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Rise, Fall and Level Statistics on r-Jacobi-Stirling Set Partitions

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Abstract: In this paper, we consider sequential representations of the recently introduced r-Jacobi-Stirling set partitions (denoted by $\mathscr{P}(n,k)$) and study various statistics on these representations. We compute an explicit formula for the generating function which counts members of $\mathscr{P}(n,k)$ where k and r are fixed according to these statistics in the case of levels, descents and ascents. In each case, we use a more-or-less uniform strategy which also yields the distribution of the statistic on those members of $\mathscr{P}(n,k)$ ending in a certain letter. Finally, we give explicit formulas for the total number of levels, descents and ascents within all of the members of $\mathscr{P}(n,k)$, providing both algebraic and combinatorial proofs.

Keywords: Set partition, Jacobi-Stirling numbers, generating function, combinatorial statistic

1 Introduction

The r-Jacobi-Stirling numbers of the second kind, denoted by $JS_r(n,k;z)$, were introduced and studied by Mihoubi and Rahim [8] and are defined for $0 \le r \le k \le n$ by the recurrence

$$JS_r(n,k;z) = JS_r(n-1,k-1;z) + k(k+z)JS_r(n-1,k;z), n > r, \tag{1}$$

with $JS_r(r,r;z) = 1$ for all $r \ge 0$. It was shown in [8] that the $JS_r(n,k;z)$ are also given explicitly as

$$JS_r(n+r,k+r;z) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{(2j+2r+z)((j+r)(j+r+z))^n}{\prod_{i=0}^{k}(j+k+2r+z-i)},$$

or equivalently as connection constants in the polynomial identities

$$(x+r(r+z))^n = \sum_{k=0}^{n} JS_r(n+r,k+r;z)x_{k,z+2r}, \qquad n \ge 0,$$

where $x_{k,z} = \prod_{i=0}^{k-1}(x - i(i+z))$. When $r = 0$, the $JS_r(n,k;z)$ reduce to what are known as the Jacobi-Stirling numbers $JS(n,k;z)$, which were introduced by Gelineau [2] and later studied [1,3,7,10]. Using (1), one can derive the generating function formula

$$\sum_{n\ge k} JS_r(n,k;z)x^n = \frac{x^k}{\prod_{j=r}^{k}(1-j(j+z)x)}, \qquad k \ge r \ge 0; \tag{2}$$

see [8, Theorem 1]. The numbers $JS(n,k;z)$ and later $JS_r(n,k;z)$ were given combinatorial interpretations as enumerators of a certain class of partitions of a set of size $2n$, denoted by $\mathscr{P}(n,k)$, where some of the blocks are labeled and allowed to be empty. Members of $\mathscr{P}(n,k)$ were termed r-Jacobi-Stirling set partitions in [8].

Here, we consider certain combinatorial aspects of the $JS_r(n,k;z)$ related to various refined enumerations of the underlying structure $\mathscr{P}(n,k)$. Recall that the level, descent and ascent statistics defined on a class of k-ary sequences record the number of indices corresponding to positions where the letter directly following the indexed letter is the same, smaller or larger, respectively. These statistics have been studied on sequential representations for a variety of discrete structures, including finite set partitions, k-ary words and compositions (see, e.g., [4,6] and references contained therein). We consider in the current paper extensions of these statistics to sequential representations of the r-Jacobi-Stirling partitions of a given size and derive various facts about their

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distributions. As a consequence, one obtains q-analogues of the sequence $JS_r(n,k;z)$ which reduce to it when $q=1$.

The organization of this paper is as follows. In the next section, we describe a sequential representation for members of $\mathscr{P}(n,k)$ which is used to define the level, descent and ascent statistics. In the third section, we find in each case a formula for the ordinary generating function of the distribution on $\mathscr{P}(n,k)$. An explicit formula for the total number of occurrences of each statistic on $\mathscr{P}(n,k)$ is computed which can be expressed in terms of the numbers $JS_r(n,k;z)$. Combinatorial proofs of these formulas may also be given. Some concluding remarks are made in the fourth section.

2 Preliminaries

We first recall the combinatorial interpretation for the r-Jacobi-Stirling numbers from [8]. Throughout, the sets $\{1,2,\ldots,n\}$ and $\{\pm 1,\pm 2,\ldots,\pm n\}$ will be denoted by $[n]$ and $[\pm n]$, respectively. Let $n,k\geq 1$ and $r,s\geq 0$ with $n\geq k\geq r$ be integers. Then an *r-Jacobi-Stirling set partition of* $[\pm n]$ into s zero blocks $A_1,\ldots,A_s$ and k nonzero blocks $B_1,\ldots,B_k$ is an ordinary set partition of $[\pm n]$ into $s+k$ blocks satisfying the following conditions:

1. The blocks $A_1,\ldots,A_s$ are ordered (i.e., labeled), whereas $B_1,\ldots,B_k$ are not.
2. Zero blocks may be empty, but nonzero blocks are nonempty.
3. If $i\in[n]$, then $\{i,-i\}\not\subseteq \cup_{j=1}^{s}A_j$.
4. We have $\{i,-i\}\subseteq B_j$ for some $1\leq j\leq k$ if and only if i has minimum absolute value of all the elements in B_j.
5. If $1\leq j\leq r$, then $\{j,-j\}\subseteq B_j$.

Note that the r-Jacobi-Stirling partitions coincide with the Jacobi-Stirling partitions (see [2]) when $r=0$ or $r=1$, and with a subset of the usual partitions of $[\pm n]$ when $s=0$.

Given n, k and r as above and a non-negative even integer m, let $\mathscr{P}(n,k)=\mathscr{P}^{(r)}(n,k,m/2)$ denote the set of r-Jacobi-Stirling partitions of $[\pm n]$ having $m/2$ zero blocks and k nonzero blocks. (We represent the s parameter in this way so that the recurrences below will be slightly simpler.) Note that

$$|\mathscr{P}(n,k)|=JS_r(n,k;m-1),$$

which can be shown by demonstrating that both quantities satisfy the same recurrence (1). For if $\{n,-n\}$ does not comprise its own (nonzero) block within a member of $\mathscr{P}(n,k)$, then there are $k(k-1)|\mathscr{P}(n-1,k)|$ possibilities if both elements belong to nonzero blocks and $2(km/2)|\mathscr{P}(n-1,k)|$ possibilities if one of $\{n,-n\}$ belongs to a zero block.

We define statistics on $\mathscr{P}(n,k)$ as follows. Suppose that the nonzero blocks $B_1,\ldots,B_k$ of $\pi\in\mathscr{P}(n,k)$ are written so that $\min B_1<\cdots<\min B_k$. We associate to each π a k-ary sequence $w_\pi=w_1w_2\cdots w_n$ whose i-th entry w_i is obtained by either reading the index of the nonzero block containing i or $-i$ if one of $\{i,-i\}$ belongs to a zero block or by taking $\max\{a,b\}$ where $i\in B_a$ and $-i\in B_b$. That is, w_i records the index of the rightmost nonzero block containing an element of absolute value i. For example, if $n=7$, $k=3$, $r=2$, $m=8$ and $\pi\in\mathscr{P}^{(2)}(7,3,4)$ has zero blocks

$$A_1=\{3,-7\},\quad A_2=\varnothing,\quad A_3=\{-4\},\quad A_4=\varnothing$$

and nonzero blocks

$$B_1=\{1,4,7,-1,-6\},\quad B_2=\{2,6,-2,-3\},\quad B_3=\{5,-5\},$$

then $w_\pi=1221321$. Note that π is not uniquely determined by w_π and that w_π is in fact what is known as a *restricted growth function*, meaning it is a function from $[n]$ onto $[k]$ satisfying

$$\max\{w_j:1\leq j\leq i+1\}\leq\max\{w_j:1\leq j\leq i\}+1$$

for all $1\leq i\leq n-1$ (see, e.g., [9,11] for details). At times, we will speak of a partition π as if it were a sequence (for instance, π ending in a given letter), in which case we are referring to a property of its associated sequence w_π.

Given $\pi\in\mathscr{P}(n,k)$ with $w_\pi=w_1w_2\cdots w_n$, we will say that π has a *level* (at index i) if $w_i=w_{i+1}$ where $1\leq i\leq n-1$. A *descent* (*ascent*, resp.) refers to an index i for which $w_i>w_{i+1}$ ($w_i<w_{i+1}$, resp.). For example, if $\pi\in\mathscr{P}(7,3)$ is as above, then π has one level (at index 2), three descents (at 3, 5 and 6) and two ascents (at 1 and 4). Note that a level of π occurs if a nonzero block B of π is such that B contains at least one element of both absolute value i and $i+1$ for some i, with no blocks to the right of B containing an element of the set $\{\pm i,\pm(i+1)\}$. Similar interpretations may be given for descents and ascents. See, e.g., the text [5] for the definitions of the analogous statistics on ordinary set partitions which are extended here to $\mathscr{P}(n,k)$.

3 Statistics on r-Jacobi-Stirling partitions

In this section, we study the distributions for the statistics recording the number of levels, descents or ascents on $\mathscr{P}(n,k)$.

3.1 Counting by levels

Throughout, m denotes a non-negative even integer which may also be regarded as an indeterminate. Given $n\geq k\geq r\geq 0$, let $P(n,k)=P_q^{(r)}(n,k,m/2)$ count members of $\mathscr{P}(n,k)$ according to the number of levels (marked by q). Let $P(n,k|i)=P_q^{(r)}(n,k,m/2|i)$ denote the restriction of $P(n,k)$ to those members of $\mathscr{P}(n,k)$ ending in i, where $1\leq i\leq k$. Put $P(n,k)=P(n,k|i)=0$ if $k<r$ or $n<k$. The $P(n,k|i)$ are determined recursively as follows.

Lemma 1. *We have*

$$P(n,k|i)=(2i-2+m)P(n-1,k)+(q-1)(2i-2+m)P(n-1,k|i), \quad 1\le i<k\le n \text{ and } k\ge r, \tag{3}$$

$$P(n,k|k)=P(n-1,k-1)+(2k-2+m)P(n-1,k)+(q-1)(2k-2+m)P(n-1,k|k), \quad n\ge k>r, \tag{4}$$

and

$$P(n,r|r)=\delta_{n,r}+(2r-2+m)P(n-1,r)+(q-1)(2r-2+m)P(n-1,r|r), \quad n\ge r\ge 1, \tag{5}$$

with $P(0,0)=\delta_{r,0}$ and $P(n,0)=0$ if $n\ge 1$ for all r.

Proof. Let $\mathscr{P}(n,k|i)$ denote the subset of $\mathscr{P}(n,k)$ whose members have last letter i. First suppose $1\le i<k$ and $n\ge k$. If $n=k$, then $\mathscr{P}(n,k)$ is empty in this case and (3) is obvious. So assume $n>k$ and let j denote the penultimate letter of $\lambda\in\mathscr{P}(n,k|i)$. Note that there are $2(i-1)+m$ choices for the positions of the letters $\pm n$ in λ since one of them must go in the i-th nonzero block, while the other goes in one of the first $i-1$ nonzero blocks or in one of the $m/2$ zero blocks. Thus, if $j\neq i$, there are $(2i-2+m)(P(n-1,k)-P(n-1,k|i))$ possibilities for the partition λ. On the other hand, if $j=i$, then a level is realized and we get $q(2i-2+m)P(n-1,k|i)$ possibilities. Combining the two previous cases gives (3). If $i=k$ with $n\ge k>r$, then there are the same number of possibilities as before with i replaced by k if it is not the case that the final letter k is the only letter k within $\lambda\in\mathscr{P}(n,k|k)$. On the other hand, if k occurs only once within such λ, then deleting it results in a member of $\mathscr{P}(n-1,k-1)$. Combining this with the previous case gives (4). Finally, if $i=k=r$ and $\lambda\in\mathscr{P}(n,r|r)$, then the cases are the same as in the proof of (4) except now if the letter k occurs only once, then there are $\delta_{n,r}$ possibilities for λ, which implies (5).

Define the generating function $P(x;k)$ by $P(x;k)=\sum_{n\ge k}P(n,k)x^n$, where $k\ge r\ge 0$. We have the following explicit formula for $P(x;k)$.

Theorem 1. *Let $k\ge r\ge 0$. Then*

$$P(x;k)=\frac{x^k}{\prod_{j=r}^{k}(1-(q-1)(2j-2+m)x)\prod_{j=r}^{k}\left(1-\sum_{i=1}^{j}\frac{(2i-2+m)x}{1-(q-1)(2i-2+m)x}\right)}, r\ge 1, \tag{6}$$

and

$$P(x;k)=\frac{x^k}{\prod_{j=1}^{k}(1-(q-1)(2j-2+m)x)\prod_{j=1}^{k}\left(1-\sum_{i=1}^{j}\frac{(2i-2+m)x}{1-(q-1)(2i-2+m)x}\right)}, r=0. \tag{7}$$

Proof. Consider the refinement $P(x;k|i)$ of $P(x;k)$ defined by $P(x;k|i)=\sum_{n\ge k}P(n,k|i)x^n$ for $1\le i\le k$. First suppose $k>r\ge 0$. Then multiplying both sides of (3) and (4) by x^n, and summing over $n\ge k$, we have

$$P(x;k|i)=(2i-2+m)xP(x;k)+(q-1)(2i-2+m)xP(x;k|i),$$
$$P(x;k|k)=xP(x;k-1)+(2k-2+m)xP(x;k)+(q-1)(2k-2+m)xP(x;k|k),$$

which is equivalent to

$$P(x;k|i)=\frac{(2i-2+m)x}{1-(q-1)(2i-2+m)x}P(x;k), \qquad 1\le i<k \text{ and } k\ge r, \tag{8}$$
$$P(x;k|k)=\frac{x}{1-(q-1)(2k-2+m)x}(P(x;k-1)+(2k-2+m)P(x;k)), k>r.$$

By the fact $P(x;k)=\sum_{i=1}^{k}P(x;k|i)$, we then have

$$P(x;k)=P(x;k)\sum_{i=1}^{k}\frac{(2i-2+m)x}{1-(q-1)(2i-2+m)x}+\frac{x}{1-(q-1)(2k-2+m)x}P(x;k-1),$$

which leads to

$$P(x;k)=\frac{x}{(1-(q-1)(2k-2+m)x)\left(1-\sum_{i=1}^{k}\frac{(2i-2+m)x}{1-(q-1)(2i-2+m)x}\right)}P(x;k-1).$$

Iterating the last equality gives

$$P(x;k)=\frac{x^{k-r}}{\prod_{j=r+1}^{k}(1-(q-1)(2j-2+m)x)\prod_{j=r+1}^{k}\left(1-\sum_{i=1}^{j}\frac{(2i-2+m)x}{1-(q-1)(2i-2+m)x}\right)}P(x;r), k>r. \tag{9}$$

We now determine the case when $k=r$. Note that by (5), we have

$$P(x;r|r)=\frac{x^r+(2r-2+m)xP(x;r)}{1-(q-1)(2r-2+m)x}, \qquad r>0.$$

The last equality combined with (8) when $k=r$ yields, by addition, the formula

$$P(x;r)=\frac{x^r}{(1-(q-1)(2r-2+m)x)\left(1-\sum_{i=1}^{r}\frac{(2i-2+m)x}{1-(q-1)(2i-2+m)x}\right)}, r>0, \tag{10}$$

upon solving for $P(x;r)$. Formula (6) now follows from (9) and (10), while (7) follows from the $r=0$ case of (9), upon noting $P(x;0)=1$.

Taking $q=1$ in Theorem 1 yields

$$\sum_{n\ge k}P_1^{(r)}(n,k,m/2)x^n=\sum_{n\ge k}JS_r(n,k;m-1)x^n=\frac{x^k}{\prod_{j=r}^{k}(1-j(j-1+m)x)}, k\ge r\ge 0,$$

which agrees with (2). Taking the partial derivative with respect to q in (6), and setting $q=1$, gives for $r\ge 1$,

$$\begin{aligned}\frac{\partial}{\partial q}P(x;k)\mid_{q=1}&=\frac{x^k}{\prod_{j=r}^{k}(1-j(j-1+m)x)}\Bigg(\sum_{j=r}^{k}(2j-2+m)x\\&\quad+\sum_{j=r}^{k}\sum_{i=1}^{j}\frac{(2i-2+m)^2x^2}{1-j(j-1+m)x}\Bigg)\\&=\frac{x^k}{\prod_{j=r}^{k}(1-j(j-1+m)x)}\Bigg(k(k-1+m)x\\&\quad-(r-1)(r-2+m)x\\&\quad+\sum_{j=r}^{k}\frac{j(2(j-1)(3m+2j-1)+3m^2)x^2}{3(1-j(j-1+m)x)}\Bigg).\end{aligned}$$

Extracting the coefficient of x^n in the last formula, and considering separately the case when $r=0$, yields the following result.

Corollary 1. *If $n > k \geq r \geq 0$, then the total number of levels in all of the r-Jacobi-Stirling partitions of $[\pm n]$ having $m/2$ zero blocks and k nonzero blocks is given by*

$$(k(k-1+m)-(1-\delta_{r,0})(r-1)(r-2+m))JS_r(n-1,k;m-1)$$
$$+\sum_{j=r}^{k}\frac{j(2(j-1)(3m+2j-1)+3m^2)}{3}\sum_{\ell=0}^{n-k-2}JS_r(n-\ell-2,k;m-1)j^{\ell}(j-1+m)^{\ell}.$$

Proof. It is also possible to provide a combinatorial proof of this result as follows. Let $\lambda \in \mathscr{P}(n,k)$ and suppose that the sequential representation of λ is given by $w = 12\cdots(r-1)rw^{(r)}\cdots kw^{(k)}$, where each $w^{(i)}$ is i-ary for $r \leq i \leq k$. Assume for now $r > 0$. Note that a level occurs each time the first letter of some $w^{(j)}$ is j. If this letter corresponds to the p-th position within w, then the elements $\pm p$ may be arranged in $2(j-1)+m$ ways. Deleting $\pm p$ results in a member of $\mathscr{P}(n-1,k)$ (on the set $[\pm n]-\{p,-p\}$) and considering all possible j gives

$$\sum_{j=r}^{k}(2j-2+m) = k(k-1+m)-(r-1)(r-2+m)$$

ways to arrange the elements $\pm p$. Thus, there are $(k(k-1+m)-(r-1)(r-2+m))JS_r(n-1,k;m-1)$ levels of the stated form, which accounts for the first term of the formula when $r > 0$. A similar argument applies if $r = 0$.

We now count levels where both letters belong to $w^{(j)}$ for some j. Suppose that $w^{(j)}$ can be written as $w^{(j)} = \alpha ii\beta$ for some $i \in [j]$, where α and β are possibly empty. Let ℓ denote the length of α, where $0 \leq \ell \leq n-k-2$. Then there are

$$\sum_{i=1}^{j}(2i-2+m)^2 = \frac{j(2(j-1)(3m+2j-1)+3m^2)}{3}$$

possibilities for arranging the elements of $[\pm n]$ corresponding to i,i in $w^{(j)}$ as i varies and $[j(j-1+m)]^{\ell}$ ways in which to arrange the elements corresponding to the letters of α. Deleting α, and the two letters that follow, results in a partition enumerated by $JS_r(n-\ell-2,k;m-1)$. Considering all possible j and ℓ thus yields the second part of the expression above for the total number of levels, which is seen to apply also when $r = 0$.

Let $P_{p,q}^{(r)}(n,k,m/2)$ denote the joint distribution on $\mathscr{P}(n,k)$ for the statistics recording the number of elements in zero blocks and levels. Reasoning as in the proof of Theorem 1 shows further that

$$\sum_{n\geq k}P_{p,q}^{(r)}(n,k,m/2)x^n = \frac{x^k}{\prod_{j=r}^{k}(1-(q-1)(2j-2+pm)x)\prod_{j=r}^{k}\left(1-\sum_{i=1}^{j}\frac{(2i-2+pm)x}{1-(q-1)(2i-2+pm)x}\right)},$$

if $r > 0$, with a comparable formula if $r = 0$. Differentiating with respect to p, setting $p = 1$ and extracting the coefficient of x^n yields the following result.

Corollary 2. *If $n > k \geq r \geq 0$, then the sum of the cardinalities of the zero blocks within all members of $\mathscr{P}(n,k)$ is given by*

$$m\sum_{j=r}^{k}\sum_{\ell=0}^{n-k-1}JS_r(n-\ell-1,k;m-1)j^{\ell+1}(j-1+m)^{\ell}.$$

Proof. One can also give a combinatorial proof as follows. Let $\mathscr{P}^*(n,k)$ denote the set of "marked" members of $\mathscr{P}(n,k)$ wherein the first zero block contains at least one negative element with one such element marked. By symmetry, it suffices to show

$$|\mathscr{P}^*(n,k)| = \sum_{j=r}^{k}j\sum_{\ell=0}^{n-k-1}JS_r(n-\ell-1,k;m-1)j^{\ell}(j-1+m)^{\ell}.$$

Let $\rho \in \mathscr{P}^*(n,k)$ have associated sequence $w = 12\cdots(r-1)rw^{(r)}\cdots kw^{(k)}$. To find $|\mathscr{P}^*(n,k)|$, we count members of $\mathscr{P}^*(n,k)$ whose marked element $-t$ is such that the $(\ell+1)$-st position of $w^{(j)}$ for some $r \leq j \leq k$ and $0 \leq \ell \leq n-k-1$ corresponds to the t-th letter of w. Then there are j choices concerning the position of the element t (as it can go in any one of j nonzero blocks), $j^{\ell}(j-1+m)^{\ell}$ possibilities for the positions of $\pm s$ for $t-\ell \leq s \leq t-1$, and $JS_r(n-\ell-1,k;m-1)$ ways in which to arrange the remaining elements of $[\pm n]$. Considering all possible j and ℓ implies $|\mathscr{P}^*(n,k)|$ is as stated, which completes the proof.

3.2 Counting by descents.

Let $Q(n,k) = Q_q^{(r)}(n,k,m/2)$ be the distribution polynomial on $\mathscr{P}(n,k)$ for the statistic recording the number of descents. Let $Q(n,k|i) = Q_q^{(r)}(n,k,m/2|i)$ denote the distribution for descents restricted to $\mathscr{P}(n,k|i)$ for $1 \leq i \leq k$. Put $Q(n,k) = Q(n,k|i) = 0$ if $k < r$ or $n < k$. The $Q(n,k|i)$ are determined recursively as follows.

Lemma 2. *We have*

$$Q(n,k|i) = (2i-2+m)\sum_{j=1}^{i}Q(n-1,k|j)+q(2i-2+m)\sum_{j=i+1}^{k}Q(n-1,k|j), \quad 1 \leq i < k \text{ and } k \geq r, \tag{11}$$

$$Q(n,k|k) = Q(n-1,k-1)+(2k-2+m)Q(n-1,k), \quad n \geq k > r, \tag{12}$$

and

$$Q(n,r|r) = \delta_{n,r}+(2r-2+m)Q(n-1,r), \quad n \geq r \geq 1, \tag{13}$$

with $Q(0,0) = \delta_{r,0}$ and $Q(n,0) = 0$ if $n \geq 1$ for all r.

Proof. Let $\lambda \in \mathscr{P}(n,k|i)$, where $1 \leq i \leq k$. If $i < k$, then $Q(n,k|i) = 0$ when $n = k$, so assume $n > k$. Considering the penultimate letter j gives

$$(2i-2+m)\sum_{j=1}^{i}Q(n-1,k|j)$$

possibilities if $j \le i$ and

$$q(2i-2+m)\sum_{j=i+1}^{k} Q(n-1,k|j)$$

possibilities if $j > i$ which implies (11), for in the latter case a descent occurs between the last two letters. If $i = k > r$, then the last letter can never create a descent and there are $Q(n-1,k-1)$ or $(2k-2+m)Q(n-1,k)$ possibilities depending on whether or not the final letter is the only one of its kind, which gives (12). If $i = k = r$, then there are $\delta_{n,r}$ possible partitions if the final r is the only r and $(2r-2+m)Q(n-1,r)$ partitions otherwise, which completes the proof.

Define $Q(x;k) = \sum_{n\ge k} Q(n,k)x^n$. We have the following explicit formula.

Theorem 2.*Let $k \ge r \ge 0$. Then*

$$Q(x;k) = \frac{x^k}{\prod_{j=r}^{k}\left(1-(2j-2+m)x-\sum_{i=1}^{j-1}\frac{(2i-2+m)xq}{\prod_{\ell=1}^{i}(1-(1-q)(2\ell-2+m)x)}\right)}, r \ge 1, \qquad (14)$$

and

$$Q(x;k) = \frac{x^k}{\prod_{j=1}^{k}\left(1-(2j-2+m)x-\sum_{i=1}^{j-1}\frac{(2i-2+m)xq}{\prod_{\ell=1}^{i}(1-(1-q)(2\ell-2+m)x)}\right)}, \qquad r = 0. \qquad (15)$$

*Proof.*Let $Q(x;k|i) = \sum_{n\ge k} Q(n,k|i)x^n$, where $1 \le i \le k$. If $k > r$, then recurrences (11) and (12) may be rewritten as

$$\begin{aligned}
Q(x;k|i) &= (2i-2+m)x\sum_{j=1}^{i} Q(x;k|j) + (2i-2+m)xq\sum_{j=i+1}^{k} Q(x;k|j)\\
&= (1-q)(2i-2+m)x\sum_{j=1}^{i} Q(x;k|j) + (2i-2+m)xqQ(x;k),\ 1 \le i < k,\\
Q(x;k|k) &= xQ(x;k-1) + (2k-2+m)xQ(x;k). \qquad (16)
\end{aligned}$$

Thus, for $1 \le i < k$, we have

$$Q(x;k|i) = \frac{(1-q)(2i-2+m)x}{1-(1-q)(2i-2+m)x}\sum_{j=1}^{i-1} Q(x;k|j) + \frac{(2i-2+m)xq}{1-(1-q)(2i-2+m)x}Q(x;k). \qquad (17)$$

Then we have

$$Q(x;k|i) = \frac{(2i-2+m)xq}{\prod_{j=1}^{i}(1-(1-q)(2j-2+m)x)}Q(x;k), 1 \le i < k. \qquad (18)$$

To show (18), note first that it holds for $i = 1$, by (17). To complete an inductive proof of (18) using (17), it suffices to show for $i > 1$,

$$\begin{aligned}
&\frac{(2i-2+m)xq}{\prod_{j=1}^{i-1}(1-(1-q)(2j-2+m)x)}\\
&= (2i-2+m)x\sum_{j=1}^{i-1}\frac{(1-q)(2j-2+m)xq}{\prod_{\ell=1}^{j}(1-(1-q)(2\ell-2+m)x)}\\
&\quad +(2i-2+m)xq,
\end{aligned}$$

which may be rewritten as

$$\sum_{j=1}^{i-1}(1-(1-q)(2j-2+m)x) - 1$$

$$= \sum_{j=1}^{i-1} -(1-q)(2j-2+m)x\prod_{\ell=j+1}^{i-1}(1-(1-q)(2\ell-2+m)x).$$

The last equality is a special case of the identity $\prod_{j=1}^{a}(1+c_j) - 1 = \sum_{j=1}^{a} c_j \prod_{\ell=j+1}^{a}(1+c_\ell)$, which can be shown by considering the first time a c_j factor is chosen when expanding the product $\prod_{j=1}^{a}(1+c_j)$.

Summing (18) over $1 \le i < k$, and using the fact $Q(x;k) = \sum_{i=1}^{k} Q(x;k|i)$, we obtain

$$Q(x;k) - Q(x;k|k) = Q(x;k)\sum_{i=1}^{k-1}\frac{(2i-2+m)xq}{\prod_{j=1}^{i}(1-(1-q)(2j-2+m)x)}.$$

By (16), we have

$$\begin{aligned}
Q(x;k) &= Q(x;k)\sum_{i=1}^{k-1}\frac{(2i-2+m)xq}{\prod_{j=1}^{i}(1-(1-q)(2j-2+m)x)}\\
&\quad + xQ(x;k-1) + (2k-2+m)xQ(x;k),
\end{aligned}$$

which leads to

$$Q(x;k) = \frac{x}{1-(2k-2+m)x-\sum_{i=1}^{k-1}\frac{(2i-2+m)xq}{\prod_{j=1}^{i}(1-(1-q)(2j-2+m)x)}}Q(x;k-1), \quad k > r. \qquad (19)$$

If $k = r$, then (13) implies

$$Q(x;r|r) = x^r + (2r-2+m)xQ(x;r), \qquad r > 0.$$

Proceeding as before gives

$$Q(x;r) = \frac{x^r}{1-(2r-2+m)x-\sum_{i=1}^{r-1}\frac{(2i-2+m)xq}{\prod_{j=1}^{i}(1-(1-q)(2j-2+m)x)}},$$

from which (14) follows from (19). On the other hand, formula (15) follows from (19) and noting $Q(x;0) = 1$.

Theorem 2 implies $Q(x;k) = \frac{x^k}{\prod_{j=r}^{k}(1-j(j-1+m)x)}$ when $q = 1$, as expected. Taking the derivative of $Q(x;k)$ with respect to q, and setting $q = 1$, yields

$$\begin{aligned}
\frac{\frac{\partial}{\partial q}Q(x;k)\mid_{q=1}}{Q(x;k)\mid_{q=1}} &= \sum_{j=r}^{k}\sum_{i=1}^{j-1}\frac{(2i-2+m)x}{1-j(j-1+m)x} - \sum_{j=r}^{k}\sum_{i=1}^{j-1}\sum_{\ell=1}^{i}\frac{(2i-2+m)(2\ell-2+m)x^2}{1-j(j-1+m)x}\\
&= x\sum_{j=r}^{k}\frac{(j-1)(j-2+m)}{1-j(j-1+m)x} - x^2\sum_{j=r}^{k}\frac{(6m+3j-5)\binom{j}{3}+m^2\binom{j}{2}}{1-j(j-1+m)x}\\
&= x\sum_{j=r}^{k}(j-1)(j-2+m) + x^2\sum_{j=r}^{k}\frac{(3j-1)\binom{j}{3}+mj(j-1)^2+m^2\binom{j}{2}}{1-j(j-1+m)x}
\end{aligned}$$

for $r > 0$, with the same holding for $r = 0$ except that both sums have lower index $j = 1$. Extracting the coefficient of x^n gives the following result.

Corollary 3. *If $n > k \geq r \geq 0$, then the total number of descents in all members of $\mathscr{P}(n,k)$ is given by*

$$\left[2\binom{k}{3} + m\binom{k}{2} - (1-\delta_{r,0})\left(2\binom{r-1}{3} + m\binom{r-1}{2}\right)\right] \times JS_r(n-1,k;m-1)$$
$$+ \sum_{j=r}^{k}\left[(3j-1)\binom{j}{3} + mj(j-1)^2 + m^2\binom{j}{2}\right]$$
$$\times \sum_{\ell=0}^{n-k-2} JS_r(n-\ell-2,k;m-1)j^{\ell}(j-1+m)^{\ell}.$$

Proof. We provide a direct proof in the $r = 0$ case as follows (where appropriate modifications can be made for the case $r > 0$). Let λ denote an arbitrary member of $\mathscr{P}(n,k)$ where $r = 0$ having associated sequence $w = 1w^{(1)} \cdots kw^{(k)}$ with each $w^{(i)}$ i-ary. We first count descents in λ where the first letter in the descent is the leftmost of its kind. Given $j \in [k]$, there are

$$\sum_{i=1}^{j-1}(2i-2+m) = (j-1)(j-2+m)$$

options concerning the arrangement of the pair of elements corresponding to the second letter of the descent if j is to be the first letter. Considering all possible j gives

$$\sum_{j=2}^{k}(j-1)(j-2+m) = 2\binom{k}{3} + m\binom{k}{2}$$

such descents arising from each member of $\mathscr{P}(n-1,k)$, which accounts for the first part of the formula above. Now suppose both letters a and b of the descent belong to $w^{(j)}$, where a is the $(\ell+1)$-st letter of $w^{(j)}$ for some $0 \leq \ell \leq n-k-2$. Then there are

$$\sum_{a=1}^{j}\sum_{b=1}^{a-1}(2a-2+m)(2b-2+m)$$

$$= (3j-1)\binom{j}{3} + mj(j-1)^2 + m^2\binom{j}{2}$$

possibilities for the arrangement of the elements associated with the letters of the descent. There are $JS_0(n-\ell-2,k;m-1)[j(j-1+m)]^{\ell}$ ways in which to arrange the remaining elements of $[\pm n]$ and considering all possible j and ℓ gives the second part of the formula and completes proof.

3.3 Counting by ascents

Let $R(n,k) = R_q^{(r)}(n,k,m/2)$ be the distribution on $\mathscr{P}_{n,k}$ for the statistic recording the number of ascents and $R(n,k|i)$ denote the restriction of $R(n,k)$ to those partitions ending in i. Similar to before, we have the following recurrence relations satisfied by the $R(n,k|i)$.

Lemma 3. *We have*

$$R(n,k|i) = q(2i-2+m)\sum_{j=1}^{i-1}R(n-1,k|j) + (2i-2+m)\sum_{j=i}^{k}R(n-1,k|j), \quad 1 \leq i < k \text{ and } k \geq r, \tag{20}$$

$$R(n,k|k) = qR(n-1,k-1) + q(2k-2+m)R(n-1,k) + (1-q)(2k-2+m)R(n-1,k|k), \quad n \geq k > r, \tag{21}$$

and

$$R(n,r|r) = q^{r-1}\delta_{n,r} + q(2r-2+m)R(n-1,r) + (1-q)(2r-2+m)R(n-1,r|r), \quad n \geq r \geq 1, \tag{22}$$

with $R(0,0) = \frac{\delta_{r,0}}{q}$ and $R(n,0) = 0$ if $n \geq 1$ for all r.

Let

$$R(x;k) = \sum_{n \geq k} R(n,k)x^n$$

and

$$R(n,k|i) = \sum_{n \geq k} R(n,k|i)x^n.$$

Then (20) and (21) may be rewritten in terms of generating functions as

$$R(x;k|i) = (2i-2+m)xR(x;k) + (q-1)(2i-2+m)x\sum_{j=1}^{i-1}R(x;k|j), 1 \leq i < k \text{ and } k \geq r, \tag{23}$$

and

$$R(x;k|k) = \frac{xqR(x;k-1) + (2k-2+m)xqR(x;k)}{1+(q-1)(2k-2+m)x}, \; k > r. \tag{24}$$

We have the following recurrence relation satisfied by $R(x;k)$.

Lemma 4. *If $k > r \geq 0$, then*

$$R(x;k) = \frac{\frac{xq}{1+(q-1)(2k-2+m)x}}{1-\frac{(2k-2+m)xq}{1+(q-1)(2k-2+m)x} - \sum_{\ell=1}^{k-1}(q-1)^{\ell-1}x^{\ell}u_{k,\ell}}R(x;k-1), \tag{25}$$

where $u_{k,\ell} = \sum_{1 \leq j_1 < j_2 < \cdots < j_\ell \leq k-1}\prod_{s=1}^{\ell}(2j_s - 2 + m)$.

Proof. Let $R(x;k|i) = \alpha_i R(x;k)$. Then (23) gives

$$\alpha_i = a_i x + (q-1)a_i x\sum_{j=1}^{i-1}\alpha_j, \qquad 1 \leq i < k \text{ and } k \geq r,$$

with $a_i = 2i-2+m$. By induction on i, we have

$$\alpha_i = \sum_{\ell \geq 1}\sum_{1 \leq j_1 < j_2 < \cdots < j_\ell = i}(q-1)^{\ell-1}x^{\ell}\prod_{s=1}^{\ell}a_{j_s}, \qquad 1 \leq i < k,$$

which leads to

$$\sum_{i=1}^{k-1}\alpha_i = \sum_{\ell \geq 1}(q-1)^{\ell-1}x^{\ell}\sum_{1 \leq j_1 < j_2 < \cdots < j_\ell \leq k-1}\prod_{s=1}^{\ell}(2j_s - 2 + m).$$

Hence,

$$R(x;k) - R(x;k|k) = R(x;k)\sum_{\ell=1}^{k-1}(q-1)^{\ell-1}x^{\ell}$$

$$\times \sum_{1\le j_1<j_2<\cdots<j_\ell\le k-1} \prod_{s=1}^{\ell}(2j_s-2+m).$$

Formula (25) now follows from (24), upon solving for $R(x;k)$.

Theorem 3. *Let $k\ge 1$ and $r\ge 0$, with $k\ge r$. Then*

$$R(x;k)=\frac{x^k q^{k-1}}{\prod_{j=r}^{k}\left[1-\frac{1}{q-1}\left(\prod_{i=0}^{j-1}(1+(q-1)(2i+m)x)-1\right)\right]}. \tag{26}$$

Proof. We seek an expression for $u_{k,\ell}$ for $k\ge 1$ and $1\le \ell\le k-1$. From the definition of the $u_{k,\ell}$, we have the recurrence

$$u_{k,\ell}=(2k-4+m)u_{k-1,\ell-1}+u_{k-1,\ell}, \qquad k>\ell\ge 1,$$

with $u_{k,0}=1$ and $u_{k,k}=0$ for all $k\ge 1$. Define $U_k(t)=\sum_{\ell=0}^{k} u_{k,\ell}t^\ell$. Then

$$U_k(t)=(2k-4+m)tU_{k-1}(t)+U_{k-1}(t), \qquad k\ge 2,$$

with $U_1(t)=1$, which leads to

$$U_k(t)=\prod_{j=0}^{k-2}(1+(2j+m)t).$$

By Lemma 4, we have

$$R(x;k)=\frac{\frac{xq}{1+(q-1)(2k-2+m)x}}{1-\frac{(2k-2+m)xq}{1+(q-1)(2k-2+m)x}-\frac{1}{q-1}(U_k((q-1)x)-1)}R(x;k-1),$$

which is equivalent to

$$R(x;k)=\frac{xq}{1-(2k-2+m)x-\frac{1+(q-1)(2k-2+m)x}{q-1}(U_k((q-1)x)-1)}R(x;k-1).$$

Since $U_{k+1}((q-1)x)=(1+(q-1)(2k-2+m)x)U_k((q-1)x)$, we get

$$R(x;k)=\frac{xq}{1-\frac{1}{q-1}(U_{k+1}((q-1)x)-1)}R(x;k-1), k>r\ge 0. \tag{27}$$

If $k=r$, then proceeding in a similar manner as before yields

$$R(x;r)=\frac{x^r q^{r-1}}{1-\frac{1}{q-1}(U_{r+1}((q-1)x)-1)}, \qquad r\ge 1. \tag{28}$$

Formula (26) now follows from iterating (27), and using (28) if $r\ge 1$, or by noting $R(x;0)=\frac{1}{q}$ if $r=0$.

Letting $q=1$ in (26) gives

$$\begin{aligned} R(x;k) &= \frac{x^k}{\prod_{j=r}^{k}\left[1-\lim_{q\to 1}\left(\frac{\prod_{i=0}^{j-1}(1+(q-1)(2i+m)x)-1}{q-1}\right)\right]} \\ &= \frac{x^k}{\prod_{j=r}^{k}\left[1-\sum_{i=0}^{j-1}(2i+m)x\right]} \\ &= \frac{x^k}{\prod_{j=r}^{k}(1-j(j-1+m)x)}, \end{aligned}$$

as expected, by L'Hôpital's rule.

Taking the derivative with respect to q, and setting $q=1$, gives

$$\frac{\partial}{\partial q}R(x;k)\mid_{q=1}=$$

$$\frac{x^k}{\prod_{j=r}^{k}(1-j(j-1+m)x)}\left(k-1+\sum_{j=r}^{k}\frac{F(j)}{1-j(j-1+m)x}\right),$$

where $F(0)=0$ and $F(j)$ for $j\ge 1$ is defined as

$$F(j)=\lim_{q\to 1}\frac{\partial}{\partial q}\left[\frac{\prod_{i=0}^{j-1}(1+(q-1)(2i+m)x)-1}{q-1}\right].$$

By two applications of L'Hôpital's rule, we have

$$\begin{aligned} F(j) &= \lim_{q\to 1}\left[\frac{(q-1)\Sigma_{i=0}^{j-1}(2i+m)x\Pi_{\ell=0,\ell\neq i}^{j-1}(1+(q-1)(2\ell+m)x)-\Pi_{i=0}^{j-1}(1+(q-1)(2i+m)x)+1}{(q-1)^2}\right] \\ &= \frac{1}{2}\frac{\partial}{\partial q}\left[\sum_{i=0}^{j-1}(2i+m)x\prod_{\ell=0,\ell\neq i}^{j-1}(1+(q-1)(2\ell+m)x)\right]_{q=1} = \frac{1}{2}\left(\sum_{i=0}^{j-1}(2i+m)x\sum_{\ell=0,\ell\neq i}^{j-1}(2\ell+m)x\right) \\ &= \frac{x^2}{2}\sum_{i=0}^{j-1}(2i+m)[(j+m)(j-1)-2i]=(3j-1)\binom{j}{3}x^2+mj(j-1)^2x^2+m^2\binom{j}{2}x^2. \end{aligned}$$

Thus, extracting the coefficient of x^n in the formula for $\frac{\partial}{\partial q}R(x;k)$ at $q=1$ implies the following result.

Corollary 4. *If $n>k\ge r\ge 0$, then the total number of ascents in all members of $\mathscr{P}(n,k)$ is given by*

$$\begin{aligned} &(k-1)JS_r(n,k;m-1) \\ &\quad+\sum_{j=r}^{k}\left[(3j-1)\binom{j}{3}+mj(j-1)^2+m^2\binom{j}{2}\right] \\ &\times\sum_{\ell=0}^{n-k-2}JS_r(n-\ell-2,k;m-1)j^\ell(j-1+m)^\ell. \end{aligned}$$

It is possible to provide a combinatorial proof of this result similar to before.

4 Perspective

In this paper, we have computed the generating functions for the distribution of the statistics on $\mathscr{P}(n,k)$ that record the number of levels, descents and ascents. Our formulas led to expressions for the average values of these statistics on $\mathscr{P}(n,k)$, which were afforded combinatorial explanations. From the formulas for the q-derivative of the respective generating functions at $q=1$, one may find asymptotic estimates for the total number of occurrences of these statistics on $\mathscr{P}(n,k)$ for a fixed k as n increases without bound, which we leave to the reader.

Furthermore, variations on the statistics defined above are possible. For example, one can say that a level occurs at index i, where $1\le i\le n-1$, if both i and $i+1$ are positioned exactly the same way with respect to the blocks of $\pi \in \mathscr{P}(n,k)$. That is, a level occurs if the elements i and $i+1$ both belong to the same block of π (either zero or nonzero), with $-i$ and $-(i+1)$ also belonging to the same block. Let $a_q(n,k)$ denote the distribution on $\mathscr{P}(n,k)$ where $r=0$ for this new definition of the level statistic. Applying the methods from the previous section, one can show

$$\sum_{n\geq k} a_q(n,k)x^n = \frac{x^k}{\prod_{j=1}^{k}\left(1-\frac{j(j-1+m)x}{1+(1-q)x}\right)}, \qquad k\geq 1,$$

which reduces to (2) when $q=1$. Similarly, one can define a comparable statistic recording descents or ascents on $\mathscr{P}(n,k)$ if one were to order all of the possible options concerning the position of the pair $\{i,-i\}$ for some i, where one may assume that a member of this pair belongs to the j-th nonzero block with the other lying in some block to its left (provided $\{i,-i\}$ does not start a new nonzero block). Thus, for example, it would be possible now for a descent or ascent to occur even in the case when i and $i+1$ belong to the same nonzero block and none of the blocks to the right of this block contain either $-i$ or $-(i+1)$. One can derive comparable expressions for the generating functions of the respective distributions on $\mathscr{P}(n,k)$. However, we found the statistics featured in the current paper to be easier to define, to possess more interesting properties and to provide a more natural extension of the typical rise, fall and level statistics on k-ary sequences than the variations described above.

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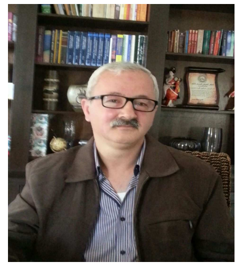

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