} d(S^2T^2x, T^2S^2y, a) \le c\,\max\{&d(T^2x, T^2S^2y, a) + d(S^2y, S^2T^2x, a), \\ &d(T^2x, S^2T^2x, a) + d(S^2y, T^2S^2y, a)\}, \end{aligned} \tag{2.4}$$

for all $x, y, a \in X$, where $0 \le c < 1/2$. Then S and T have a unique common fixed point.

Remark 2. *Assuming S and T are idempotent mappings (A mapping $S : X \to X$ is called an idempotent if $S^2 = S$) of X in Corollary 2.1, we obtain an extension of Theorem 1.11 in the complete 2-metric space setting.*

In the following we establish a general common fixed point theorem for three mappings satisfying a rational inequality in a complete 2-metric space.

Theorem 2.3. *Let S, T and T_1 be three mappings of a complete 2-metric space (X,d) such that for all $x, y, a \in X$, either*

$$\begin{aligned} d(S^mx, T^my, a) &\le K'\left[d(T_1^mx, S^mx, a) + d(T_1^my, T^my, a)\right] \\ &+ K\left[\frac{d(T_1^mx, S^mx, a)d(T_1^my, T^my, a) + d(T_1^mx, T^my, a)d(T_1^my, S^mx, a)}{d(T_1^mx, S^mx, a) + d(T_1^my, T^my, a)}\right], \end{aligned} \tag{2.5}$$

with $d(T_1^m x, S^m x, a) + d(T_1^m y, T^m y, a) \neq 0$, *where* $K < 1$ *and* $3K' < 1$, *or*

$$d(S^m x, T^m y, a) = 0 \ \text{ if } \quad d(T_1^m x, S^m x, a) + d(T_1^m y, T^m y, a) = 0. \tag{2.6}$$

Suppose that the range of T_1^m *contains the range of* S^m *and* T^m. *If either*

(b_1) T_1^m *is sequentially continuous,* T_1 *is* m*-weak** commuting with* S *and* T *is m-rotative w. r. t.* T_1, *or*

(b_2) T_1^m *is sequentially continuous,* T_1 *is* m*-weak** commuting with* T *and* S *is* m*-rotative w. r. t.* T_1, *or*

(b_3) S^m *is sequentially continuous,* S *is* m*-weak** commuting with* T_1 *and* T *is* m*-rotative w. r. t.* S, *or*

(b_4) T^m *is sequentially continuous,* T *is* m*-weak** commuting with* T_1 *and* S *is* m*-rotative w. r. t.* T,

then S, T *and* T_1 *have a unique common fixed point* z. *Further,* z *is the unique common fixed point of* S *and* T_1 *and of* T *and* T_1.

Proof. Let x_0 be an arbitrary point in X. Since the range of T_1^m contains the range of S^m, let x_1 be a point in X such that $S^m x_0 = T^m x_1$. Since the range of T_1^m contains the range of T^m, we can choose a point x_2 such that $T^m x_1 = T_1^m x_2$. In general, having chosen the point x_{2n} such that

$$S^m x_{2n} = T_1^m x_{2n+1} \ \text{ and } \ T^m x_{2n+1} = T_1^m x_{2n+2} \quad \text{for } n = 1, 2, 3, \dots .$$

Put

$$d_{2n-1} = d(T^m x_{2n-1}, S^m x_{2n}, a) \ \text{ and } \ d_{2n} = d(S^m x_{2n}, T^m x_{2n+1}, a)$$

for $n = 1, 2, 3, \dots$. Now, we distinguish three cases:

(i) Let $d_{2n-1} \neq 0$ and $d_{2n} \neq 0$ for $n = 1, 2, 3, \dots$, we have

$$d_{2n-1} + d_{2n} = d(T_1^m x_{2n}, S^m x_{2n}, a) + d(T_1^m x_{2n+1}, T^m x_{2n+1}, a) \neq 0$$

Using inequality (2.5), we obtain

$$\begin{aligned} &d_{2n} = d(S^m x_{2n}, T^m x_{2n+1}, a) \leq K'[d_{2n-1} + d_{2n}] \\ &+K\left[\frac{d_{2n-1}d_{2n} + d(T^m x_{2n-1}, T^m x_{2n+1}, a) d(S^m x_{2n}, S^m x_{2n}, a)}{d_{2n-1} + d_{2n}}\right] \end{aligned}$$

$$\text{i.e., } \ d_{2n} \leq K'\big(d_{2n-1} + d_{2n}\big) + K\left(\frac{d_{2n-1}d_{2n}}{d_{2n-1} + d_{2n}}\right).$$

Then,

$$(1 - K')d_{2n}^2 \leq K' d_{2n-1}^2 + 2K' d_{2n-1} d_{2n}.$$

Now the positive root of the quadratic equation

$$(1-K')d_{2n}^2 - 2K'd_{2n-1}d_{2n} - K'd_{2n-1}^2 = 0$$

is

$$d_{2n} = \frac{K' + \sqrt{K'}}{1-K'} d_{2n-1} < \frac{2K'}{1-K'} d_{2n-1},$$

which implies that $d_{2n} \leq d_{2n-1}$, since $3K' < 1$. Then

$$d(S^m x_{2n}, T^m x_{2n+1}, a) \leq d(T^m x_{2n-1}, S^m x_{2n}, a) \tag{2.7}$$

for $n = 1, 2, 3, \ldots$. Similarly, it can be obtained that $d_{2n-1} \leq d_{2n-2}$. So

$$d(T^m x_{2n-1}, S^m x_{2n}, a) \leq d(S^m x_{2n-2}, T^m x_{2n-1}, a)$$

It follows that the sequence

$$\{S^m x_0, T^m x_1, S^m x_2, \ldots, T^m x_{2n-1}, S^m x_{2n}, T^m x_{2n+1}, \ldots\} \tag{2.8}$$

is a Cauchy sequence in the complete 2-metric space X and so has a limit w in X. Hence the sequences

$$\{S^m x_{2n}\} = \{T_1^m x_{2n+1}\} \quad \text{and} \quad \{T^m x_{2n-1}\} = \{T_1^m x_{2n}\}$$

converge to the point w because they are subsequences of the sequence (2.8). Suppose first of all that T_1^m is sequentially continuous, then the sequences $\{T_1^{2m} x_{2n}\}$ and $\{T_1^m S^m x_{2n}\}$ converge to the point $T_1^m w$.

If T_1 is m-weak** commuting with S, we then have

$$\begin{aligned} &d(S^m T_1^m x_{2n}, T_1^m w, a) \leq d(S^m T_1^m x_{2n}, T_1^m S^m x_{2n}, a) + d(T_1^m S^m x_{2n}, T_1^m w, a) \\ &\leq d(S^m x_{2n}, T_1^m x_{2n}, a) + d(T_1^m S^m x_{2n}, T_1^m w, a), \end{aligned}$$

which implies by letting n tends to infinity that the sequence $\{S^m T_1^m x_{2n}\}$ also converges to $T_1^m w$. We now claim that $T^m w = T_1^m w$. Suppose not. Then we have $d(T_1^m w, T^m w, a) > 0$ and using inequality (2.5), we obtain

$$\begin{aligned} d(S^m T_1^m x_{2n}, T^m w, a) \;\leq\; & K'\big[d(T_1^{2m} x_{2n}, S^m T_1^m x_{2n}, a) + d(T_1^m w, T^m w, a)\big] \\ & + K\left(\frac{\alpha_1 + \alpha_2}{\alpha_3}\right), \end{aligned} \tag{2.9}$$

where

$$\alpha_1 = d(T_1^{2m} x_{2n}, S^m T_1^m x_{2n}, a) d(T_1^m w, T^m w, a),$$

$$\alpha_2 = d(T_1^{2m} x_{2n}, T^m w, a) d(T_1^m w, S^m T_1^m x_{2n}, a)$$

and

$$\alpha_3 = d(T_1^{2m}x_{2n}, S^m T_1^m x_{2n}, a) + d(T_1^m w, T^m w, a)$$

Letting $n \to \infty$ in (2.9), we deduce that

$$d(T_1^m w, T^m w, a) \leq K' d(T_1^m w, T^m w, a).$$

i.e., $(1 - K')d(T_1^m w, T^m w, a) \leq 0$, a contradiction since $3K' < 1$.

Now suppose that $S^m w \neq T^m w$, then we have

$$\begin{aligned} d(S^m w, T^m w, a) &\leq K'\big[d(T_1^m w, S^m w, a) + d(T_1^m w, T^m w, a)\big] \\ &+ K\left(\frac{\lambda_1 + \lambda_2}{\lambda_3}\right) \leq K' d(S^m w, T^m w, a). \end{aligned}$$

Therefore, $(1 - K')d(S^m w, T^m w) \leq 0$, a contradiction since $3K' < 1$. Thus

$$T_1^m w = S^m w = T^m w,$$

where,

$$\lambda_1 = d(T_1^m w, S^m w, a)d(T_1^m w, T^m w, a), \quad \lambda_2 = d(T_1^m w, T^m w, a)d(T_1^m w, S^m w, a),$$

and

$$\lambda_3 = d(T_1^m w, S^m w, a) + d(T_1^m w, T^m w, a).$$

A similar conclusion is achieved if T_1 is m-weak**commuting with T. Let us now suppose that S^m is sequentially continuous instead of T_1^m. Then the sequences $\{S^{2m}x_{2n}\}$ and $\{S^m T_1^m x_{2n}\}$ converge to the point $S^m w$. Since S is m-weak** commuting with T_1, we have that the sequence $\{T_1^m S^m x_{2n}\}$ also converges to $S^m w$. Since the range of T_1^m contains the range of S^m, there exists a point w', such that

$$T_1^m w' = S^m w.$$

Then if $T^m w\prime \neq S^m w = T_1^m w\prime$, we obtain

$$\begin{aligned} &d(S^{2m}x_{2n}, T^m w', a) \leq K'\big[d(T_1^m S^m x_{2n}, S^{2m}x_{2n}, a) + d(T_1^m w', T^m w', a)\big] \\ &+K\left(\frac{\lambda_1' + \lambda_2'}{\lambda_3'}\right), \end{aligned}$$

where,

$$\lambda_1' = d(T_1^m S^m x_{2n}, S^{2m}x_{2n}, a)d(T_1^m w', T^m w', a),$$
$$\lambda_2' = d(T_1^m S^m x_{2n}, T^m w', a)d(T_1^m w', S^{2m}x_{2n}, a),$$

and

$$\lambda_3' = d(T_1^m S^m x_{2n}, S^{2m}x_{2n}, a) + d(T_1^m w', T^m w', a).$$

Letting $n \to \infty$, we deduce that

$$
\begin{aligned}
d(S^m w, T^m w', a) \;\leq\; & K'\big[d(S^m w, S^m w, a) + d(T_1^m w', T^m w', a)\big] \\
& + K\left[\frac{d(S^m w, S^m w, a)d(T_1^m w', T^m w', a)}{d(S^m w, S^m w, a) + d(T_1^m w', T^m w', a)}\right. \\
& + \left.\frac{d(S^m w, T^m w', a)d(T_1^m w', S^m w, a)}{d(S^m w, S^m w, a) + d(T_1^m w', T^m w', a)}\right] \\
& \leq K' d(S^m w, T^m w', a)
\end{aligned}
$$

i.e., $(1-K')d(S^m w, T^m w') \leq 0$, which is a contradiction since $3K' < 1$. Thus

$$S^m w = T^m w' = T_1^m w'.$$

Now suppose that $S^m w' \neq T^m w' = T_1^m w'$, then

$$
\begin{aligned}
& d(S^m w', T^m w', a) \leq K'\big[d(T_1^m w', S^m w', a) + d(T_1^m w', T^m w', a)\big] \\
& + K\left[\frac{d(T_1^m w', S^m w', a)d(T_1^m w', T^m w', a)}{d(T_1^m w', S^m w', a) + d(T_1^m w', T^m w', a)}\right. \\
& + \left.\frac{d(T_1^m w', T^m w', a)d(T_1^m w', S^m w', a)}{d(T_1^m w', S^m w', a) + d(T_1^m w', T^m w', a)}\right].
\end{aligned}
$$

Letting $n \to \infty$, we deduce that

$$d(S^m w', T^m w', a) \leq K' d(T_1^m w', T^m w', a),$$

which is a contradiction since $3K' < 1$. Thus

$$S^m w' = T^m w' = T_1^m w'.$$

A similar conclusion is obtained if one assumes that T^m is sequentially continuous and T is m-weak** commuting with T_1.

(ii) $d_{2n-1} = 0$ for some n. Then

$$T_1^m x_{2n} = T^m x_{2n-1} = S^m x_{2n}.$$

We claim that $T_1^m x_{2n} = T^m x_{2n}$, since otherwise if $d(T_1^m x_{2n}, T^m x_{2n}, a) > 0$, inequality

(2.5) implies,

$$\begin{aligned}
0 &< d(T_1^m x_{2n}, T^m x_{2n}, a) \\
&= d(S^m x_{2n}, T^m x_{2n}, a) \\
&\le K'\big[d(T_1^m x_{2n}, S^m x_{2n}, a) + d(T_1^m x_{2n}, T^m x_{2n}, a)\big] \\
&+ K\bigg[\frac{d(T_1^m x_{2n}, S^m x_{2n}, a) d(T_1^m x_{2n}, T^m x_{2n}, a)}{d(T_1^m x_{2n}, S^m x_{2n}, a) + d(T_1^m x_{2n}, T^m x_{2n}, a)} \\
&+ \frac{d(T_1^m x_{2n}, T^m x_{2n}, a)\, d(T_1^m x_{2n}, S^m x_{2n}, a)}{d(T_1^m x_{2n}, S^m x_{2n}, a) + d(T_1^m x_{2n}, T^m x_{2n}, a)}\bigg] \\
&= K'\big[d_{2n-1} + d(T_1^m x_{2n}, T^m x_{2n}, a)\big] \\
&+ K\bigg[\frac{d_{2n-1} d(T_1^m x_{2n}, T^m x_{2n}, a) + d(T_1^m x_{2n}, T^m x_{2n}, a) d_{2n-1}}{d_{2n-1} + d(T_1^m x_{2n}, T^m x_{2n}, a)}\bigg],
\end{aligned}$$

i.e., $0 < d(T_1^m x_{2n}, T^m x_{2n}, a) \le K' d(T_1^m x_{2n}, T^m x_{2n}, a)$,
that is $0 < (1-K')d(T_1^m x_{2n}, T^m x_{2n}, a) \le 0$, which is a contradiction. Thus

$$T_1^m x_{2n} = S^m x_{2n} = T^m x_{2n}.$$

(iii) Let $d_{2n} = 0$ for some $n := 1, 2, 3, \ldots$. Then $T_1^m x_{2n+1} = S^m x_{2n} = T^m x_{2n+1}$ and reasoning as in (ii), we have

$$T_1^m x_{2n+1} = S^m x_{2n+1} = T^m x_{2n+1}.$$

Therefore in all cases, there exists a point w such that

$$T_1^m w = S^m w = T^m w.$$

If T_1 m-weak** commutes with S, we have

$$d(S^m T_1 w, T_1 S^m w, a) \le d(S T_1^m w, T_1^m S w, a) \le d(S^m w, T_1^m w, a) = 0,$$

which implies that

$$S^m T_1 w = T_1 S^m w, \quad S T_1^m w = T_1^m S w \quad \text{and so} \quad T_1^m S w = S^{m+1} w. \tag{2.10}$$

Thus $d(T_1^m Sw, S^m Sw, a) + d(T_1^m w, T^m w, a) = 0$, and using condition (2.6), we deduce that

$$S T_1^m w = S^m S w = T^m w = T_1^m w.$$

It follows that $T_1^m w = z$ is fixed point of S. Further,

$$d(T_1^m T_1 w, S^m T_1 w, a) + d(T_1^m w, T^m w, a) = 0.$$

Using the condition (2.6), we obtain

$$T_1 z = S^m T_1 w = T^m w = z.$$

Now, using inequality (2.5), on the assumption that $T^m z \neq z$, we have

$$\begin{aligned}
&d(z, T^m z, a)\\
&= d(S^m z, T^m z, a)\\
&\leq K'\big[d(T_1^m z, S^m z, a) + d(T_1^m z, T^m z, a)\big]\\
&+ K\left[\frac{d(T_1^m z, S^m z, a)d(T_1^m z, T^m z, a) + d(T_1^m z, T^m z, a)d(T_1^m z, S^m z, a)}{d(T_1^m z, S^m z, a) + d(T_1^m z, T^m z, a)}\right],
\end{aligned}$$

i.e., $d(z, T^m z, a) \leq K' d(z, T^m z, a)$,
that is $(1 - K')d(z, T^m z, a) \leq 0$, a contradiction since $3K' < 1$. Thus $z = T^m z$.

Now using the m-rotative of T w. r. t. T_1(or w. r. t. S), we have

$$d(Tz, z, a) = d(Tz, T_1^m z, a) \leq d(T_1 z, T^m z, a) = d(z, z, a) = 0,$$

so z is a common fixed point of T_1, S and T.

If one assumes that T_1 is m-weak** commuting with T and rotatively of S w. r. t. T_1 (or w. r. t. S) the proof is of course similar.

Now, suppose that z' be another common fixed point of T_1 and S. Then

$$d(T_1^m z', S^m z', a) + d(T_1^m z, T^m z, a) = 0,$$

and condition (2.6) implies that

$$z' = Sz' = S^m z' = T^m z = z.$$

We can prove similarly that z is the unique common fixed point of T_1 and T. This completes the proof of our theorem. □

Assuming $S = T$ in Theorem 2.3, we have the following corollary.

Corollary 2.4. *Let S and T_1 be two mappings of a complete 2-metric space (X, d) into itself such that for all $x, y, a \in X$, either*

$$\begin{aligned}
&d(S^m x, S^m y, a)\\
&\leq K'\big[d(T_1^m x, S^m x, a) + d(T_1^m y, S^m y, a)\big]\\
&+ K\left[\frac{d(T_1^m x, S^m x, a)d(T_1^m y, S^m y, a) + d(T_1^m x, S^m y, a)d(T_1^m y, S^m x, a)}{d(T_1^m x, S^m x, a) + d(T_1^m y, S^m y, a)}\right].
\end{aligned}$$

If $d(T_1^m x, S^m x, a) + d(T_1^m y, S^m y, a) \neq 0$, *where* $K < 1$ *and* $3K' < 1$, *or*

$$d(S^m x, S^m y, a) = 0 \text{ if } d(T_1^m x, S^m x, a) + d(T_1^m y, S^m y, a) = 0.$$

*If the range of T_1^m contains the range of S^m, if T_1 is m-weak** commuting with S and if T_1^m or S^m is sequentially continuous, then S and T_1 have a unique common fixed point.*

Remark 3. *If we put $m = 2$ in Theorem 2.3, we obtain an extension of the result of Rathore et al (see [24]) in the complete 2-metric space. Assuming $T_1^m = I$ (where, I be identity mapping on X), $m = 2$ and dropping the rotatively of T (or S), we obtain a generalization of Corollary 2 of [24].*

Remark 4. *It is still an open problem to extend Theorem 2.3 by considering more than three mappings in 2-metric space setting using the notion of m-weak** commuting mappings. In [2, 10, 11] the authors have extended some theorems by considering more than three mappings but they used another spaces (not 2-metric spaces) also, they have applied another types of mappings.*

Remark 5. *It is still an open problem to generalize our main results in this paper to the class of compatible or weakly compatible mappings. For more details about these types of mappings we refer to [1, 4, 6, 7, 13, 14, 22, 28] and others.*

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